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PROBABILITY THEORY AND STOCHASTIC PROCESSES

Moment Inequality for the Martingale Square Function

by

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Summary. Consider the sequence $(C_n)_{n\geq 1}$ of positive numbers defined by $C_1 = 1$ and $C_{n+1} = 1 + C_n^2/4$, $n = 1, 2, \ldots$. Let M be a real-valued martingale and let $S(M)$ denote its square function. We establish the bound

$$\mathbb{E}|M_n| \leq C_n \mathbb{E} S_n(M), \qquad n = 1, 2, \ldots,$$

and show that for each n, the constant C_n is the best possible.

1. Introduction. Square function inequalities play an important role in both classical and noncommutative probability theory, harmonic analysis, potential theory and many other areas of mathematics. The purpose of this paper is to establish a sharp bound between the first moments of a martingale and its square function, with a constant depending on the length of the martingale.

We start with some definitions. Throughout the paper, $(\Omega, \mathcal{F}, \mathbb{P})$ will be a given probability space, filtered by a nondecreasing family $(\mathcal{F}_n)_{n=0}^{\infty}$ of sub-σ-fields of $\mathcal{F}$. Let $M = (M_n)_{n\geq 1}$ be a real-valued martingale adapted to $(\mathcal{F}_n)_{n\geq 1}$ and let $dM = (dM_n)_{n\geq 1}$ stand for its difference sequence:

$$dM_1 = M_1, \qquad dM_n = M_n - M_{n-1}, \quad n = 2, 3, \ldots.$$

A martingale M is called *simple* if for any $n = 1, 2, \ldots$ the random variable M_n takes only a finite number of values. For any nonnegative integer n, let $S_n(M)$ be given by

$$S_n(M) = \Big(\sum_{k=1}^{n} |dM_k|^2 \Big)^{1/2}.$$

Then one defines the *square function* $S(M)$ by $S(M) = \lim_{n\to\infty} S_n(M)$.

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For $p > 0$, let $\|M\|_p = \sup_n \|M_n\|_p = \sup_n(\mathbb{E}|M_n|^p)^{1/p}$. We will be interested in inequalities between the moments of M and $S(M)$. Such estimates (without using the martingale concept or word) appeared for the first time in the classical papers of Khintchine [9], Littlewood [10], Marcinkiewicz [11] and Paley [14]. For more recent results in this direction, we refer the interested reader to the survey [3] by Burkholder or the monograph [13] by the author.

For example, the inequality

$$c_p\|M\|_p \le \|S(M)\|_p \le C_p\|M\|_p \qquad \text{if } 1 < p < \infty, \tag{1.1}$$

valid for all martingales, was proved by Burkholder in [2]. Later, Burkholder refined his proof and showed that (cf. [3]) the inequality holds with $c_p^{-1} = C_p = p^* - 1$, where $p^* = \max\{p, p/(p-1)\}$. Furthermore, the constant c_p is optimal for $p \ge 2$, C_p is the best for $1 < p \le 2$ and the proof carries over to the case of martingales taking values in a separable Hilbert space. The right inequality of (1.1) does not hold for general martingales if $p \le 1$, nor does the left one if $p < 1$. It was shown by the author in [12] that $c_1 = 1/2$ is the best constant. In the remaining cases the optimal values of c_p and C_p are not known. Let us mention here a related result of Cox [5], who identified the best constant in the corresponding weak type inequality: we have

$$\mathbb{P}(S(M) \ge 1) \le \sqrt{e}\,\|M\|_1 \tag{1.2}$$

(see also Bollobás [1]). Our objective is to compare the first moments of M_n and $S_n(M)$ for each fixed n, and the novelty lies in the sharp dependence of the constants on n. Here is the precise statement of our main result.

Theorem 1.1. *Let $(C_n)_{n\ge1}$ be the sequence of numbers given by $C_1 = 1$ and $C_{n+1} = 1 + C_n^2/4$, $n = 1, 2, \ldots$. Then for any real martingale M and any $n \ge 1$,*

$$\|M_n\|_1 \le C_n\|S_n(M)\|_1. \tag{1.3}$$

For each n the constant C_n is the best possible.

Unfortunately, there seems to be no explicit formula for the sequence $(C_n)_{n\ge1}$. However, an easy analysis shows that this sequence increases to 2; thus, letting $n \to \infty$ in (1.3) we obtain the inequality $\|M\|_1 \le 2\|S(M)\|_1$, proved by the author in [12]. It is worth mentioning here the following version of (1.3) in the reverse direction: in the proof of (1.2), Cox [5] actually showed the more exact estimate

$$\mathbb{P}(S_n(M) \ge 1) \le \left(\frac{n}{n-1}\right)^{(n-1)/2} \mathbb{E}|M_n|, \qquad n \ge 1.$$

A few words about the method of proof are in order. The technique used in this paper has its roots in the theory of moments, introduced by Kemperman in [8]. This approach, when applicable, always leads to a sharp inequality and provides an example of a martingale attaining equality or nearly so. The argument rests on the construction of an appropriate sequence of special functions and is closely related to a method invented by Burkholder in [4] (see the discussion in Section 2). In the literature, there are several papers in which the method of moments has been successfully implemented. We refer the reader to the works of Cox [5], [6], Cox and Kemperman [7] and Kemperman [8]. The main problem is that the technique has the drawback of computational complexity, which sometimes makes it difficult to push the calculations through. This happens also in our case, and to overcome this difficulty, we slightly modify the method, which enables us to simplify the technicalities.

We have organized the paper as follows. The next section contains the description of the approach which is used in the proof of Theorem 1.1. In Section 3 we exploit this method: we introduce a family of special functions and establish appropriate statements about them. This enables us to deduce the desired bound (1.3). The final part of the paper is devoted to proving the optimality of the constant C_n.

2. On the method of proof. Let $V : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be a given function, let n be a fixed integer and suppose that we are interested in showing that for any simple martingale M,

$$\mathbb{E}V(M_n, S_n(M)) \leq 0. \tag{2.1}$$

For instance, the choice $V(x,y) = |x| - C_n|y|$ leads to moment estimates studied in this paper. Conditioning on M_1 if necessary, we may and will assume that the starting variable of M is constant almost surely. To study (2.1), we introduce a family of auxiliary functions. Namely, for any $k = 1, 2, \ldots$, let us define $U_k^V : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ by

$$U_k^V(x,y) = \sup\Big\{\mathbb{E}V\Big(M_k, \sqrt{y^2 - x^2 + S_k^2(M)}\Big)\Big\}, \tag{2.2}$$

where the supremum is taken over all simple martingales M starting from x. In the language of these functions, (2.1) can be rephrased as $U_n^V(x,|x|) \leq 0$ for all x. In particular, only the case $y = |x|$ seems to be of importance. However, the inductive step below requires the analysis of U_n^V on its whole domain. Observe that $U_1^V(x,y) = V(x,y)$, since $M_1 \equiv x$ and $\sqrt{y^2 - x^2 + S_1^2(M)} \equiv y$. Moreover, we have

$$U_2^V(x,y) = \sup\{\mathbb{E}V(x+D, \sqrt{y^2+D^2}) : \mathbb{E}D = 0\} \tag{2.3}$$

and, conditioning on M_2,

$$U_{k+1}^V(x,y) = \sup\big\{\mathbb{E}U_k^V(x+D, \sqrt{y^2+D^2}) : \mathbb{E}D = 0\big\} \tag{2.4}$$

for $k = 2, 3, \ldots$. Both equalities (2.3) and (2.4) involve the evaluation of $\sup \mathbb{E}h(D)$ over all centered random variables D, where h is a given function. This is a standard problem of the theory of moments (see Kemperman [8]) and can be solved "graphically" as follows: the required supremum is equal to the height, at location $x = 0$, of the upper boundary of the convex hull of the graph of h. However, for $V(x,y) = |x| - Cy$ the iterative computations of the heights become complicated and simplifying the above approach becomes desirable.

To describe the appropriate modification, let us note the following property of the sequence $(U_k^V)_{k\geq 1}$. Namely, if M is a martingale satisfying $M_1 \equiv x$, then by (2.3) and (2.4),

$$\begin{aligned} U_n^V(x,|x|) &= \mathbb{E}U_n^V(M_1, S_1(M)) \\ &\geq \mathbb{E}U_{n-1}^V(M_2, S_2(M)) \geq \ldots \geq \mathbb{E}U_1^V(M_n, S_n(M)) \\ &= \mathbb{E}V(M_n, S_n(M)). \end{aligned} \tag{2.5}$$

Thus, if we have $U_n^V(x,|x|) \leq 0$ for all x, we indeed get (2.1). The idea is that one may search for other functional sequences (in place of $(U_k^V)_{k\geq 1}$), for which the above chain of inequalities holds true (in the last line, we allow the bound "$\geq$", instead of equality).

Specifically, we have the following statement.

Theorem 2.1. *Let $V : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be a given function and let $n \geq 1$ be a fixed integer. Suppose that $(U_k)_{k=1}^n$ is a sequence of real-valued functions on $\mathbb{R} \times [0,\infty)$ which satisfy the following three conditions:*

(i) *$U_n(x,|x|) \leq 0$ for all $x \in \mathbb{R}$.*
(ii) *$U_1(x,y) \geq V(x,y)$ for all $x \in \mathbb{R}$ and $y \geq 0$.*
(iii) *For each $k = 1, \ldots, n-1$ there is a function $A_k : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ such that if $x, d \in \mathbb{R}$ and $y \geq 0$, then*

$$U_{k+1}(x,y) + A_k(x,y)d \geq U_k\big(x+d, \sqrt{y^2+d^2}\big).$$

Then (2.1) holds true.

Proof. Fix $k \in \{1, \ldots, n-1\}$. Apply (iii) to $x = M_{n-k}$, $y = S_{n-k}(M)$ and $d = M_{n+1-k} - M_{n-k}$, and take the conditional expectation of both sides with respect to $\mathcal{F}_{n-k}$. We obtain

$$U_{k+1}(M_{n-k}, S_{n-k}(M)) \geq \mathbb{E}[U_k(M_{n-k+1}, S_{n-k+1}(M)) \mid \mathcal{F}_{n-k}]$$

and thus, integrating both sides, we get

$$\mathbb{E}U_{k+1}(M_{n-k}, S_{n-k}(M)) \geq \mathbb{E}U_k(M_{n-k+1}, S_{n-k+1}(M)).$$

Combining this with (ii), we see that the chain (2.5) (with inequality in the last line and U_k^V replaced by U_k) is valid. It remains to apply (i) and the claim follows. ∎

This methodology is closely related to the approach invented by Burkholder in [4] (see also [2] for a related technique concerning martingale transforms). Let us say a few words about this interesting connection. Suppose that we are given a function $V : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ and we want to establish the inequality (2.1) for all simple martingales and *for all values of* n. To handle this problem, we apply the formula (2.2) for each n, thus obtaining an infinite sequence $(U_n^V)_{n \geq 1}$. It follows directly from the definition that the sequence is nondecreasing: indeed, if M is any simple martingale, then the sequence $(M_1, M_2, \ldots, M_{n-1}, M_n, M_n, M_{n+1}, M_{n+2}, \ldots)$ is also a simple martingale, so by the definition of U_{n+1}^V,

$$U_{n+1}^V(x, y) \geq \mathbb{E} V(M_n, \sqrt{y^2 - x^2 + S_n^2(M)}).$$

Thus, taking the supremum over all M gives monotonicity. Therefore it makes sense to speak about the limit

$$U^V(x, y) = \lim_{n \to \infty} U_n^V(x, y).$$

Now, if U^V is finite on $\mathbb{R} \times [0, \infty)$, one easily checks that a version of (2.5) holds true (all U_k^V's are replaced by U^V and there is inequality in the last line). These observations lead us to the following analogue of Theorem 2.1, which can be found, in a slightly different form, in [4].

THEOREM 2.2. *Let $V : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a given function. Suppose that U is a real-valued function on $\mathbb{R} \times [0, \infty)$ which satisfies the following three conditions:*

(i) $U(x, |x|) \leq 0$ *for all* $x \in \mathbb{R}$.
(ii) $U(x, y) \geq V(x, y)$ *for all* $x \in \mathbb{R}$ *and* $y \geq 0$.
(iii) *There is a measurable function* $A : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ *such that if* $x, d \in \mathbb{R}$ *and* $y \geq 0$, *then*

$$U(x, y) + A(x, y)d \geq U\big(x + d, \sqrt{y^2 + d^2}\big).$$

Then (2.1) *holds true for all simple martingales* M *and all integers* n.

For further details, examples and the proof of the above statement, we refer the reader to the works [4], [12] and Chapter 8 in [13].

3. Proof of (1.3). For the sake of clarity, we have decided to split this section into two parts. The first subsection contains the proofs of three technical facts which are needed later; in the second part, we introduce appropriate special functions and establish (1.3).

3.1. Technical lemmas. Throughout this subsection, we assume that C is a fixed number in the interval $[1, 2)$.

Lemma 3.1. *For any $x, d \in \mathbb{R}$ such that $|x| \le \sqrt{C-1}$ we have*

$$|x+d| - \sqrt{C}\sqrt{x^2+1+Cd^2} \le \sqrt{C-1}\,(-1+xd).$$

Proof. By continuity, we may assume that $C > 1$ and $|x| < \sqrt{C-1}$. Observe that it suffices to show the weaker bound

$$x + d - \sqrt{C}\sqrt{x^2+1+Cd^2} - \sqrt{C-1}\,(-1+xd) \le 0. \tag{3.1}$$

Indeed, having done this, we substitute $-x, -d$ in place of x, d, obtaining

$$-x - d - \sqrt{C}\sqrt{x^2+1+Cd^2} - \sqrt{C-1}\,(-1+xd) \le 0,$$

and the two inequalities above yield the desired statement. To prove (3.1), consider its left-hand side as a function of d and denote it by $F(d)$. Note that F tends to $-\infty$ as $d \to \pm\infty$: this follows at once from the assumption $|x| < \sqrt{C-1}$. Thus, it suffices to check that $F(d) \le 0$ for all d such that $F'(d) = 0$. It is straightforward to verify that the latter equation is equivalent to

$$d = \left(\frac{x^2+1}{C(C^2-(1-\sqrt{C-1}\,x)^2)}\right)^{1/2}(1-\sqrt{C-1}\,x)$$

and the inequality $F(d) \le 0$ can be rewritten in the form

$$\sqrt{C-1} + x \le \left(\frac{x^2+1}{C}[C^2-(1-\sqrt{C-1}\,x)^2]\right)^{1/2}.$$

However, we have

$$C^2 - (1-\sqrt{C-1}\,x)^2 = \sqrt{C-1}\,(\sqrt{C-1}+x)(C+1-\sqrt{C-1}\,x),$$

so squaring the above inequality and dividing by $\sqrt{C-1}+x$, we obtain the equivalent bound

$$\sqrt{C-1} + x \le \frac{x^2+1}{C}\sqrt{C-1}\,(C+1-\sqrt{C-1}\,x).$$

This estimate, in turn, can be transformed into

$$(x-\sqrt{C-1})(\sqrt{C-1}\,x-1)^2 \le 0,$$

which is of course valid. ∎

Lemma 3.2. *For any $y \ge 0$, $d \in \mathbb{R}$ and $x \ge \sqrt{C-1}\,y$ we have*

$$|x+d| - C\sqrt{y^2+d^2} \le x - 2\sqrt{C-1}\,y + (C-1)d.$$

Proof. If $d \le -x$, the bound is equivalent to

$$-Cd - C\sqrt{y^2+d^2} \le 2(x - \sqrt{C-1}\,y),$$

which holds true: the left-hand side is nonpositive, while the expression on the right is nonnegative. Suppose then that $d > -x$. Then the desired estimate takes the form

$$(2-C)d - C\sqrt{y^2+d^2} \le -2\sqrt{C-1}\,y.$$

Clearly, the left-hand side, considered as a function of d, is concave. A straightforward analysis of its derivative shows that this function attains its maximum at $d = (2-C)y/\big(2\sqrt{C-1}\big)$, and the maximal value is precisely the right-hand side. ∎

LEMMA 3.3. *Assume that the numbers $x, y \geq 0$ and $d \in \mathbb{R}$ satisfy the conditions $x \geq \sqrt{C-1}\,y$ and $|x+d| \leq (C/2)\sqrt{y^2+d^2}$. Then*

$$-\frac{C}{2}\sqrt{(C^2/4+1)(y^2+d^2)-(x+d)^2}-(C-1)d \leq x-2\sqrt{C-1}\,y. \tag{3.2}$$

Proof. If d is nonnegative, then the left-hand side is not larger than $-(C/2)\sqrt{y^2+d^2} \leq -Cy/2$ and the right-hand side is at least $-\sqrt{C-1}\,y \geq -Cy/2$, so the assertion is valid. Let us turn to the case when $d \leq 0$. Assume first that $x \leq Cy/2$; then the discriminant of

$$d \mapsto (C^2/4+1)(y^2+d^2)-(x+d)^2 \tag{3.3}$$

is nonpositive and hence the left-hand side of (3.2) is concave as a function of d (denoted by $G(d)$). However, $G(0) \leq x-2\sqrt{C-1}\,y$, as we have already proved above, and

$$G'(0) = \frac{C}{2}\,\frac{x}{\sqrt{(C^2/4+1)y^2-x^2}}-(C-1).$$

Since $x \geq \sqrt{C-1}\,y$, we obtain

$$G'(0) \geq \frac{C\sqrt{C-1}}{\sqrt{(C-2)^2+4}}-(C-1),$$

and the expression on the right is nonnegative: after some straightforward manipulations, this is equivalent to $(2-C)^3 \geq 0$. Hence $G(d) \leq G(0)$ for $d \leq 0$, which is exactly what we need.

It remains to deal with the case $d \leq 0$ and $x > Cy/2$. The assumption $|x+d| \leq (C/2)\sqrt{y^2+d^2}$ is equivalent to saying that $d \in [d_-, d_+]$, where

$$d_\pm = \frac{-x \pm \frac{C}{2}\sqrt{(1-C^2/4)y^2+x^2}}{1-C^2/4}.$$

This time the discriminant of the binomial (3.3) turns out to be nonnegative, so the function $G(d)$ (the left-hand side of (3.2)) is convex. Thus, all we need is the bound $G(d_\pm) \leq x-2\sqrt{C-1}\,y$. Let us first handle the upper bound for $G(d_-)$. We have $x+d_- \leq 0$, so

$$\begin{aligned} G(d_-)-x+2\sqrt{C-1}\,y &= (2-C)d_-+2\sqrt{C-1}\,y \\ &= \frac{4\big[\,x+\frac{C}{2}\sqrt{(1-C^2/4)y^2+x^2}\,\big]}{2+C}+2\sqrt{C-1}\,y \end{aligned}$$

and the latter expression is an increasing function of x. Thus, we will be done if we show $G(d_-) \le x - 2\sqrt{C-1}\,y$ for $x = Cy/2$. This amounts to proving that

$$\left(\frac{4C}{C+2} + 2\sqrt{C-1}\right)y \le 0,$$

or equivalently $(C-2)(C^2+C-2) \le 0$, which is evident.

Finally, we turn to the upper bound for $G(d_+)$. We have $x + d_+ \ge 0$ and hence

$$\begin{aligned} G(d_+) - x + 2\sqrt{C-1}\,y &= -Cd_+ - 2(x - \sqrt{C-1}\,y) \\ &= \frac{4C\big[\,x - \frac{C}{2}\sqrt{(1-C^2/4)y^2 + x^2}\,\big]}{4-C^2} - 2(x - \sqrt{C-1}\,y). \end{aligned}$$

The latter expression, considered as a function of x, is nonincreasing: indeed, its derivative equals

$$\begin{aligned} \frac{4C}{4-C^2} - \frac{2C^2}{(4-C^2)\sqrt{(1-C^2/4)y^2/x^2+1}} - 2 &\le \frac{4C}{4-C^2} - \frac{C^3}{4-C^2} - 2 \\ &= C - 2 \le 0, \end{aligned}$$

where the first bound above follows from the assumption $x > Cy/2$. Thus, we will be done if we show $G(d_+) \le x - 2\sqrt{C-1}\,y \le 0$ for $x = Cy/2$. This is equivalent to $(2\sqrt{C-1} - C)y \le 0$, or $(C-2)^2 \ge 0$. ■

3.2. A family of special functions. Let $1 \le C \le 2$ be a fixed number. Consider the function $U^C : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ given by

$$U^C(x,y) = \begin{cases} -(C/2)\sqrt{(C^2/4+1)y^2 - x^2} & \text{if } |x| \le Cy/2, \\ |x| - Cy & \text{if } |x| > Cy/2. \end{cases} \tag{3.4}$$

Note that U^2 is precisely the special function used by the author in [12] in the proof of the estimate $\|M\|_1 \le 2\|S(M)\|_1$. We will also need an auxiliary function A^C on $\mathbb{R} \times [0,\infty)$, defined by

$$A^C(x,y) = \begin{cases} (C/2)x/\sqrt{(C^2/4+1)y^2 - x^2} & \text{if } |x| \le Cy/2, \\ (C^2/4)\operatorname{sgn} x & \text{if } |x| > Cy/2. \end{cases}$$

Let $(C_n)_{n\ge1}$ be the sequence introduced in Section 1 and let $n \ge 1$ be fixed. For any $k = 1,\dots,n$, let $U_k = U^{C_{n+1-k}}$; furthermore, for $k = 1,\dots,n-1$, let $A_k = A^{C_{n-k}}$. Finally, put $V(x,y) = |x| - C_n y$. We will show that the sequence $(U_k)_{k=1}^n$ has all the necessary properties listed in Theorem 2.1.

Lemma 3.4. *The conditions* (i) *and* (ii) *of Theorem* 2.1 *are satisfied.*

Proof. The property (i) is trivial: we have $U_n(x,|x|) = U^{C_1}(x,|x|) = 0$. The condition (ii) also has a simple proof. Indeed, for $|x| \ge C_n y/2$ we get equality, so we may assume that $|x| < C_n y/2$. Furthermore, we may restrict ourselves to nonnegative x. Rewrite the majorization in the form

$$C_n y \geq x + \frac{C_n}{2}\sqrt{(C_n^2/4+1)y^2 - x^2} \tag{3.5}$$

and observe that the right-hand side is increasing as a function of $x \in [0, C_n y/2]$: its derivative equals

$$1 - \frac{C_n}{2}\frac{1}{\sqrt{(C_n^2/4+1)y^2/x^2 - 1}} \geq 1 - \frac{C_n^2}{4} \geq 0.$$

Hence, it suffices to show (3.5) for $x = C_n y/2$; but then both sides are equal. ∎

We turn to the third condition of Theorem 2.1.

LEMMA 3.5. *For any $k = 1, \ldots, n-1$, any $x, d \in \mathbb{R}$ and any $y \geq 0$ we have*

$$U_k\big(x + d, \sqrt{y^2 + d^2}\big) \leq U_{k+1}(x, y) + A_k(x, y)d. \tag{3.6}$$

Proof. Denote $C = C_{n+1-k}$, so that $C_{n-k} = 2\sqrt{C-1}$. The function U_k is defined by the right-hand side of (3.4), while the formulas for U_{k+1} and A_k read

$$U_{k+1}(x,y) = \begin{cases} -\sqrt{C-1}\sqrt{Cy^2 - x^2} & \text{if } |x| \leq \sqrt{C-1}\,y, \\ |x| - 2\sqrt{C-1}\,y & \text{if } |x| > \sqrt{C-1}\,y, \end{cases}$$

and

$$A_k(x,y) = \begin{cases} \sqrt{C-1}\,x/\sqrt{Cy^2 - x^2} & \text{if } |x| \leq \sqrt{C-1}\,y, \\ C - 1 & \text{if } |x| > \sqrt{C-1}\,y. \end{cases}$$

Suppose first that $x \leq \sqrt{C-1}\,y$. If $|x + d| \leq (C/2)\sqrt{y^2 + d^2}$, then

$$\begin{aligned} U_n(x + d, \sqrt{y^2 + d^2}) &= -\frac{C}{2}\sqrt{(C^2/4+1)(y^2+d^2) - (x+d)^2} \\ &\leq -\sqrt{C-1}\sqrt{C(y^2+d^2) - (x+d)^2} \end{aligned}$$

(simply square both sides to verify the latter bound). The discriminant of the quadratic function $d \mapsto C(y^2 + d^2) - (x+d)^2$ is nonpositive (because of the assumption $x \leq \sqrt{C-1}\,y$), so the function

$$H(d) = -\sqrt{C-1}\sqrt{C(y^2+d^2) - (x+d)^2}$$

is concave. Thus $H(d) \leq H(0) + H'(0)d$, or

$$U_n(x + d, \sqrt{y^2 + d^2}) \leq -\sqrt{C-1}\sqrt{Cy^2 - x^2} + \sqrt{C-1}\,\frac{xd}{\sqrt{Cy^2 - x^2}},$$

which is precisely (3.6).

Next, assume that $x \le \sqrt{C-1}\,y$ and $|x+d| > (C/2)\sqrt{y^2+d^2}$. The bound (3.6) becomes

$$|x+d| - C\sqrt{y^2+d^2} \le \sqrt{C-1}\sqrt{Cy^2-x^2}\left(-1+\frac{xd}{Cy^2-x^2}\right).$$

By homogeneity, we may assume that $Cy^2 - x^2 = 1$. Then the above inequality is precisely the assertion of Lemma 3.1. Therefore, all we need is the verification of the assumption $|x| \le \sqrt{C-1}$ appearing in the statement of the lemma. But this follows from

$$x^2 = Cx^2 - (C-1)x^2 \le C(C-1)y^2 - (C-1)x^2 = C-1.$$

Finally, we turn to the case $|x| \ge \sqrt{C-1}\,y$. Since $U_n(x,y) = U_n(-x,y)$ and $A_n(x,y) = -A_n(-x,y)$, we may restrict ourselves to nonnegative x. Now, if $|x+d| > (C/2)\sqrt{y^2+d^2}$, the inequality (3.6) is precisely the assertion of Lemma 3.2. On the other hand, if $|x+d| \le (C/2)\sqrt{y^2+d^2}$, then the claim follows from Lemma 3.3. ■

4. Sharpness. To prove that for a given n the constant C_n cannot be replaced by a smaller number, one could try to construct appropriate examples. However, we will use a different approach which rests on the properties of the abstract special functions U of Section 2.

Let $n \ge 1$ be a fixed integer and let $\alpha < 1$ be a given constant. Consider the sequence $(\alpha_k)_{k=1}^n$ given by $\alpha_1 = \alpha$, $\alpha_{k+1} = 1 + \alpha_k^2/4$, $k = 1, \ldots, n-1$. Of course, then $\alpha_n < C_n$; furthermore, by the proper choice of α, we may make α_n as close to C_n as we wish. For any $1 \le k \le n$, let

$$U_k^\alpha(x,y) = \sup\left\{\mathbb{E}|M_k| - \alpha_n \mathbb{E}\sqrt{y^2 - x^2 + S_k^2(M)}\right\},$$

where the supremum is taken over all simple martingales starting from x (we stress here that the constant above is α_n, not α_k). Note that $U_k^\alpha(x,y) = U_k^\alpha(-x,y)$: this follows from the trivial fact that if M is a martingale starting from x, then $-M$ is a martingale starting from $-x$, and the two sequences have the same square function. As we have observed in Section 2, U_k^α satisfies

$$U_k^\alpha(x,y) = \sup\{\mathbb{E}U_{k-1}^\alpha(x+D, \sqrt{y^2+D^2}) : \mathbb{E}D = 0\},$$

and hence, for any centered random variable D we have

$$U_k^\alpha(x,y) \ge \mathbb{E}U_{k-1}^\alpha(x+D, \sqrt{y^2+D^2}). \tag{4.1}$$

We will prove that

$$U_k^\alpha(x,y) \ge |x| - \alpha_{n+1-k}y \quad \text{for } k = 1, \ldots, n, \tag{4.2}$$

which will immediately yield the claim: indeed, in particular this will give $U_n^\alpha(1,1) \ge 1 - \alpha > 0$, and will imply that no constant smaller than C_n suffices in (1.3).

To show (4.2), we use induction. Fix x, y; actually, by the symmetry of U_k^α, we may assume that $x \geq 0$. For $k = 1$ the bound is trivial. Next, fix $1 \leq k \leq n-1$ and consider a centered random variable D which takes values in the set $\{t_1, t_2\}$, where $t_1 = (2 - \alpha_{n-k+1})y/(2\sqrt{\alpha_{n-k+1} - 1}) > 0$ and t_2 is a negative number. By (4.1) and the inductive assumption, we get

$$\begin{aligned} U_{k+1}^\alpha(x, y) &\geq \beta_1 U_k^\alpha\left(x + t_1, \sqrt{y^2 + t_1^2}\right) + \beta_2 U_k^\alpha\left(x + t_2, \sqrt{y^2 + t_2^2}\right) \\ &\geq \beta_1\left[x + t_1 - \alpha_{n-k+1}\sqrt{y^2 + t_1^2}\right] + \beta_2\left[|x + t_2| - \alpha_{n-k+1}\sqrt{y^2 + t_2^2}\right], \end{aligned}$$

where $\beta_1 = -t_2/(t_1 - t_2)$ and $\beta_2 = t_1/(t_1 - t_2)$. Now let $t_2 \to -\infty$: as a result, we obtain

$$\begin{aligned} U_{k+1}^\alpha(x, y) &\geq x + t_1 - \alpha_{n-k+1}\sqrt{y^2 + t_1^2} + t_1(1 - \alpha_{n-k+1}) \\ &= x - 2\sqrt{\alpha_{n-k+1} - 1}\, y = x - \alpha_{n-k} y. \end{aligned}$$

This completes the proof.

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