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Actuarial Approach to Option Pricing in a Fractional Black–Scholes Model with Time-Dependent Volatility

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Summary. We study actuarial methods of option pricing in a fractional Black–Scholes model with time-dependent volatility. We interpret the option as a potential loss and we show that the fair premium needed to insure this loss coincides with the expectation of the discounted claim payoff under the average risk neutral measure.

1. Introduction. In [1] Bladt and Rydberg introduced a new method of option pricing by replacing the option pricing problem with an equivalent insurance problem. Their idea is to discount risk free and stochastic future prices according to the risk free interest rate and the so-called *expected rate of return*, respectively. With these discounted prices they calculated the price (fair insurance premium) of a European call option as the expected value of the difference between the actual price and the strike price (in present values) when exercising the option. Later on the above actuarial approach was used to option pricing in many papers (see e.g. [3, 4, 16]).

In the present paper we apply the actuarial approach to option pricing in a fractional Black–Scholes model with time-dependent volatility. Let $(B_t^H)_{t\in[0,T]}$ be a fractional Brownian motion (fBm) with Hurst parameter $H \in (1/2, 1)$ defined on some probability space $(\Omega, \mathcal{F}, P)$, i.e. a continuous zero-mean Gaussian process with covariance

$$EB_t^H B_s^H = \tfrac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \qquad s,t \in [0,T].$$

We consider a continuous market model consisting of stock whose price S evolves under the underlying measure P according to the stochastic differ-

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ential equation

(1.1) $$S_t = S_0 + \int_0^t \mu(u) S_u\, du + \int_0^t \sigma(u) S_u\, dB_u^H, \qquad t \in [0,T],$$

with some deterministic functions $\mu, \sigma : [0,T] \to \mathbb{R}$, and of bond whose price B is given by

(1.2) $$B_t = \exp(rt), \qquad t \in [0,T],$$

for some $r \geq 0$. For σ constant the above model was considered by Valkeila [15] and Sottinen and Valkeila [13, 14]. Since B^H is not a semimartingale and there is no martingale measure on the market described by (1.1), (1.2), the classical option pricing methods do not apply. To overcome this difficulty Valkeila [15] and Sottinen and Valkeila [13, 14] proposed to replace the martingale measure by a measure Q equivalent to P under which the discounted stock price is given by a geometric fractional Brownian motion. This measure resembles the martingale measure in the sense that it has the property that

(1.3) $$B_t^{-1} E_Q(S_t) = S_0, \qquad t \in [0,T].$$

In [15] it is proved that such a measure Q, called the *average risk neutral measure*, exists and is unique. In [15] and [13, 14] the measure Q is used to derive an option pricing formula for a European call option. A drawback of the above papers is the lack of economic justification of the pricing formula based on Q.

In the present paper we first use a Girsanov-type theorem for Wiener integrals with respect to fBm to extend the results of Valkeila [15] and Sottinen and Valkeila [13, 14] on existence and uniqueness of the average risk neutral measure to the model (1.1), (1.2) with time-dependent volatility. Then we prove that for a general functional $F : C[0,T] \times \mathbb{R} \to \mathbb{R}$ the fair insurance premium of the option of the form $F_T = F(S,K)$ is equal to

$$C(F_T) = B_T^{-1} E_Q(\tilde{F}(S,K)),$$

where $\tilde{F} : C[0,T] \times \mathbb{R} \to \mathbb{R}$ is some modified functional determined by F. It is worth noting that $\tilde{F} = F$ if $r = 0$, and if $r > 0$ then $\tilde{F} = F$ in the important case of European options of the form $F_T(S,K) = f(S_T, K)$ with $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $f(cx, cy) = cf(x,y)$ for $c \in \mathbb{R}^+$, $x, y \in \mathbb{R}$. This class of options includes call and put options, chooser options, binary options and others (see Remark 3.4). In particular, the price of the European call option with expiration time T and strike price K is given by $C((S_T - K)^+) = B_T^{-1} E_Q(S_T - K)^+$, which leads to the analogue of the Black–Scholes formula:

(1.4) $$C((S_T - K)^+) = S_0 \Phi(y_1) - K e^{-rT} \Phi(y_2),$$

where

$$y_1 = \frac{\log \frac{S_0}{K} + rT + \frac{1}{2}\langle \sigma, \sigma \rangle_T^H}{\sqrt{\langle \sigma, \sigma \rangle_T^H}}, \qquad y_2 = \frac{\log \frac{S_0}{K} + rT - \frac{1}{2}\langle \sigma, \sigma \rangle_T^H}{\sqrt{\langle \sigma, \sigma \rangle_T^H}}$$

and

$$\langle \sigma, \sigma \rangle_t^H = H(2H-1) \int_0^t \int_0^t \sigma(u)\sigma(s)|u-s|^{2H-2}\, ds\, du, \qquad t \in [0,T]. \tag{1.5}$$

2. Preliminaries

2.1. Integration with respect to fBm. Since B^H is not a semimartingale, the classical Itô integration theory cannot be used to define the stochastic integral with respect to B^H. However, the integral can be easily defined if we restrict ourselves to deterministic integrands $f : [0,T] \to \mathbb{R}$ such that

$$\|f\|_{\mathbb{L}^{1/H}_{[0,T]}} = \Big(\int_0^T |f(s)|^{1/H}\, ds \Big)^H < \infty.$$

To define it, let us denote by $\mathcal{E}$ the set of all step functions, i.e. functions of the form $f(t) = \sum_{i=1}^n f_i \mathbf{1}_{(t_{i-1}-t_i]}(t)$, where $0 = t_0 < t_1 < \cdots < t_n = T$, $f_i \in \mathbb{R}$. For $f \in \mathcal{E}$ we define the integral with respect to B^H by

$$\int_0^T f(s)\, dB_s^H := \sum_{i=1}^n f_i (B_{t_i}^H - B_{t_{i-1}}^H).$$

It is easy to see that $EB_t^H B_s^H = H(2H-1) \int_0^t \int_0^s |u-v|^{2H-2}\, du\, dv$, which implies that for every $f, g \in \mathcal{E}$ we have

$$E\Big(\int_0^T f(s)\, dB_s^H \int_0^T g(s)\, dB_s^H \Big) = \langle f, g \rangle_T^H, \tag{2.1}$$

where $\langle f, g \rangle_T^H = H(2H-1) \int_0^T \int_0^T f(t)g(s)|t-s|^{2H-2}\, ds\, dt$. In [9] it is proved that $\langle f, g \rangle_T^H$ is an inner product in $\mathbb{L}^{1/H}_{[0,T]}$, whereas in [6] it is proved that for every $f \in \mathbb{L}^{1/H}_{[0,T]}$,

$$\sqrt{\langle f, f \rangle_T^H} \le C_H \|f\|_{\mathbb{L}^{1/H}_{[0,T]}}. \tag{2.2}$$

Therefore we can define the stochastic integral (usually called the *Wiener integral*) as follows. Let $f \in \mathbb{L}^{1/H}_{[0,T]}$ and let $\{f_n\}$ be a sequence of step functions such that $f_n \to f$ in $\mathbb{L}^{1/H}_{[0,T]}$. Then by (2.1) and (2.2) the sequence $\{\int_0^T f_n(s)\, dB_s^H\}$ is Cauchy in $\mathbb{L}^2(\Omega)$. Since $\mathbb{L}^2(\Omega)$ is complete, we may define $\int_0^T f(s)\, dB_s^H \in \mathbb{L}^2(\Omega)$ as the limit of $\{\int_0^T f_n(s)\, dB_s^H\}$ as $n \to \infty$.

To construct the stochastic integral with respect to the fBm of non-deterministic integrands we follow Ruzmaikina [10] (see also [2], [12]). Let C^α be the space of α-Hölder continuous functions on $[0,T]$. Since $B^H \in C^\alpha$ for $\alpha < H$ with probability one, it follows that the Stieltjes integral $\int_0^T X(s,\omega)\, dB_s^H(\omega)$ exists for almost all $\omega \in \Omega$ and $X(\cdot,\omega) \in C^\gamma$ with $\gamma > 1-\alpha$.

2.2. Girsanov-type theorem. Consider the so-called *fundamental martingale*, i.e. the process

$$M_t^H = \int_0^t k(t,s)\, dB_s^H, \tag{2.3}$$

where $k(t,s) = \kappa_H^{-1} s^{1/2-H}(t-s)^{1/2-H}$, $\kappa_H = 2HB(3/2-H, H+1/2)$ and B is the Euler beta function. It is obvious that M^H is a centered Gaussian process. Let $w_t^H = \lambda_H^{-1} t^{2-2H}$, $t \in [0,T]$, where $\lambda_H = (2H\Gamma(3-2H)\Gamma(H+1/2))/(\Gamma(3/2-H))$ and Γ stands for the gamma function. It follows from Theorem 3.1 in [8] that M^H has independent increments and $E(M_t^H)^2 = w_t^H$. In particular, M^H is a martingale. It is easy to verify that the process

$$W_t = \left(\frac{\lambda_H}{2-2H}\right)^{1/2} \int_0^t s^{H-1/2}\, dM_s^H, \qquad t \in [0,T],$$

is a centered continuous martingale with quadratic variation $[W]_t = t$. Therefore, by Lévy's theorem W is a Brownian motion and we also have

$$M_t^H = \left(\frac{2-2H}{\lambda_H}\right)^{1/2} \int_0^t s^{1/2-H}\, dW_s, \qquad t \in [0,T]. \tag{2.4}$$

We now recall the Girsanov-type theorem for Wiener integrals with respect to the fBm. In what follows, $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the σ-field generated by W.

THEOREM 2.1 ([5]). *Let M^H be the fundamental martingale defined by (2.3) and let $r_H = (2-2H)B(3/2-H, 3/2-H)$. Suppose that f, c are such that*

$$\rho_H(t) = r_H^{-1} t^{2H-1} \frac{d}{dt} \int_0^t s^{1/2-H}(t-s)^{1/2-H} \frac{c(s)}{f(s)}\, ds, \qquad t \in [0,T], \tag{2.5}$$

is well defined for almost every t with respect to the Lebesgue measure and

$$\int_0^T \rho_H^2(s)\, dw_s^H < \infty. \tag{2.6}$$

If we define Q on $\mathcal{F}_T$ by

$$\frac{dQ}{dP} = \exp\Big(-\int_0^T \rho_H(s)\, dM_s^H - \frac{1}{2}\int_0^T \rho_H^2(s)\, dw_s^H\Big), \tag{2.7}$$

then the process $\int_0^{\cdot} f(s)\, dB_s^H + \int_0^{\cdot} c(s)\, ds$ has the same finite-dimensional distributions under Q as $\int_0^{\cdot} f(s)\, dB_s^H$ under P.

THEOREM 2.2. *Under the assumptions of Theorem 2.1, Q defined by (2.7) is the only measure equivalent to P such that the process $\int_0^{\cdot} f(s)\, dB_s^H + \int_0^{\cdot} c(s)\, ds$ has the same law under Q as $\int_0^{\cdot} f(s)\, dB_s^H$ under P.*

Proof. Suppose that there exist two measures Q_1 and Q_2 equivalent to P such that $Y = \int_0^{\cdot} f(s)\, dB_s^H + \int_0^{\cdot} c(s)\, ds$ under Q_1 and under Q_2 has the same finite-dimensional distribution as $X = \int_0^{\cdot} f(s)\, dB_s^H$ under P. It is easy to check that

$$\int_0^t k(t,s)\frac{c(s)}{f(s)}\, ds = \int_0^t \rho_H(s)\, dw_s^H, \quad t \in [0,T].$$

Hence, by (2.3), $M^H + \int_0^{\cdot} \rho_H(s)\, dw_s^H = \int_0^{\cdot} k(\cdot,s) f^{-1}(s)\, dY_s$ has under both Q_1 and Q_2 the same finite-dimensional distribution as $M^H = \int_0^{\cdot} k(\cdot,s) f^{-1}(s)\, dX_s$ under P. From this and (2.4) it follows that the process $\tilde{W}$ defined as

$$\tilde{W}_t = W_t + \Big(\frac{\lambda_H}{2-2H}\Big)^{1/2} \int_0^t s^{H-1/2} \rho_H(s)\, dw_s^H, \quad t \in [0,T],$$

is a Brownian motion under both Q_1 and Q_2. By the Radon–Nikodym theorem there exists a random variable Z_T such that $Q_1(A) = \int_A Z_T\, dQ_2$ for all $A \in \mathcal{F}_T$. Let

$$Z_t = E_{Q_2}(Z_T \mid \mathcal{F}_t), \quad t \in [0,T].$$

Since Z is a martingale under Q_2, it follows that

$$Z_t = 1 + \int_0^t H_s\, d\tilde{W}_s, \quad t \in [0,T],$$

for some progressively measurable process $\{H_t\}_{t\in[0,T]}$. Note that, since $\tilde{W}$ is a Q_1-martingale, for all $A \in \mathcal{F}_t$ we have

$$\begin{aligned}\int_A Z_t \tilde{W}_t\, dQ_2 &= \int_A E_{Q_2}(Z_T \mid \mathcal{F}_t)\tilde{W}_t\, dQ_2 = \int_A Z_T \tilde{W}_t\, dQ_2 = \int_A \tilde{W}_t\, dQ_1 \\ &= \int_A E_{Q_1}(\tilde{W}_T \mid \mathcal{F}_t)\, dQ_1 = \int_A \tilde{W}_T\, dQ_1 = \int_A Z_T \tilde{W}_T\, dQ_2.\end{aligned}$$

Therefore $Z_t\tilde{W}_t = E_{Q_2}(Z_T\tilde{W}_T \mid \mathcal{F}_t)$, so $Z\tilde{W}$ is also a Q_2-martingale. As a consequence,

$$\int_0^t H_s\,ds = [Z,\tilde{W}]_t = 0, \qquad t \in [0,T].$$

It follows that $H_t = 0$ for a.e. $t \in [0,T]$. Thus $Z_T = 1$ and $Q_1(A) = Q_2(A)$ for all $A \in \mathcal{F}_T$, which completes the proof. ■

In general, the verification that assumption (2.6) in Theorem 2.1 holds true may cause difficulties. We shall give a simple (but convenient for our purposes) sufficient condition for (2.6) to hold.

LEMMA 2.3. *If f, c are absolutely continuous on $[0,T]$ and $f(t) \neq 0$ for all $t \in [0,T]$ then ρ_H given by* (2.5) *is well defined. Moreover, if for some $C \in \mathbb{R}$,*

$$\left|\frac{d}{dt}\left(\frac{c(t)}{f(t)}\right)\right| \le Ct^{2H-2}, \qquad t \in (0,T), \tag{2.8}$$

then (2.6) *is satisfied.*

Proof. First observe that ρ_H is well defined, because from the rules of fractional calculus (see Theorem 14.8 in [11]) it follows that the function $t \mapsto \int_0^t F(x)(t-x)^{1/2-H}dx$, where $F(t) = t^{1/2-H}c(t)/f(t)$ for $t \in [0,T]$, is differentiable a.e. with respect to the Lebesgue measure. We now show that (2.8) implies (2.6). Since

$$\begin{aligned}
\rho_H(t) &= r_H^{-1} t^{2H-1} \frac{d}{dt} \int_0^t s^{1/2-H}(t-s)^{1/2-H} \frac{c(s)}{f(s)}\,ds \\
&= r_H^{-1} t^{2H-1} \frac{d}{dt}\left(t^{2-2H} \int_0^1 u^{1/2-H}(1-u)^{1/2-H} \frac{c(ut)}{f(ut)}\,du \right) \\
&= r_H^{-1}\bigg((2-2H) \int_0^1 u^{1/2-H}(1-u)^{1/2-H} \frac{c(ut)}{f(ut)}\,du \\
&\qquad + t\int_0^1 u^{1/2-H}(1-u)^{1/2-H} \frac{d}{dt}\left(\frac{c(ut)}{f(ut)}\right) du \bigg),
\end{aligned}$$

using the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$ we obtain

$$\int_0^T \rho_H^2(s)\,dw_s^H$$
$$\le 2r_H^{-2}\lambda_H^{-1}\bigg[(2-2H)^3 \int_0^T t^{1-2H} \left(\int_0^1 u^{1/2-H}(1-u)^{1/2-H} \frac{c(ut)}{f(ut)}\,du \right)^2 dt$$

$$+ (2-2H)\int_0^T t^{3-2H}\Big(\int_0^1 u^{1/2-H}(1-u)^{1/2-H}\frac{d}{dt}\Big(\frac{c(ut)}{f(ut)}\Big)\,du\Big)^2 dt\Big]$$
$$= I_1 + I_2.$$

Clearly $I_1 \le C_1 \int_0^T t^{1-2H} dt = (C_1/(2-2H))T^{2-2H}$ for some $C_1 \in \mathbb{R}$. Furthermore,

$$I_2 \le 2r_H^{-2}\lambda_H^{-1}(2-2H)\int_0^T t^{3-2H}\Big(\int_0^1 u^{1/2-H}(1-u)^{1/2-H}\Big|\frac{d}{dt}\Big(\frac{c(ut)}{f(ut)}\Big)\Big|\,du\Big)^2 dt$$
$$\le 2r_H^{-2}\lambda_H^{-1}(2-2H)C^2\int_0^T t^{3-2H}\Big(\int_0^1 u^{1/2-H}(1-u)^{1/2-H}u^{2H-1}t^{2H-2}\,du\Big)^2 dt$$
$$= C_2\int_0^T t^{2H-1}dt = (C_2/2H)T^{2H},$$

and the proof is complete. ■

2.3. Absolute continuity of $\langle\sigma,\sigma\rangle^H$. The following lemma gives conditions ensuring absolute continuity of $\langle\sigma,\sigma\rangle^H$ defined by (1.5).

Lemma 2.4. *If f is an absolutely continuous function on $[0,T]$ and $H \in (1/2,1)$ then $g:[0,T]\to\mathbb{R}$ given by*

$$g(s) = 2f(s)\int_0^s f(r)(s-r)^{2H-2}\,dr, \qquad s\in[0,T],$$

is also absolutely continuous and for every $t\in[0,T]$,

$$\int_0^t g(s)\,ds = \int_0^t\int_0^t f(s)f(r)|s-r|^{2H-2}\,dr\,ds.$$

Proof. Since f is absolutely continuous, $f(r) = f(0) + \int_0^r f'(u)\,du$. Hence

$$\int_0^s f(r)(s-r)^{2H-2}\,dr = \frac{f(0)}{2H-1}s^{2H-1} + \int_0^s\Big(\int_0^r f'(u)\,du\Big)(s-r)^{2H-2}\,dr. \tag{2.9}$$

Using Fubini's theorem we obtain

$$\int_0^s\Big(\int_0^r f'(u)\,du\Big)(s-r)^{2H-2}\,dr = \int_0^s f'(u)\Big(\int_u^s (s-r)^{2H-2}\,dr\Big)du$$
$$= \int_0^s f'(u)\Big(\int_u^s (r-u)^{2H-2}\,dr\Big)du$$
$$= \int_0^s\int_0^r f'(u)(r-u)^{2H-2}\,du\,dr,$$

which implies that g is absolutely continuous. Using Fubini's theorem once again we have

$$\begin{aligned}
2\int_0^t f(s)\Big(\int_0^s f(r)(s-r)^{2H-2}\,dr\Big)ds &= \int_0^t f(s)\Big(\int_0^s f(r)(s-r)^{2H-2}\,dr\Big)ds \\
&\quad + \int_0^t f(r)\Big(\int_0^r f(s)(r-s)^{2H-2}\,ds\Big)dr \\
&= \int_0^t f(s)\Big(\int_0^s f(r)(s-r)^{2H-2}\,dr\Big)ds \\
&\quad + \int_0^t f(s)\Big(\int_s^t f(r)(r-s)^{2H-2}\,dr\Big)ds \\
&= \int_0^t\int_0^t f(s)f(r)|s-r|^{2H-2}\,dr\,ds,
\end{aligned}$$

which is our claim. ∎

Observe that from Lemma 2.4 it follows that if σ is absolutely continuous on $[0,T]$ then

$$\langle\sigma,\sigma\rangle_t^H = \int_0^t g(s)\,ds, \qquad t\in[0,T], \tag{2.10}$$

where

$$g(t) = 2H(2H-1)\sigma(t)\int_0^t \sigma(s)(t-s)^{2H-2}\,ds, \qquad t\in[0,T]. \tag{2.11}$$

Therefore $\langle\sigma,\sigma\rangle^H$ is also absolutely continuous.

3. Main results. We consider the model described by (1.1) and (1.2) with $\mu\in C^1[0,T]$ and $\sigma\in C^1[0,T]$ such that $\sigma(t)>0$ for $t\in[0,T]$. We assume that there are no dividends and no transaction costs.

By Lemma 3.1 and Theorem 4 in [10] the unique pathwise solution of (1.1) is given by

$$S_t = S_0\exp\Big(\int_0^t \mu(s)\,ds + \int_0^t \sigma(s)\,dB_s^H\Big), \qquad t\in[0,T]. \tag{3.1}$$

Definition ([1]). The *expected rate of return* ν of S is defined by the formula

$$\exp\Big(\int_0^t \nu(s)\,ds\Big) = S_0^{-1}ES_t, \qquad t\in[0,T].$$

Remark 3.1. By (3.1),

$$S_0^{-1} E S_t = \exp\Big(\int_0^t \mu(s)\,ds\Big) E\Big(\exp\int_0^t \sigma(s)\,dB_s^H\Big), \quad t \in [0,T].$$

Since

$$E\exp\Big(\int_0^t \sigma(s)\,dB_s^H\Big) = \exp\Big(\frac{\langle\sigma,\sigma\rangle_t^H}{2}\Big) \tag{3.2}$$

(see e.g. [7]), it follows from (2.10) that

$$\nu(t) = \mu(t) + \frac{g(t)}{2}, \quad t \in [0,T], \tag{3.3}$$

where g is given by (2.11).

Our aim is to calculate the price $C(F_T)$ for options of the form $F_T = F(S,K)$, where $F : C[0,T] \times \mathbb{R} \to \mathbb{R}$. Following Bladt and Rydberg [1] we interpret the price as the fair insurance premium.

Let

$$\tilde{S}_t = S_t \exp\Big(-\int_0^t \nu(s)\,ds\Big), \quad t \in [0,T], \quad \tilde{K} = Ke^{-rT}.$$

Definition ([1]). Assume that $F_T(\tilde{S},\tilde{K})$ is integrable. The *fair premium* of the option $F_T = F(S,K)$ is given by

$$C(F_T) = E(F_T(\tilde{S},\tilde{K})).$$

Definition ([15]). Let Q be a measure equivalent to P. We say that Q is an *average risk neutral measure* if under Q the discounted stock price process is given by

$$B_t^{-1} S_t = S_0 \exp\Big(\int_0^t \sigma(s)\,d\tilde{B}_s^H - \frac{\langle\sigma,\sigma\rangle_t^H}{2}\Big), \quad t \in [0,T],$$

where $\tilde{B^H}$ is a Q-fractional Brownian motion.

Remark 3.2. Note that by (3.2) the average risk neutral measure satisfies (1.3).

The main result of the paper is the following theorem.

Theorem 3.3.

(i) *The average risk neutral measure exists and is unique.*

(ii) *For any* $F : C[0,T] \times \mathbb{R} \to \mathbb{R}$,

$$C(F_T) = B_T^{-1} E_Q(\tilde{F}(S,K)), \tag{3.4}$$

where $\tilde{F} : C[0,T] \times \mathbb{R} \to \mathbb{R}$ *is such that*

$$\tilde{F}_T(S,K) = e^{rT} F_T(Se^{-r\cdot}, Ke^{-rT}).$$

Proof. (i) Note that the Q is the average risk neutral measure if and only if the process

$$\int_0^t \sigma(s)\, dB_s^H + \int_0^t \mu(s) ds - rt + \frac{\langle \sigma, \sigma \rangle_t^H}{2}, \qquad t \in [0,T], \tag{3.5}$$

has the same finite-dimensional distribution under Q as $\int_0^{\cdot} \sigma(s)\, dB_s^H$ under P. Let

$$c(t) = \mu(t) - r + H(2H-1)\sigma(t) \int_0^t \sigma(s)(t-s)^{2H-2}\, ds, \qquad t \in [0,T].$$

Fix $t \in (0,T)$. By Lemma 2.4 the function c is absolutely continuous and

$$\int_0^t c(s)\, ds = \int_0^t \mu(s)\, ds - rt + \frac{\langle \sigma, \sigma \rangle_t^H}{2}. \tag{3.6}$$

We have

$$\begin{aligned}\left| \frac{d}{dt}\left(\frac{c(t)}{\sigma(t)} \right) \right| &= \left| \frac{c(t)\sigma'(t) - c'(t)\sigma(t)}{\sigma^2(t)} \right| \\ &\le \sup_{t \in [0,T]} \left| \frac{c(t)\sigma'(t)}{\sigma^2(t)} \right| + \left(\sup_{t\in[0,T]} \frac{1}{|\sigma(t)|} \right) |c'(t)|\end{aligned}$$

and

$$\begin{aligned} c'(t) &= \mu'(t) + H(2H-1)\Big[\sigma'(t) \int_0^t \sigma(r)(t-r)^{2H-2}\, dr \\ &\qquad\qquad + \sigma(t) \frac{d}{dt}\Big(\int_0^t \sigma(r)(t-r)^{2H-2} \Big) \Big] \\ &\le \sup_{t\in[0,T]} |\mu'(t)| + \sup_{t\in[0,T]} |\sigma'(t)| \sup_{t\in[0,T]} |\sigma(t)| HT^{2H-1} \\ &\quad + H(2H-1) \Big(\sup_{t\in[0,T]} |\sigma(t)| \Big) \frac{d}{dt}\Big(\int_0^t \sigma(r)(t-r)^{2H-2} \Big).\end{aligned}$$

By (2.9),

$$\begin{aligned}\frac{d}{dt}\Big(\int_0^t \sigma(r)(t-r)^{2H-2} \Big) &= \sigma(0)t^{2H-2} + \int_0^t \sigma'(u)(t-u)^{2H-2}\, du \\ &\le \sigma(0)t^{2H-2} + \sup_{t\in[0,T]} \left| \frac{\sigma'(t)}{2H-1} \right| T^{2H-1}.\end{aligned}$$

Hence

$$|c'(t)| \le 3K_1 + |\sigma(0)| H(2H-1) \sup_{t\in[0,T]} |\sigma(t)| t^{2H-2},$$

where $K_1 = \max(\sup_{t\in[0,T]} |\mu'(t)|, \sup_{t\in[0,T]} |\sigma'(t)| \sup_{t\in[0,T]} |\sigma(t)| H T^{2H-1})$, which implies that

$$\left|\frac{d}{dt}\left(\frac{c(t)}{\sigma(t)}\right)\right| \le K_2 + K_3 t^{2H-2} \le \left(\frac{K_2}{T^{2H-2}} + K_3\right) t^{2H-2}$$

with

$$K_2 = \sup_{t\in[0,T]}\left|\frac{c(t)\sigma'(t)}{\sigma^2(t)}\right| + 3K_1 \sup_{t\in[0,T]}\left|\frac{1}{\sigma(t)}\right| \quad \text{and} \quad K_3 = \sup_{t\in[0,T]}\left|\frac{\sigma(0)}{\sigma(t)}\right|.$$

Assertion (i) now follows from Lemma 2.3 and Theorems 2.1 and 2.2. Since ν admits the decomposition (3.3), in much the same way as in the proof of (3.6) one can show that

$$\int_0^t \nu(s)\,ds = \int_0^t \mu(s)\,ds + \frac{\langle\sigma,\sigma\rangle_t^H}{2}.$$

Since the process defined by (3.5) has the same distribution under Q as $\int_0^\cdot \sigma(s)\,dB_s^H$ under P, $F_T(Se^{-r\cdot}, Ke^{-rT})$ has the same distribution under Q as $F_T(\tilde{S}, \tilde{K})$ under P, which completes the proof. ∎

Remark 3.4. (a) If $r = 0$ then $\tilde{F} = F$.

(b) If $F_T(S,K) = f(S_T, K)$ for some $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that $f(cx, cy) = cf(x,y)$, $c \in \mathbb{R}^+$, $x, y \in \mathbb{R}$, then $\tilde{F} = F$ for $r \ge 0$. The class of such options includes for instance the call option ($f(x,y) = (x-y)^+$), the put option ($f(x,y) = (y-x)^+$), the chooser option for which

$$f(x,y) = \begin{cases} (x-y)^+ & \text{if } C((S_{T_0} - K)^+) \ge C((K - S_{T_0})^+), \\ (y-x)^+ & \text{otherwise,} \end{cases}$$

for some fixed $T_0 < T$, the binary option cash-or-nothing ($f(x,y) = a\mathbf{1}_{\{x>y\}}$ or $f(x,y) = a\mathbf{1}_{\{x\le y\}}$), the binary option asset-or-nothing ($f(x,y) = x\mathbf{1}_{\{x>y\}}$ or $f(x,y) = x\mathbf{1}_{\{x\le y\}}$).

Corollary 3.5 (Fractional Black–Scholes formula). *The fair insurance premium of a European call option with expiration time T and strike price K is given by* (1.4).

Proof. By (3.3),

$$\begin{aligned} &C((S_T - K)^+) \\ &\quad = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(S_0\exp\left(\sqrt{\langle\sigma,\sigma\rangle_T^H}\,y + rT - \frac{\langle\sigma,\sigma\rangle_T^H}{2}\right) - K\right)^+ e^{-y^2/2}\,dy \\ &\quad = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{y_0}^{\infty}\left(S_0\exp\left(\sqrt{\langle\sigma,\sigma\rangle_T^H}\,y + rT - \frac{\langle\sigma,\sigma\rangle_T^H}{2}\right) - K\right) e^{-y^2/2}\,dy, \end{aligned}$$

where

$$y_0 = \frac{\ln\frac{K}{S_0} - rT + \frac{\langle\sigma,\sigma\rangle_T^H}{2}}{\sqrt{\langle\sigma,\sigma\rangle_T^H}}.$$

Hence

$$\begin{aligned}
&C_T((S_T-K)^+)\\
&\quad = S_0\frac{1}{\sqrt{2\pi}}\int_{y_0}^{\infty}\exp\Big(-\frac{\langle\sigma,\sigma\rangle_T^H}{2} + y\sqrt{\langle\sigma,\sigma\rangle_T^H} - \frac{y^2}{2}\Big)\,dy\\
&\qquad - e^{-rT}K(1-\Phi(y_0))\\
&\quad = S_0\frac{1}{\sqrt{2\pi}}\int_{y_0}^{\infty}\exp\Big(-\frac{1}{2}\Big(y-\sqrt{\langle\sigma,\sigma\rangle_T^H}\Big)^2\Big)\,dy - e^{-rT}K(1-\Phi(y_0))\\
&\quad = S_0\frac{1}{\sqrt{2\pi}}\int_{y_0-\sqrt{\langle\sigma,\sigma\rangle_T^H}}^{\infty} e^{-y^2/2}\,dy - e^{-rT}K(1-\Phi(y_0))\\
&\quad = S_0\Big(1-\Phi\Big(y_0-\sqrt{\langle\sigma,\sigma\rangle_T^H}\Big)\Big) - e^{-rT}K(1-\Phi(y_0))\\
&\quad = S_0\Phi\Big(\sqrt{\langle\sigma,\sigma\rangle_T^H} - y_0\Big) - e^{-rT}K(\Phi(-y_0))\\
&\quad = S_0\Phi(y_1) - Ke^{-rT}\Phi(y_2),
\end{aligned}$$

which completes the proof. ■

References

[1] M. Bladt and T. H. Rydberg, *An actuarial approach to option pricing under the physical measure and without market assumptions*, Insurance Math. Econom. 22 (1998), 65–73.

[2] R. M. Dudley and R. Norvaiša, *An Introduction to p-Variation and Young Integrals*, Maphysto Lecture Notes 1, Aarhus, 1998.

[3] H. Gu, J.-R. Liang, J. Wang, L.-J. Lu, W.-Y. Qiu and F.-Y. Ren, *Fractional Fokker–Planck equation and Black–Scholes formula in composite-diffusive regime*, J. Statist. Phys. 146 (2012), 205–216.

[4] H. Gu, J.-R. Liang and Y.-X. Zhang, *On a time-changed geometric Brownian motion and its application in financial market*, Acta Phys. Polon. B 43 (2012), 1667–1681.

[5] M. L. Kleptsyna, A. Le Breton and M. C. Roubaud, *Parameter estimation and optimal filtering for fractional type stochastic systems*, Statist. Inference Stoch. Process 3 (2000), 173–182.

[6] J. Mémin, Y. Mishura and E. Valkeila, *Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion*, Statist. Probab. Lett. 51 (2001), 197–206.

[7] Y. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, Springer, Berlin, 2008.

[8] I. Norros, E. Valkeila and J. Virtamo, *An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions*, Bernoulli 5 (1999), 571–587.

[9] V. Pipiras, V. Murad and S. Taqqu, *Are classes of deterministic integrands for fractional Brownian motion on an interval complete?*, Bernoulli 7 (2001), 873–897.

[10] A. A. Ruzmaikina, *Stieltjes integrals of Hölder continuous functions with applications to fractional Brownian motion*, J. Statist. Phys. 100 (2000), 1049–1069.

[11] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives and Some of Their Applications*, Nauka i Tekhnika, Minsk, 1987.

[12] L. Słomiński and B. Ziemkiewicz, *Inequalities for the $\mathbb{L}^p$ norms of integrals with respect to a fractional Brownian motion*, Statist. Probab. Lett. 73 (2005), 79–90.

[13] T. Sottinen and E. Valkeila, *Fractional Brownian motion as a model in finance*, preprint 302, Univ. of Helsinki Dept. Math., 2001.

[14] T. Sottinen and E. Valkeila, *On arbitrage and replication in the fractional Black–Scholes pricing model*, Statistics and Decisions 21 (2003), 137–151.

[15] E. Valkeila, *On some properties of geometric fractional Brownian motions*, preprint 224, Univ. of Helsinki, Dept. Math., 1999.

[16] H. Xue and Q. Y. Li, *An actuarial approach to the minimum or maximum option pricing in fractional Brownian motion environment*, in: 2nd IEEE Int. Conf. Inf. and Financial Engineering, 2010, 216–219.

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