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*CONVEX ANALYSIS*

# On $\Phi^{\gamma(\cdot,\cdot)}$-subdifferentiable and $[\Phi+\gamma]$-subdifferentiable Functions

by

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**Summary.** Let $X$ be an arbitrary set. Let $\Phi$ be a family of real-valued functions defined on $X$. Let $\gamma : X \times X \to \mathbb{R}$. Set $[\Phi+\gamma] = \{\phi(\cdot)+\gamma(\cdot,x) \mid \phi \in \Phi,\ x \in X\}$. We give conditions guaranteeing the equivalence of $\Phi^{\gamma(\cdot,\cdot)}$-subdifferentiability and $[\Phi+\gamma]$-subdifferentiability.

Let $X$ be an arbitrary set. Let $\Phi$ be a family of real-valued functions defined on $X$. Let $f$ be a real-valued function defined on $X$. We recall (see for example Pallaschke–Rolewicz (1997), Rubinov (2000), Singer (1997)) that a function $\phi_0 \in \Phi$ is a *$\Phi$-subgradient* of the function $f$ at a point $x_0$ if

$$f(x) - f(x_0) \geq \phi_0(x) - \phi_0(x_0) \tag{1}$$

for all $x \in X$.

The set of all $\Phi$-subgradients of $f$ at $x_0$ is called the *$\Phi$-subdifferential* of $f$ at $x_0$ and denoted by $\partial_\Phi f|_{x_0}$. Of course $\partial_\Phi f|_\cdot$ is a multifunction mapping $X$ into subsets of $\Phi$, $\partial_\Phi f|_\cdot : X \to 2^\Phi$. If $\partial_\Phi f|_x \neq \emptyset$ for all $x \in X$ we say that $f$ is *$\Phi$-subdifferentiable.*

Let $\gamma : X \times X \to \mathbb{R}$ and $f : X \to \mathbb{R}$. We say that a function $\phi_0 \in \Phi$ is a *$\Phi^{\gamma(\cdot,\cdot)}$-subgradient* of $f$ at a point $x_0$ if

$$f(x) - f(x_0) \geq \phi_0(x) - \phi_0(x_0) + \gamma(x, x_0) \tag{2}$$

for all $x \in X$. The set of all $\Phi^{\gamma(\cdot,\cdot)}$-subgradients of $f$ at $x_0$ is called the *$\Phi^{\gamma(\cdot,\cdot)}$-subdifferential* of $f$ at $x_0$ and denoted by $\partial_\Phi^{\gamma(\cdot,\cdot)} f|_{x_0}$. If $\partial_\Phi^{\gamma(\cdot,\cdot)} f|_x \neq \emptyset$ for all $x \in X$ we say that $f$ is *$\Phi^{\gamma(\cdot,\cdot)}$-subdifferentiable.*

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EXAMPLE 1. Let $(X, \|\cdot\|)$ be a normed space and let $\Phi = X^*$ be its conjugate. Let $\gamma(\cdot,\cdot) \equiv 0$. Then a $\Phi^{\gamma(\cdot,\cdot)}$-subgradient is a subgradient in the classical sense (see for example Rockafellar (1970)).

EXAMPLE 2. Let $(X, \|\cdot\|)$ be a normed space and $\Phi = X^*$. Let $\gamma(x,y) = -\varepsilon\|x-y\|$, where $\varepsilon > 0$. Then a $\Phi^{\gamma(\cdot,\cdot)}$-subgradient is an $\varepsilon$-subgradient (Ekeland–Lebourg (1975)).

EXAMPLE 3. Let $(X, \|\cdot\|)$ be a normed space and $\Phi = X^*$. Suppose that

$$\liminf_{x \to x_0} \frac{\gamma(x, x_0)}{\|x - x_0\|} \geq 0.$$

Then a $\Phi^{\gamma(\cdot,\cdot)}$-subgradient is an approximate subgradient of $f$ at $x_0$ (see Ioffe (1984), (1986), (1989), (1990), (2000), Mordukhovich (1976), (1980), (1988)).

EXAMPLE 4. Let $X$ be an arbitrary set. Let $\Phi$ be a family of real-valued functions defined on $X$. Let $\gamma(\cdot,\cdot) \equiv 0$. Then a $\Phi^{\gamma(\cdot,\cdot)}$-subgradient is a $\Phi$-subgradient in the sense of $\Phi$-convex analysis (see for example Pallaschke–Rolewicz (1997), Rubinov (2000), Singer (1997)).

EXAMPLE 5. Let $(X, d_X)$ be a metric space. Let $\Phi$ be a family of real-valued continuous functions defined on $X$. Let $\gamma(x,y) = \alpha(d_X(x,y))$, where $\alpha(\cdot)$ is a real-valued function. Then a $\Phi^{\gamma(\cdot,\cdot)}$-subgradient is a strong $\Phi$-subgradient with modulus $\alpha(\cdot)$ if $\alpha(\cdot) \geq 0$ (Rolewicz (1998), (2003)), and it is a weak $\Phi$-subgradient with modulus $\alpha(\cdot)$ if $\alpha(\cdot) \leq 0$ (Rolewicz (2000a,b)).

A multifunction $\Gamma : X \to 2^{\Phi}$ is called *$n$-cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotone* if, for arbitrary $x_0, x_1, \ldots, x_n = x_0 \in X$ and $\phi_{x_i} \in \Gamma(x_i)$, $i = 1, \ldots, n$, we have

$$\sum_{i=1}^{n} [\phi_{x_{i-1}}(x_{i-1}) - \phi_{x_{i-1}}(x_i) - \gamma(x_i, x_{i-1})] \geq 0.$$

A multifunction $\Gamma : X \to 2^{\Phi}$ is called *cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotone* if it is $n$-cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotone for $n = 2, 3, \ldots$.

For cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotone multifunctions the following extension of the Rockafellar Theorem can be shown:

THEOREM 6 (Rolewicz (2006)). *Let $X$ be an arbitrary set. Let $\Phi$ be a family of real-valued functions defined on $X$. Let $\gamma : X \times X \to \mathbb{R}$. Let $\Gamma$ be a cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotone multifunction. Suppose that $\Gamma(x) \neq \emptyset$ for all $x \in X$. Then there is a $\Phi^{\gamma(\cdot,\cdot)}$-subdifferentiable function $f$ such that $\Gamma(x)$ is contained in the $\Phi^{\gamma(\cdot,\cdot)}$-subdifferential of $f$,*

$$\Gamma(x) \subset \partial_{\Phi}^{\gamma(\cdot,\cdot)} f|_x.$$

Define

$$[\Phi + \gamma] = \{\phi(\cdot) + \gamma(\cdot, x) \mid \phi \in \Phi,\ x \in X\}. \tag{3}$$

It is natural to ask if it is possible to deduce Theorem 6 from Proposition 1.1.11 of Pallaschke–Rolewicz (1997) on existence, for each cyclic monotone multifunction $\Gamma$, of a function such that $\Gamma(x)$ is contained in its $[\Phi+\gamma]$-subdifferential.

For this purpose in this note we investigate the relation between $\Phi^{\gamma(\cdot,\cdot)}$-subdifferentiable and $[\Phi+\gamma]$-subdifferentiable functions.

The following is easy to see:

PROPOSITION 7. *Let $X$ be an arbitrary set. Let $\Phi$ be a family of real-valued functions defined on $X$. Let $\gamma : X \times X \to \mathbb{R}$ be such that $\gamma(x,x) = 0$ for all $x \in X$. Let $f : X \to \mathbb{R}$. Then a $\Phi^{\gamma(\cdot,\cdot)}$-subgradient $\phi_0$ of $f$ at a point $x_0$ is simultaneously a $[\Phi+\gamma]$-subgradient of $f$ at $x_0$.*

*Proof.* By definition

$$f(x) - f(x_0) \geq \phi_0(x) - \phi_0(x_0) + \gamma(x, x_0) \tag{2}$$

for all $x \in X$. Since $\gamma(x,x) = 0$, in particular $\gamma(x_0, x_0) = 0$, (2) can be rewritten as

$$\begin{aligned} f(x) - f(x_0) &\geq \phi_0(x) - \phi_0(x_0) + \gamma(x, x_0) - \gamma(x_0, x_0) \\ &= [\phi_0(x) + \gamma(x, x_0)] - [\phi_0(x_0) + \gamma(x_0, x_0)], \end{aligned} \tag{2'}$$

i.e. $[\phi_0(x) + \gamma(x, x_0)] \in [\Phi + \gamma]$ is a subgradient of $f$ at $x_0$. ∎

The converse is not true as follows from

EXAMPLE 8. Let $X = [-1, 1]$, let $\Phi$ consist of constant functions only and let $\gamma(y, x) = (y - x)^2$. Let $f(x) = \max[(x-1)^2, (x+1)^2]$. At any point $x_0$ the function $f$ has the $[\Phi+\gamma]$-subgradient

$$\phi_{x_0}(x) = \begin{cases} (x-1)^2 & \text{for } x_0 < 0, \\ (x+1)^2 & \text{for } x_0 \geq 0. \end{cases}$$

On the other hand, a $\Phi^{\gamma(\cdot,\cdot)}$-subgradient of $f$ exists at no $x_0 \neq 0$.

As a consequence of Example 8 we see that there are $[\Phi+\gamma]$-subdifferentiable functions which are not $\Phi^{\gamma(\cdot,\cdot)}$-subdifferentiable.

The aim of this note is to obtain conditions which guarantee that every $[\Phi+\gamma]$-subdifferentiable function is $\Phi^{\gamma(\cdot,\cdot)}$-subdifferentiable.

We say that a function $\gamma(\cdot,\cdot)$ is *$\Phi$-subdifferentiable with respect to the first variable* if for every $x_1$ the function $\gamma(\cdot, x_1)$ is $\Phi$-subdifferentiable, i.e. for every $y \in X$ there exists a $\Phi$-subgradient $\phi_y$ of $\gamma(y, x_1)$ at $y$. In other words, for any $z \in X$,

$$\gamma(z, x_1) - \gamma(y, x_1) \geq \phi_y(z) - \phi_y(y) + \gamma(z, y). \tag{4}$$

Proposition 9. *Let $X$ be an arbitrary set. Let $\Phi$ be a linear family of real-valued functions defined on $X$. Let $\gamma : X \times X \to \mathbb{R}$ be such that $\gamma(x,x) = 0$ for all $x \in X$. Suppose that $\gamma$ is $\Phi$-subdifferentiable with respect to the first variable. If $\phi$ is a $[\Phi+\gamma]$-subgradient of a function $f$ at $x_0$, then there is a $\psi \in \Phi$ such that $\psi(\cdot)$ is a $\Phi^{\gamma(\cdot,\cdot)}$-subgradient of $f$ at $x_0$.*

*Proof.* By definition if $\phi \in [\Phi+\gamma]$, there are $\psi \in \Phi$ and $x_1 \in X$ such that $\phi(\cdot) = \psi(\cdot) + \gamma(\cdot, x_1)$. Since $\phi(\cdot)$ is a subgradient of $f$ at $x_0$, for all $x \in X$ we have

$$f(x) - f(x_0) \geq \phi(x) + \gamma(x, x_1) - \phi(x_0) - \gamma(x_0, x_1). \tag{5}$$

Since $\gamma$ is $\Phi$-subdifferentiable with respect to the first variable, putting $z = x$, $y = x_0$ we deduce from (4) that there is a $\Phi$-subgradient $\phi_{x_0}$ at $x_0$ such that for any $x \in X$,

$$\gamma(x, x_1) - \gamma(x_0, x_1) \geq \phi_{x_0}(x) - \phi_{x_0}(x_0) + \gamma(x, x_0). \tag{4'}$$

Therefore

$$f(x) - f(x_0) \geq \phi(x) + \phi_{x_0}(x) - \phi(x_0) - \phi_{x_0}(x_0) + \gamma(x, x_0). \tag{6}$$

Thus $\psi(\cdot) = \phi(\cdot) + \phi_{x_0}(\cdot)$ is a $\Phi^{\gamma(\cdot,\cdot)}$-subgradient of $f$ at $x_0$. ∎

As an obvious consequence we obtain

Corollary 10. *Let $X$ be an arbitrary set. Let $\Phi$ be a linear family of real-valued functions defined on $X$. Let $\gamma : X \times X \to \mathbb{R}$ be such that $\gamma(x,x) = 0$ for all $x \in X$. Suppose that $\gamma$ is $\Phi$-subdifferentiable with respect to the first variable. Then every $[\Phi+\gamma]$-subdifferentiable function $f$ is $\Phi^{\gamma(\cdot,\cdot)}$-subdifferentiable.*

It is interesting to find the form of functions $\gamma(\cdot,\cdot)$ $\Phi$-subdifferentiable with respect to the first variable.

Let $X$ be a linear space over the reals. Let $\gamma(x,y) = \alpha(x-y)$, where $\alpha : X \to \mathbb{R}$. Putting $y = 0$ we trivially get

Proposition 11. *Let $X$ be a linear space over the reals. Let $\Phi$ be a linear family of real-valued functions defined on $X$. Let $\alpha : X \to \mathbb{R}^+$. If $\gamma$ is $\Phi$-subdifferentiable with respect to the first variable, then the function $\alpha(\cdot)$ is $\Phi$-subdifferentiable.*

The converse is true under some additional condition. We say that a family $\Phi$ of real-valued functions defined on a linear space $X$ over the reals is *shift invariant* if for all $\phi \in \Phi$ and $z \in X$ there are $\phi_z \in \Phi$ and $c_z \in \mathbb{R}$ such that

$$\phi(x+z) = \phi_z(x) + c_z. \tag{7}$$

Example 12. Let $X$ be a linear space. Let $\Phi$ be a family of linear functionals. Then $\Phi$ is shift invariant.

EXAMPLE 13. Let $X$ be a linear space. Let $\Phi$ be the family of all polynomial functionals of order $n$. Then $\Phi$ is shift invariant.

EXAMPLE 14. Let $X$ be a normed space. Let $\Phi$ be the family of all continuous polynomial functionals of order $n$. Then $\Phi$ is shift invariant.

EXAMPLE 15. Let $X = \mathbb{R}^m$. Let $\Phi$ be the family of all trigonometric polynomials of order $n$. Then $\Phi$ is shift invariant.

PROPOSITION 16. *Let $X$ be a linear space over the reals. Let $\Phi$ be a shift invariant family. Let $\gamma(x,z) = \alpha(x-z)$, where $\alpha : X \to \mathbb{R}^+$. If $\alpha$ is $\Phi$-subdifferentiable, then $\gamma$ is $\Phi$-subdifferentiable with respect to the first variable.*

*Proof.* Since $\alpha$ is $\Phi$-subdifferentiable, there is $\phi^{x-z}(\cdot) \in \Phi$ such that

$$\gamma(y,z) - \gamma(x,z) = \alpha(y-z) - \alpha(x-z) \geq \phi^{x-z}(y-z) - \phi^{x-z}(x-z). \tag{8}$$

Since the family $\Phi$ is shift invariant, there are $\phi_z \in \Phi$ and $c_z \in \mathbb{R}$ such that

$$\phi^{x-z}(u+z) = \phi_z(u) + c_z. \tag{7'}$$

Therefore (8) can be rewritten as

$$\gamma(y,z) - \gamma(x,z) = \alpha(y-z) - \alpha(x-z) \geq \phi_z(y) - \phi_z(x), \tag{8}$$

i.e. $\gamma$ is $\Phi$-subdifferentiable with respect to the first variable. ∎

Let $\Phi$ be a linear shift invariant family of linear functionals defined on $X$. Let $\gamma(x,y) = \alpha(x-y)$, where $\alpha : X \to \mathbb{R}$. Suppose that $\gamma$ is $\Phi$-subdifferentiable with respect to the first variable. In this case the formula (4′) can be rewritten in the form

$$\alpha(x-x_1) - \alpha(x_0-x_1) \geq \phi_{x_0}(x) - \phi_{x_0}(x_0) + \alpha(x-x_0). \tag{9}$$

Since $\phi_{x_0}$ is linear this can be rewritten as

$$\alpha(x-x_1) - \alpha(x_0-x_1) - \alpha(x-x_0) \geq \phi_{x_0}(x-x_0). \tag{10}$$

We put $t = x_0 - x_1$, $s = x - x_0$. It is easy to see that $t + s = x - x_1$ and $x_0 = t + x_1$. Let

$$\Psi(t,s) = \phi_{t+x_1}(s).$$

Then (10) can be rewritten in the form

$$\alpha(t+s) - \alpha(t) - \alpha(s) \geq \Psi(t,s), \tag{11}$$

where $\Psi(t,\cdot)$ is linear (then homogeneous) with respect to the second variable. Therefore by the result of Baron and Kominek (2003) (Corollary 2; see also Choczewski (2001) and Choczewski *et al.* (2000)) we obtain

PROPOSITION 17. *Let $X$ be a linear space over the reals. Let $\Phi$ be a linear family of linear functionals defined on $X$. Let $\gamma(x,y) = \alpha(x-y)$,* *where* $\alpha : X \to \mathbb{R}^+$*. Then any* $\gamma$ *that is* $\Phi$*-subdifferentiable with respect to the first variable is of the form*

$$\gamma(x, y) = B(x - y, x - y), \tag{12}$$

*where* $B(\cdot, \cdot)$ *is bilinear and symmetric.*

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