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Some Remarks on Indicatrices of Measurable Functions

by

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Summary. We show that for a wide class of σ-algebras $\mathcal{A}$, indicatrices of $\mathcal{A}$-measurable functions admit the same characterization as indicatrices of Lebesgue-measurable functions. In particular, this applies to functions measurable in the sense of Marczewski.

Let $f : X \to Y$ be a function. The function $s(f) : Y \to \textsc{Card}$ defined by the formula $s(f)(y) = |f^{-1}[\{y\}]$ is called the (*Banach*) *indicatrix* of f. For $f, g : X \to Y$, we say that f is *equivalent* to g if there exists a bijection $\varphi : X \to X$ such that $f = g \circ \varphi$. Obviously, this is equivalent to saying that $s(f) = s(g)$.

Morayne and Ryll-Nardzewski show in [5] that a function $f : [0,1] \to [0,1]$ is equivalent to a Lebesgue-measurable one if, and only if, either $s(f) > 0$ on a perfect set $P \subseteq [0,1]$ or there exists $y \in [0,1]$ such that $s(f)(y) = \mathfrak{c}$. In fact, they prove a more general statement. Namely, the same is true for the class of functions which are measurable with respect to the σ-algebra $\mathcal{A}$ generated by the Borel sets and a σ-ideal $\mathcal{I}$ with Borel base containing an uncountable set. They also ask about a characterization of indicatrices of other important classes of functions.

A characterization of indicatrices of continuous functions was given by Kwiatkowska in [4]. Also, Komisarski, Michalewski and Milewski in [3] characterized (under the axiom of $\mathbf{\Sigma}^1_1$-determinacy) indicatrices of Borel functions.

The purpose of this note is to generalize the characterization of Morayne and Ryll-Nardzewski to other classes of measurable functions. We say that a set $X \subseteq [0,1]$ is *Marczewski-measurable* if for every perfect set $P \subseteq [0,1]$

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there exists a perfect set $Q \subseteq P$ such that $Q \subseteq X$ or $Q \cap X = \emptyset$. The Marczewski-measurable sets form a σ-algebra; a function $f : [0,1] \to [0,1]$ is *Marczewski-measurable* if the pre-image of every open set is Marczewski-measurable. By Marczewski's theorem (see [7]) this is equivalent to saying that for every perfect set P there exists a perfect set $Q \subseteq P$ such that $f \restriction Q$ is continuous.

We begin by showing that indicatrices of Marczewski-measurable functions admit the same characterization as those of Lebesgue-measurable ones. It is known that the algebra of Marczewski-measurable sets is not of the form considered in [5]. Then we try to isolate the properties of Marczewski-measurable sets and functions used in the proof to obtain a more general result.

For a family $\mathcal{A}$ of sets, let $\mathcal{H}(\mathcal{A}) = \{A \in \mathcal{A} : \forall B \subseteq A \ \ B \in \mathcal{A}\}$. Observe that if $\mathcal{A}$ is a σ-algebra, then $\mathcal{H}(\mathcal{A})$ is a σ-ideal.

The following lemma is a slight modification of an argument from [5]. The main difference is that we do not use the assumption of Borel base of the ideal.

LEMMA 1. *Let $\mathcal{A}$ be a σ-algebra containing* Bor *such that $\mathcal{H}(\mathcal{A})$ contains a set of size $\mathfrak{c}$. Let $f : [0,1] \to [0,1]$ be a function such that $f[[0,1]]$ contains a perfect set. Then f is equivalent to an $\mathcal{A}$-measurable function.*

Proof. Let P be a perfect set contained in the image of f; we may always assume that $|f[[0,1]] \setminus P| = \mathfrak{c}$. Let $\Psi : [0,1] \to P$ be a Borel isomorphism and let $M \in \mathcal{H}(\mathcal{A})$ be a set of cardinality $\mathfrak{c}$ such that $|[0,1] \setminus M| = \mathfrak{c}$. Observe that Ψ is $\mathcal{A}$-measurable.

Let $s(f) : [0,1] \to \text{CARD}$ be the indicatrix of f and let $\{M_y : y \in [0,1]\}$ be a partition of M such that $|M_y| = s(f)(y) - 1$ for $y \in \Psi[[0,1] \setminus M]$ (this is meaningful, because $s(f)(y) > 0$ for $y \in P$ and we allow M_y to be empty) and $|M_y| = s(f)(y)$ otherwise. Such a partition can be found because for continuum many $y \in [0,1]$ we stipulate that $|M_y| > 0$, so $\sum_{y \in [0,1]} |M_y| = \mathfrak{c}$. Define $g : [0,1] \to [0,1]$ in the following way:

$$g(x) = \begin{cases} \Psi(x) & \text{for } x \notin M, \\ y & \text{for } x \in M_y. \end{cases}$$

Clearly, g is equivalent to f because they have the same indicatrix, and g is $\mathcal{A}$-measurable, as

$$\{x \in [0,1] : g(x) \neq \Psi(x)\} \subseteq M \in \mathcal{H}(\mathcal{A}). \ \blacksquare$$

Using exactly the same argument as in [5], one can prove the following.

LEMMA 2. *Let $\mathcal{A}$ be a σ-algebra containing* Bor *such that $\mathcal{H}(\mathcal{A})$ contains a set of size $\mathfrak{c}$. Let $f : [0,1] \to [0,1]$ be a function constant on a set of cardinality $\mathfrak{c}$. Then f is equivalent to an $\mathcal{A}$-measurable function.* $\blacksquare$

THEOREM 3. *A function $f:[0,1]\to[0,1]$ is equivalent to a Marczewski-measurable one if, and only if, either $f[[0,1]]$ contains a perfect set, or there exists $y\in[0,1]$ such that $|f^{-1}[\{y\}]|=\mathfrak{c}$. In particular, each Lebesgue measurable function is equivalent to a Marczewski-measurable one, and vice versa.*

Proof. It is folklore that the algebra of Marczewski-measurable sets satisfies the assumptions of Lemmas 1 and 2, which shows sufficiency of this condition.

To prove the necessity, we can assume that f is itself Marczewski-measurable. Then there exists a perfect set P such that $f\restriction P$ is continuous. If $f[P]$ is uncountable, then it contains a perfect set. Otherwise, there exists $y\in f[P]$ such that the set $f^{-1}[\{y\}]$ is of size continuum. ∎

One can easily see that the argument above is more general than for Marczewski-measurable functions. The assumptions needed for sufficiency of the characterization (i.e. the assumptions of Lemmas 1 and 2) are very general (as long as the extensions of Bor are concerned). To prove the necessity, we only used the fact that a Marczewski-measurable function is continuous on a perfect set.

Let us say that a class of functions $\mathcal{F}$ from a Polish space to $[0,1]$ has the *Weak Continuous Restriction Property* (WCRP for short) if every $f\in\mathcal{F}$ is continuous on a perfect set. This is a weaker property than the Continuous Restriction Property considered in [6], where the perfect set is required not to belong to a given σ-ideal. It is also a weaker version of a suitable instance of the Sierpiński condition considered in [1].

Let us point out that some natural reformulations of the WCRP are in fact equivalent.

PROPOSITION 4 (folklore). *The following conditions are equivalent for $f:X\to[0,1]$, where X is a Polish space:*

(1) *$f\restriction P$ is continuous for some perfect set P,*
(2) *$f\restriction B$ is continuous for some uncountable Borel set B,*
(3) *$f\restriction P$ is Borel for some perfect set P,*
(4) *$f\restriction B$ is Borel for some uncountable Borel set B.* ∎

As an immediate generalization of Theorem 3 we obtain the following.

THEOREM 5. *Let $\mathcal{A}$ be a σ-algebra of subsets of a Polish space X containing $\mathsf{Bor}(X)$ such that $\mathcal{H}(\mathcal{A})$ contains a set of size $\mathfrak{c}$. Assume that the class of $\mathcal{A}$-measurable functions has WCRP. Then a function $f:X\to X$ is equivalent to an $\mathcal{A}$-measurable one if, and only if, either $f[X]$ contains a perfect set, or there exists $y\in X$ such that $|f^{-1}[\{y\}]|=\mathfrak{c}$.*

Proof. Analogous to the proof of Theorem 3. ∎

An important class of algebras satisfying the assumptions of Theorem 5 are the algebras of sets decided by popular forcing notions. We can interpret the Marczewski-measurable sets as sets decided by the Sacks forcing $\mathbb{S}$ (i.e. sets X such that the set of conditions in $\mathbb{S}$ which either miss X or are included in X is dense). It is folklore that if we replace the Sacks forcing by the forcing notion of Laver, Mathias, Miller or Silver, the functions measurable with respect to the corresponding σ-algebra have WCRP. Also, each of the respective ideals ([1]) contains a set of size $\mathfrak{c}$ (this follows from the results of [2]). In particular, in the case of Mathias forcing, we obtain the following.

COROLLARY 6. *Let $\mathcal{A}$ be the σ-algebra of completely Ramsey subsets of 2^ω. Then a function $f : 2^\omega \to 2^\omega$ is equivalent to an $\mathcal{A}$-measurable one if, and only if, either $f[2^\omega]$ contains a perfect set, or there exists $y \in 2^\omega$ such that $|f^{-1}[\{y\}]| = \mathfrak{c}$.* ■

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([1]) In the case of these forcing notions, the ideal of hereditarily measurable sets coincides with the ideal of sets missed by a dense set of conditions.