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PARTIAL DIFFERENTIAL EQUATIONS

On the Convective Cahn–Hilliard Equation

by

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Summary. The author studies the convective Cahn–Hilliard equation. Some results on the existence of classical solutions and asymptotic behavior of solutions are established. The instability of the traveling waves is also discussed.

1. Introduction. In this paper, we investigate the convective Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} + \gamma \frac{\partial^4 u}{\partial x^4} = \frac{\partial^2 A(u)}{\partial x^2} + \frac{\partial B(u)}{\partial x}, \qquad x \in I = (0,1), \tag{1.1}$$

where $A(u) = \gamma_2 u^3 + \gamma_1 u^2 - u$, $B(u) = -\frac{1}{4}u^4 + \frac{1}{2}u^2$, and $\gamma > 0, \gamma_1, \gamma_2$ are constants.

On the basis of physical considerations, as usual the equation (1.1) is supplemented with the natural boundary value conditions

$$u(0,t) = u(1,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(1,t) = 0, \qquad t > 0, \tag{1.2}$$

reasonable for the thin film equation or the Cahn-Hilliard equation (see [1, 2, 4]), and the initial value condition

$$u(x,0) = u_0(x). \tag{1.3}$$

The equation (1.1) arises naturally as a continuous model for the formation of facets and corners in crystal growth (see [6], [12]). Here $u(x,t)$ denotes the slope of the interface. The convective term $(u^3 - u)\partial u/\partial x$ (see [6]) stems from the effect of kinetics (the finite rate of atoms or molecules attachment

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to the crystal surface) that provides an independent flux of the order parameter, similar to the effect of an external field in spinodal decomposition of a driven system.

During the past years, many authors have paid much attention to the Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} + k\Delta^2 u = \Delta A(u), \qquad k > 0 \tag{1.4}$$

(see [3, 5, 14]). However, only a few papers are devoted to the generalized Cahn–Hilliard equation. It was K. H. Kwek [7] who first studied the equation (1.1) for the case with convection, namely, $B(u) = u$. Based on the discontinuous Galerkin finite element method, he proved the existence of classical solutions.

This paper is organized as follows: We first discuss the existence and asymptotic behavior of classical solutions. Then we discuss the instability of traveling waves.

2. Existence. In this section, we prove the global existence of solutions. From the classical approach, it is not difficult to conclude that the problem (1.1)–(1.3) admits a unique classical solution local in time. So, it is sufficient to make an a priori estimate.

THEOREM 2.1. *If* $\gamma_2 > 0$, *then for any initial data* $u_0 \in H^6(I)$ *with* $u_0(0) = u_0(1) = D^2u_0(0) = D^2u_0(1) = 0$ *and* $T > 0$, *the problem* (1.1)–(1.3) *has a unique global classical solution.*

Proof. Multiplying both sides of (1.1) by u and then integrating the resulting relation with respect to x over $(0,1)$, integrating by parts, and using the boundary conditions, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx + \gamma\int_0^1 (D^2u)^2\,dx = -\int_0^1 A'(u)(Du)^2\,dx + \int_0^1 \left(\frac{1}{4}u^4 - \frac{1}{2}u^2\right)Du\,dx. \tag{2.1}$$

Since $\gamma_2 > 0$, a simple calculation shows that $A'(u) \ge -C_0$, $C_0 > 0$ and $\int_0^1 \left(\frac{1}{4}u^4 - \frac{1}{2}u^2\right)Du\,dx = 0$, so it follows from (2.1) that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx + \gamma\int_0^1 (D^2u)^2\,dx \le C_0\int_0^1 (Du)^2\,dx$$

$$\le C_0\Big(\int_0^1 (D^2u)^2\,dx\Big)^{1/2}\Big(\int_0^1 u^2\,dx\Big)^{1/2} \le \frac{\gamma}{2}\int_0^1 (D^2u)^2\,dx + C_1\int_0^1 u^2\,dx.$$

The Gronwall inequality implies that

$$\int_0^1 u^2\,dx \le C, \qquad 0<t<T, \tag{2.2}$$

$$\int_0^t \|D^2u(s)\|^2\,ds \le C, \qquad 0<t<T. \tag{2.3}$$

Next, multiplying both sides of (1.1) by D^2u and then integrating the resulting relation with respect to x over $(0,1)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (Du)^2\,dx + \gamma\int_0^1 (D^3u)^2\,dx \\ = -\int_0^1 D^2A(u)D^2u\,dx - \int_0^1 \Big(\frac14\,u^4-\frac12\,u^2\Big)D^3u\,dx.\end{aligned} \tag{2.4}$$

Note that

$$\begin{aligned}D^2A(u) &= A'(u)D^2u + A''(u)(Du)^2\\ &= (3\gamma_2u^2+2\gamma_1u-1)D^2u + (6\gamma_2u+2\gamma_1)(Du)^2.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (Du)^2\,dx &+ \gamma\int_0^1 (D^3u)^2\,dx + \gamma_2\int_0^1 u^2(D^2u)^2\,dx\\
&= -\int_0^1 (2\gamma_2u^2+2\gamma_1u-1)(D^2u)^2\,dx - \int_0^1 6\gamma_2u(Du)^2D^2u\,dx\\
&\quad -\int_0^1 2\gamma_1(Du)^2D^2u\,dx - \int_0^1\Big(\frac14\,u^4-\frac12\,u^2\Big)D^3u\,dx\\
&\le C\int_0^1 (D^2u)^2\,dx + \frac{\gamma_2}{2}\int_0^1 u^2(D^2u)^2\,dx + C\int_0^1 (Du)^4\,dx\\
&\quad + \frac{\gamma}{8}\int_0^1 (D^3u)^2\,dx + C\int_0^1\Big(\frac14\,u^4-\frac12\,u^2\Big)^2\,dx.\end{aligned}$$

On the other hand, by (2.2), we have

$$\begin{aligned}\int_0^1 (Du)^2dx = -\int_0^1 uD^2u\,dx &\le \Big(\int_0^1 u^2\,dx\Big)^{1/2}\Big(\int_0^1 (D^2u)^2\,dx\Big)^{1/2}\\ &\le C\Big(\int_0^1 (D^2u)^2\,dx\Big)^{1/2},\end{aligned}$$

and

$$\int_0^1 (D^2u)^2 dx = -\int_0^1 DuD^3u dx \le \Big(\int_0^1 (Du)^2 dx\Big)^{1/2}\Big(\int_0^1 (D^3u)^2\, dx\Big)^{1/2}.$$

Summing up, we see that

$$\int_0^1 (Du)^2 dx \le C\Big(\int_0^1 (Du)^2\, dx\Big)^{1/4}\Big(\int_0^1 (D^3u)^2\, dx\Big)^{1/4}.$$

Thus

$$\int_0^1 (Du)^2 dx \le C\Big(\int_0^1 (D^3u)^2\, dx\Big)^{1/3}, \tag{2.5}$$

$$\int_0^1 (D^2u)^2\, dx \le C\Big(\int_0^1 (D^3u)^2\, dx\Big)^{2/3}. \tag{2.6}$$

In addition, by the Nirenberg inequality,

$$\int_0^1 (Du)^4\, dx \le C\Big(\int_0^1 (D^3u)^2\, dx\Big)^{1/4}\Big(\int_0^1 (Du)^2\, dx\Big)^{7/4}.$$

Using (2.5), we obtain

$$\int_0^1 (Du)^4\, dx \le C\Big(\int_0^1 (D^3u)^2\, dx\Big)^{5/6}. \tag{2.7}$$

By (2.6), (2.7) and using the Hölder inequality, we have

$$C\int_0^1 (D^2u)^2\, dx \le \frac{\gamma}{8}\int_0^1 (D^3u)^2\, dx + C,$$

$$C\int_0^1 (Du)^4\, dx \le \frac{\gamma}{8}\int_0^1 (D^3u)^2\, dx + C.$$

On the other hand, by the Nirenberg inequality,

$$\sup|u| \le C\Big(\int_0^1 (D^3u)^2\, dx\Big)^{1/12}\Big(\int_0^1 u^2\, dx\Big)^{5/12} \le C\Big(\int_0^1 (D^3u)^2\, dx\Big)^{1/12}.$$

Hence

$$C\int_0^1 \Big(\frac{1}{4}\, u^4 - \frac{1}{2}\, u^2\Big)^2\, dx \le \frac{\gamma}{8}\int_0^1 (D^3u)^2\, dx + C.$$

Summing up, we see that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (Du)^2\, dx + \frac{\gamma}{2}\int_0^1 (D^3u)^2\, dx + \frac{\gamma_2}{2}\int_0^1 u^2\, (D^2u)^2\, dx \le C.$$

The Gronwall inequality implies that

$$\int_0^1 (Du)^2 dx \le C, \qquad 0 < t < T. \tag{2.8}$$

By Sobolev's imbedding theorem it follows from (2.2) and (2.8) that

$$\sup |u| \le C. \tag{2.9}$$

Multiplying both sides of (1.1) by D^4u and then integrating the resulting relation with respect to x over $(0,1)$, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (D^2u)^2\,dx + \gamma\int_0^1 (D^4u)^2\,dx = \int_0^1 D^2A(u)D^4u\,dx - \int_0^1 (u^3-u)DuD^4u\,dx.$$

By the Nirenberg inequality,

$$\|Du\|_\infty \le (\|D^4u\|^{3/8}\|u\|^{5/8} + \|u\|),$$

hence,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 (D^2u)^2\,dx + \gamma\int_0^1 (D^4u)^2\,dx \\
&\quad \le \left|\int_0^1 A'(u)D^2uD^4u\,dx\right| + \left|\int_0^1 A''(u)(Du)^2D^4u\,dx\right| + \left|\int_0^1 (u^3-u)DuD^4u\,dx\right| \\
&\quad \le C\|D^2u\|\|D^4u\| + C\|Du\|_\infty\|Du\|\|D^4u\| + C\|Du\|\|D^4u\| \\
&\quad \le \frac{\gamma}{2}\|D^4u\|^2 + C\|D^2u\|^2,
\end{aligned}$$

and by the Gronwall inequality,

$$\int_0^1 (D^2u)^2\,dx \le C, \qquad 0 < t < T, \tag{2.10}$$

$$\int_0^t \|D^4u\|^2\,ds \le C, \qquad 0 < t < T. \tag{2.11}$$

By (2.8) and (2.9), we have $\partial B(u)/\partial x \in L^2(Q_T)$. On the other hand, by (2.2) and (2.8)–(2.10), we have $\partial^2 A(u)/\partial x^2 \in L^2(Q_T)$. Then using the equation (1.1) itself, we obtain $\partial u/\partial t \in L^2(Q_T)$.

Define the linear space

$$X = \left\{u \in H^{4,1}(Q_T);\ u(0,t) = u(1,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(1,t) = 0,\right.$$
$$\left. u(x,0) = u_0(x)\right\}$$

where

$$H^{4,1}(Q_T) = \left\{u;\ \frac{\partial u}{\partial t} \in L^2(Q_T),\ \frac{\partial^i u}{\partial x^i} \in L^2(Q_T),\ 0 \le i \le 4\right\},$$

and the associated operator $T : X \to X$, $u \mapsto w$, where w is determined by the following linear problem:

$$\begin{aligned}
&\frac{\partial w}{\partial t} + \gamma\frac{\partial^4 w}{\partial x^4} = \frac{\partial^2 A(u)}{\partial x^2} + \frac{\partial B(u)}{\partial x}, \qquad x \in I = (0,1),\\
&w(0,t) = w(1,t) = \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(1,t) = 0, \qquad t > 0,\\
&w(x,0) = u_0(x).
\end{aligned}$$

From the discussions above and by the contraction mapping principle, T has a unique fixed point u, which is the desired solution of the problem (1.1)–(1.3).

Further regularity of the solution is obtained by the use of a bootstrap argument. Since $u \in H^{4,1}(Q_T)$, we have

$$Du \in L^\infty(Q_T), \qquad D^2u \in L^2(0,T;L^\infty(I)).$$

It follows, by a direct calculation, that $f(x,t) \equiv D^2A(u(x,t)) + DB(u(x,t))$, then

$$Df \in L^2(Q_T), \qquad D^2f \in L^2(Q_T).$$

By [8], we know that if $f \in L^2(0,T;L^2(I))$, and $v_0 \in H^2(I), v_0|_{\partial I} = 0$ then the initial boundary value problem

$$\begin{aligned}
&\frac{\partial v}{\partial t} + \gamma D^4 v = f,\\
&v = D^2 v = 0,\\
&v(x,0) = v_0
\end{aligned}$$

has a unique solution $v \in H^{4,1}(Q_T)$. Now it is easy to see that taking

$$f(x,t) = D^4A(u(x,t)) + D^3B(u(x,t)), \qquad v_0 = D^2u_0$$

yields $v = D^2u \in H^{4,1}(Q_T)$. This implies that $f = (\partial/\partial t)(D^2A(u) + DB(u)) \in L^2(Q_T)$, assuming that $u_t(0,0) = u_t(1,0) = 0$. Hence

$$v = \frac{\partial u}{\partial t} \in H^{4,1}(Q_T)$$

and by interpolation theory, this implies that

$$\frac{\partial u}{\partial t}, D^4 u \in C(\overline{Q}_T).$$

This completes the proof of the existence of a classical solution.

THEOREM 2.2. *Under the conditions of Theorem* 2.1, *if*

$$\gamma > \frac{1}{2}\left(\frac{\gamma_1^2}{3\gamma_2} + 1\right),$$

then the classical solution u *of* (1.1)–(1.3) *satisfies*

$$\|u\|^2 \le e^{-C_1 t - C_2}, \qquad C_1, C_2 > 0.$$

Proof. Multiplying both sides of (1.1) by u and then integrating the resulting relation with respect to x over $(0,1)$, integrating by parts, and using the boundary conditions, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx + \gamma\int_0^1 (D^2u)^2\,dx = -\int_0^1 A'(u)(Du)^2\,dx + \int_0^1 \left(\frac{1}{4}u^4 - \frac{1}{2}u^2\right) Du\,dx. \tag{2.12}$$

Since $\gamma_2 > 0$, a simple calculation shows that $A'(u) \ge -C_0 = -\gamma_1^2/3\gamma_2 - 1$, $C_0 > 0$ and $\int_0^1 \left(\frac{1}{4}u^4 - \frac{1}{2}u^2\right) Du\,dx = 0$, and it follows from (2.1) that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx + \gamma\int_0^1 (D^2u)^2\,dx \le \left(\frac{\gamma_1^2}{3\gamma_2} + 1\right)\int_0^1 (Du)^2\,dx.$$

By the Poincaré inequality,

$$\int_0^1 (Du)^2 dx \le \frac{1}{2}\int_0^1 (D^2u)^2 dx + \frac{1}{2}\left(\int_0^1 Du\,dx\right)^2 = \frac{1}{2}\int_0^1 (D^2u)^2\,dx.$$

Hence

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx + \gamma\int_0^1 (D^2u)^2\,dx \le \frac{1}{2}\left(\frac{\gamma_1^2}{3\gamma_2} + 1\right)\int_0^1 (D^2u)^2\,dx.$$

Since $\gamma > \frac{1}{2}\left(\frac{\gamma_1^2}{3\gamma_2} + 1\right)$ and

$$\int_0^1 u^2\,dx \le \int_0^1 (Du)^2\,dx \le \int_0^1 (D^2u)^2\,dx,$$

we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx + C\int_0^1 u^2\,dx \le 0.$$

Therefore

$$\|u\|^2 \le e^{-C_1 t - C_2}, \qquad C_1, C_2 > 0.$$

This completes the proof.

3. Instability of traveling waves. In this section, we study the instability of the traveling wave solutions of the equation (1.1). For simplicity we set $\gamma = 1$, $\gamma_2 = 1$, i.e.

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = \frac{\partial^2}{\partial x^2}(u^3 + \gamma_1 u^2 - u) - (u^3 - u)\frac{\partial u}{\partial x}. \tag{3.1}$$

Our main result is as follows:

THEOREM 3.1. *All the traveling wave solutions $\varphi(x-ct)$ of the equation* (3.1) *satisfying $\varphi \in L^\infty(\mathbb{R})$, $\varphi^{(n)} \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ $(n = 1,2,3,4)$ are nonlinearly unstable in the space $H^2(\mathbb{R})$. Here $\varphi^{(n)}$ denotes the nth derivative of φ.*

By [6], the equation (3.1) has a solution satisfying

$$\lim_{z\to+\infty} \varphi(z) = A_1 + B_1, \qquad \lim_{z\to-\infty} \varphi(z) = A_1 - B_1,$$

where A_1, B_1 are constants. In fact, we set

$$\varphi = A_1 + B_1 \tanh[\lambda B_1(x - ct)]. \tag{3.2}$$

Substituting (3.2) in (3.1), one finds that

$$\lambda^3 - \frac{1}{2}\lambda - \frac{1}{24} = 0, \tag{3.3}$$

and the following relations hold for A_1 and B_1:

$$A_1 = -\frac{2\gamma_1\lambda}{6\lambda+1}, \quad B_1^2 = \frac{2\lambda+1}{2(\lambda+1/3)} + \frac{2\gamma_1^2\lambda^2(6\lambda-1)}{(6\lambda+1)^2(\lambda+1/3)}.$$

It is sufficient to take $B > 0$. Since $f(\lambda) = \lambda^3 - \frac{1}{2}\lambda - \frac{1}{24}$ satisfies $f(0) < 0$ and $f(1) > 0$, by the intermediate value theorem there exists a $\lambda \in (0,1)$ such that (3.3) holds.

We note that $\varphi = A_1 + B_1 \tanh[\lambda B_1(x-ct)]$ satisfies the conditions of Theorem 3.1.

To prove Theorem 3.1, we first consider an evolution equation

$$\frac{\partial u}{\partial t} = Lu + F(u), \tag{3.4}$$

where L is a linear operator that generates a strongly continuous semigroup e^{tL} on a Banach space X, and F is a strongly continuous operator such that $F(0) = 0$. In [10] the authors considered the whole problem only on X, that is to say, the nonlinear operator maps X into X. However, many equations have nonlinear terms that include derivatives and therefore F maps into a larger Banach space Z. Hence, Strauss and Wang proved the following lemma.

LEMMA 3.1 ([11]). *Assume the following*:

(i) *$X \subset Z$ are Banach spaces and $\|u\|_Z \le C_1\|u\|_X$ for $u \in X$.*
(ii) *L generates a strongly continuous semigroup e^{tL} on Z, mapping Z into X for $t > 0$, and $\int_0^1 \|e^{tL}\|_{Z\to X}\,dt = C_4 < \infty$.*

(iii) *The spectrum of* L *on* X *meets the right half-plane* $\{\mathrm{Re}\,\lambda > 0\}$.
(iv) $F : X \to Z$ *is continuous and there are* $\varrho_0 > 0$, $C_3 > 0$, $\alpha > 1$ *such that* $\|F(u)\|_Z < C_3 \|u\|_X^\alpha$ *for* $\|u\|_X < \varrho_0$.

Then the zero solution of (3.4) *is nonlinearly unstable in the space* X.

In this section, we are going to use Lemma 3.1 to prove Theorem 3.1.

DEFINITION 3.1. A traveling wave solution $\varphi(x-ct)$ of the equation (3.1) is said to be *nonlinearly unstable* in the space X if there exist $\varepsilon_0, C_0 > 0$, a sequence $\{u_n\}$ of solutions of (3.1), and a sequence $t_n > 0$ such that $\|u_n(0) - \varphi(x)\|_X \to 0$ but $\|u_n(t_n) - \varphi(\cdot - ct_n)\|_X \geq \varepsilon_0$.

If $\varphi(x-ct) \in H^2(\mathbb{R})$ is a traveling wave solution of (3.1), then letting $w(x,t) = u(x,t) - \varphi(x-ct)$, we have

$$w_t + \partial_x^4 w - (3\varphi^2 + 2\gamma_1\varphi - 1)\partial_x^2 w - (12\varphi\varphi' + 4\gamma_1\varphi' - \varphi^3 + \varphi)\partial_x w \\ - (6\varphi\varphi'' + 6\varphi'^2 + 2\gamma_1\varphi'' - 3\varphi^2\varphi' + \varphi')w = F(w) \tag{3.5}$$

where

$$\begin{aligned} F(w) = &- \varphi' w^3 + (3\varphi'' - 3\varphi\varphi')w^2 - w^3\partial_x w - 3\varphi^2 w\partial_x w - 3\varphi w^2 \partial_x w \\ &+ w\partial_x w + 12\varphi' w\partial_x w + 2\gamma_1 w\partial_x^2 w + 6\varphi w\partial_x^2 w + 2\gamma_1(\partial_x w)^2 \\ &+ 6\varphi(\partial_x w)^2 + 6w(\partial_x w)^2 + 3w^2\partial_x^2 w, \end{aligned}$$

with initial value

$$w(x,0) = w_0(x) \equiv u_0(x) - \varphi(x). \tag{3.6}$$

So the stability of traveling wave solutions of (3.1) is translated into the stability of the zero solution of (3.5). In order to prove Theorem 3.1, taking $Z = L^2(\mathbb{R})$, $X = H^2(\mathbb{R})$, we need to prove that the four conditions of Lemma 3.1 are satisfied by the associated equation (3.5). Condition (i) is satisfied, by our choice of Z and X.

Denote the linear partial differential operator in (3.5) by

$$\begin{aligned} L = &- (\partial_x^4 + \partial_x^2 + \beta\partial_x) + [(3\varphi^2 + 2\gamma_1\varphi)\partial_x^2 + (12\varphi\varphi' + 4\gamma_1 - \varphi^3 + \varphi + \beta)\partial_x \\ &+ (6\varphi\varphi'' + 6\varphi'^2 + 2\gamma_1\varphi'' - 3\varphi^2\varphi' + \varphi')] \\ = &\, L_0 + [(3\varphi^2 + 2\gamma_1\varphi)\partial_x^2 + (12\varphi\varphi' + 4\gamma_1 - \varphi^3 + \varphi + \beta)\partial_x \\ &+ (6\varphi\varphi'' + 6\varphi'^2 + 2\gamma_1\varphi'' - 3\varphi^2\varphi' + \varphi')] \end{aligned}$$

with

$$L_0 = -(\partial_x^4 + \partial_x^2 + \beta\partial_x).$$

Then (3.5) may be rewritten in the form (3.4),

$$w_t = Lw + F(w).$$

Note that F maps $H^2(\mathbb{R})$ into $L^2(\mathbb{R})$. Using the Sobolev embedding theorem, we have

$$\|F(w)\|_{L^2} \le C\|w\|_{H^2}^2 \quad \text{for } \|w\|_{H^2} < 1, \tag{3.7}$$

for some $C > 0$. So, condition (iv) is satisfied.

To prove (ii), we need the following two lemmas.

LEMMA 3.2. *Let $L_0 = -(\partial_x^4 + \partial_x^2 + \beta\partial_x)$ for any real constant β. Then*

$$\|e^{tL_0}\|_{H^m\to H^m} \le e^{t/4} \quad \textit{for } m \in \mathbb{R}^+,\ 0 \le t < \infty, \tag{3.8}$$

$$\|e^{tL_0}\|_{L^2\to H^2} \le a(t) \equiv 5t^{-1/4} \quad \textit{for } 0 < t \le 1. \tag{3.9}$$

Proof. We write $u(x,t) = e^{tL_0}u_0(x)$. By Fourier transformation,

$$\widehat{u}(\xi,t) = e^{-t(\xi^4-\xi^2+i\beta\xi)}\widehat{u}_0(\xi).$$

We have

$$\|u\|_{H^m}^2 \equiv \int_{-\infty}^{\infty} (1+\xi^2)^m|\widehat{u}(\xi,t)|^2\,d\xi = \int_{-\infty}^{\infty} (1+\xi^2)^m e^{-2t(\xi^4-\xi^2)}|\widehat{u}_0(\xi)|^2\,d\xi$$

$$\le \sup_{\xi\in\mathbb{R}} e^{-2t(\xi^4-\xi^2)} \int_{-\infty}^{\infty} (1+\xi^2)^m|\widehat{u}_0(\xi)|^2\,d\xi = e^{t/2}\|u_0\|_{H^m}^2.$$

Hence

$$\|e^{tL_0}\|_{H^m\to H^m} \le e^{t/4}.$$

On the other hand, letting $s = \xi^2$, we have

$$\|u\|_{H^2}^2 \le \sup_{s\in\mathbb{R}^+} f(s) \int_{-\infty}^{\infty} |\widehat{u}_0(\xi)|^2\,d\xi$$

with $f(s) = (1+s)^2e^{-2t(s^2-s)}$, $t > 0$. An elementary computation shows that

$$\sup_{s>0} f(s) \le \left(\frac{3}{2} + \frac{1}{\sqrt{2}}\,t^{-1/2}\right)e^{t/2}.$$

Thus

$$\|u(x,t)\|_{H^2} \le \left(\frac{3}{2} + \frac{1}{\sqrt{2}}\,t^{-1/2}\right)^{1/2} e^{t/4}\|u_0\|_{L^2}$$

and

$$\|e^{tL_0}\|_{L^2\to H^2} \le \left(\frac{3}{2} + \frac{1}{\sqrt{2}}\,t^{-1/2}\right)^{1/2} e^{t/4} \le 5t^{-1/4} \quad \text{for } 0 < t \le 1,$$

since $e^{t/4} \le e^{1/4} < 2$. Thus Lemma 3.2 has been proved.

LEMMA 3.3. *Let L be as above with $\varphi, \varphi', \varphi'' \in L^\infty(\mathbb{R})$. Then*

$$\|e^{tL}\|_{L^2\to H^2} \le C_1t^{-1/4} \quad \textit{for } 0 < t \le 1, \tag{3.10}$$

$$\|e^{tL}\|_{H^2\to H^2} \le C_2 < \infty \quad \textit{for } 0 < t \le 1. \tag{3.11}$$

Proof. Consider the initial value problem

$$u_t = Lu = L_0 u + (3\varphi^2 + 2\gamma_1\varphi)\partial_x^2 u + (12\varphi\varphi' + 4\gamma_1 - \varphi^3 + \varphi + \beta)\partial_x u$$
$$+ (6\varphi\varphi'' + 6\varphi'^2 + 2\gamma_1\varphi'' - 3\varphi^2\varphi' + \varphi')u,$$
$$u(x,0) = u_0(x).$$

Then $u(x,t) = e^{tL}u_0(x)$, $t \geq 0$, $x \in \mathbb{R}$. Thus

$$u(x,t) = e^{tL_0}u_0 + \int_0^t e^{(t-\tau)L_0}[(3\varphi^2 + 2\gamma_1\varphi)\partial_x^2 u + (12\varphi\varphi' + 4\gamma_1 - \varphi^3 + \varphi + \beta)\partial_x u + (6\varphi\varphi'' + 6\varphi'^2 + 2\gamma_1\varphi'' - 3\varphi^2\varphi' + \varphi')u]\, d\tau.$$

Set $A = \|\varphi\|_{L^\infty}$, $B = \|\varphi'\|_{L^\infty}$, $C = \|\varphi''\|_{L^\infty}$ and $M = A^3 + 3A^2 + (2|\gamma_1| + 1)A + 3A^2B + 12AB + 6AC + 6B^2 + B + 2|\gamma_1|C + 4|\gamma_1| + |\beta|$.

$$\begin{aligned}
\|u(t)\|_{H^2} &\leq \|e^{tL_0}\|_{L^2\to H^2}\|u_0\|_{L^2} \\
&+ \int_0^t \|e^{(t-\tau)L_0}\|_{L^2\to H^2}(3\|\varphi\|_{L^\infty}^2 + 2|\gamma_1|\|\varphi\|_{L^\infty})\|\partial_x^2 u\|_{L^2}\, d\tau \\
&+ \int_0^t \|e^{(t-\tau)L_0}\|_{L^2\to H^2}(12\|\varphi\|_{L^\infty}\|\varphi'\|_{L^\infty} + 4|\gamma_1| \\
&\qquad + \|\varphi\|_{L^\infty}^3 + \|\varphi\|_{L^\infty} + |\beta|)\|\partial_x u\|_{L^2}\, d\tau \\
&+ \int_0^t \|e^{(t-\tau)L_0}\|_{L^2\to H^2}(6\|\varphi\|_{L^\infty} + 2|\gamma_1|)\|\varphi''\|_{L^\infty}\|u\|_{L^2}\, d\tau \\
&+ \int_0^t \|e^{(t-\tau)L_0}\|_{L^2\to H^2}(6\|\varphi'\|_{L^\infty}^2 + 3\|\varphi\|_{L^\infty}^2\|\varphi'\|_{L^\infty} + \|\varphi'\|_{L^\infty})\|u\|_{L^2}\, d\tau \\
&\leq a(t)\|u_0\|_{L^2} + M\int_0^t a(t-\tau)\|u(\tau)\|_{H^2}\, d\tau,
\end{aligned} \tag{3.12}$$

where $a(t)$ is defined in Lemma 3.2 and we use $u(t)$ to denote $u(\cdot,t)$.

By iteration,

$$\|u(t)\|_{H^2} \leq a(t)\|u_0\|_{L^2} + M\int_0^t a(t-\tau) \times \Big[a(\tau)\|u_0\|_{L^2} + M\int_0^\tau a(\tau - s)\|u(s)\|_{H^2}\, ds\Big]\, d\tau \tag{3.13}$$

$$= a(t)\|u_0\|_{L^2} + M\int_0^t a(t-\tau)a(\tau)\|u_0\|_{L^2}\,d\tau$$
$$+ M^2\int_0^t\int_0^\tau a(t-\tau)a(\tau-s)\|u(s)\|_{H^2}\,ds\,d\tau.$$

The second term on the right of (3.13) is

$$\text{(3.14)}\quad \int_0^t a(t-\tau)a(\tau)\|u_0\|_{L^2}\,d\tau$$
$$= M\|u_0\|_{L^2}\int_0^t 5(t-\tau)^{-1/4}5\tau^{-1/4}\,d\tau$$
$$= 25M\|u_0\|_{L^2}\int_0^t t^{-1/2}\left(1-\frac{\tau}{t}\right)^{-1/4}\left(\frac{\tau}{t}\right)^{-1/4}d\tau$$
$$= 25MC_3t^{1/2}\|u_0\|_{L^2} \quad \text{for } 0<t\le 1,$$

where $C_3 = \int_0^1(1-r)^{-1/4}r^{-1/4}\,dr$. By exchanging the order of integration in the third term on the right side of (3.13), we get

$$\int_0^t\int_0^\tau a(t-\tau)a(\tau-s)\|u(s)\|_{H^2}\,ds\,d\tau = \int_0^t\Big[\int_s^t a(t-\tau)a(\tau-s)\,d\tau\Big]\|u(s)\|_{H^2}\,ds.$$

Now

$$\text{(3.15)}\quad \int_s^t a(t-\tau)a(\tau-s)\,d\tau = 25\int_s^t(t-\tau)^{-1/4}(\tau-s)^{-1/4}\,d\tau$$
$$= 25C_3(t-s)^{1/2} \le 25C_3 \quad \text{for } 0<s\le t\le 1.$$

Therefore (3.12)–(3.15) imply

$$\text{(3.16)}\quad \|u(t)\|_{H^2} \le [a(t)+25C_3M]\|u_0\|_{L^2}$$
$$+ 25C_3M^2\int_0^t\|u(s)\|_{H^2}\,ds \quad \text{for } 0<t\le 1.$$

Let $v(t) = \int_0^t\|u(s)\|_{H^2}\,ds$. Then

$$\frac{dv(t)}{dt} \le [a(t)+25C_3M]\|u_0\|_{L^2} + 25C_3M^2v(t) \quad \text{for } 0<t\le 1.$$

Multiplying both sides of the above inequality by $e^{-25C_3M^2t}$ and integrating the resulting relation with respect to t over $(0,t)$, we obtain

$$v(t) \le e^{25C_3M^2t}\int_0^t e^{-25C_3M^2s}[a(s)+25C_3M]\,ds\,\|u_0\|_{L^2}.$$

Observing that $v(t)=\int_0^t \|u(s)\|_{H^2}\,ds$, and substituting the above inequality into (3.16), we get

$$\|u(t)\|_{H^2} \le C_1 t^{-1/4}\|u_0\|_{L^2} \quad \text{for } 0<t\le 1, \tag{3.17}$$

where $C_1>0$. Thus (3.10) has been proven. To prove (3.11), replacing the first term on the right side of (3.12) by $\|e^{tL_0}\|_{H^2\to H^2}\|u_0\|_{H^2}$ and using (3.8), we have

$$\|u(t)\|_{H^2} \le e^{t/4}\|u_0\|_{H^2} + M\int_0^t a(t-\tau)\|u(\tau)\|_{H^2}\,d\tau \quad \text{for } 0<t\le 1. \tag{3.18}$$

Similarly iterating and computing as above, we obtain

$$\|u(t)\|_{H^2} \le [2+25C_3M]\exp[25C_3M^2]\|u_0\|_{H^2} \equiv C_2\|u_0\|_{H^2}. \tag{3.19}$$

Hence (3.11) is proven and the proof of Lemma 3.3 is finished. By Lemma 3.3, condition (ii) is proved.

We now proceed to verify condition (iii) of Lemma 3.1. Observe that if $u(x,t)$ satisfies

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} = &-\frac{\partial^4 u}{\partial x^4}-\frac{\partial^2 u}{\partial x^2}-\beta\frac{\partial u}{\partial x}+(3\varphi^2+2\gamma_1\varphi)\frac{\partial^2 u}{\partial x^2}\\ &+(12\varphi\varphi'+4\gamma_1-\varphi^3+\varphi+\beta)\frac{\partial u}{\partial x}\\ &+(6\varphi\varphi''+6\varphi'^2+2\gamma_1\varphi''-3\varphi^2\varphi'+\varphi')u,\end{aligned}$$

then $u(x,s+t)$ also satisfies the above equation. By uniqueness of solution, we prove easily that e^{tL} satisfies three conditions of the definition, hence L generates a strongly continuous semigroup on the Banach space $H^2(\mathbb{R})$ (see [13]). By Fourier transformation, the essential spectrum of L_0 on $H^2(\mathbb{R})$ is

$$\sigma(L_0) \supset \{-\xi^4+\xi^2-i\beta\xi;\ \xi\in\mathbb{R}\}.$$

The curve $\lambda=-\xi^4+\xi^2-i\beta\xi$ meets the vertical lines $\operatorname{Re}\lambda=\alpha$ for $-\infty<\alpha\le 1/4$ because $-\infty<-\xi^4+\xi^2\le 1/4$.

We now prove that the same curve lies in the essential spectrum of L.

LEMMA 3.4. *The essential spectrum of L on $H^2(\mathbb{R})$ contains that of L_0.*

Proof. Let $\xi\in\mathbb{R}$ and let $\lambda=P(\xi)=-\xi^4+\xi^2-i\beta\xi$. To prove that $\lambda\in\sigma(L)$, we use Theorem 4.4 in [9] stating that if there exists a sequence $\{\eta_n\}\subset H^2(\mathbb{R})$ with

$$\|\eta_n\|_{H^2}=1, \quad \|(L-\lambda)\eta_n\|_{H^2}\to 0,$$

and $\{\eta_n\}$ does not have a strongly convergent subsequence in $H^2(\mathbb{R})$, then $\lambda \in \sigma(L)$.

Now let $\eta_0 \not\equiv 0$ be a C^∞ function with compact support in $(0,\infty)$. Define

$$\eta_n(x) = c_n e^{i\xi x}\eta_0(x/n)/\sqrt{n}, \qquad n = 1, 2, \ldots,$$

where c_n is chosen so that $\|\eta_n\|_{H^2} = 1$. Then

$$\|\eta_n\|_{L^2} = c_n\|\eta_0\|_{L^2} \quad \text{and} \quad 1 = \|\eta_n\|_{H^2} \le kc_n$$

for some positive constant k. Hence $c_n \ge 1/k > 0$. Since $\|\eta_n\|_{L^\infty} \to 0$ but $\|\eta_n\|_{L^2}$ is bounded away from zero, $\{\eta_n\}$ can have no convergent subsequence in $L^2(\mathbb{R})$.

It remains to show that $\|(L-\lambda)\eta_n\|_{H^2} \to 0$. We write

$$\begin{aligned} L - \lambda = L_0 - \lambda &+ (3\varphi^2 + 2\gamma_1\varphi)\partial_x^2 + (12\varphi\varphi' + 4\gamma_1 - \varphi^3 + \varphi + \beta)\partial_x \\ &+ (6\varphi\varphi'' + 6\varphi'^2 + 2\gamma_1\varphi'' - 3\varphi^2\varphi' + \varphi') \end{aligned}$$

A simple calculation shows that

$$\begin{aligned} (L_0 - \lambda)\eta_n(x) &= e^{i\xi x}\sum_{1\le s\le 4} P^{(s)}(\xi)c_n\eta_0^{(s)}(x/n)/(s!n^{1/2+s}), \\ \partial(L_0 - \lambda)\eta_n(x) &= i\xi(L_0-\lambda)\eta_n(x) \\ &\quad + e^{i\xi x}\sum_{1\le s\le 4} P^{(s)}(\xi)c_n\eta_0^{(s+1)}(x/n)/(s!n^{3/2+s}), \end{aligned}$$

and

$$\begin{aligned} &\partial^2(L_0 - \lambda)\eta_n(x) \\ &\quad = -\xi^2(L_0-\lambda)\eta_n(x) + 2i\xi e^{i\xi x}\sum_{1\le s\le 4} P^{(s)}(\xi)c_n\eta_0^{(s+1)}(x/n)/(s!n^{3/2+s}) \\ &\qquad + e^{i\xi x}\sum_{1\le s\le 4} P^{(s)}(\xi)c_n\eta_0^{(s+2)}(x/n)/(s!n^{5/2+s}). \end{aligned}$$

Thus

$$\begin{aligned} &\|(L_0-\lambda)\eta_n(x)\|_{H^2} \\ &\quad \le (1+|\xi|^2)\sum_{1\le s\le 4} |P^{(s)}(\xi)|c_n\|\eta_0^{(s)}(x/n)\|_{L^2}/(s!n^{1/2+s}) \\ &\qquad + 2|\xi|\sum_{1\le s\le 4} |P^{(s)}(\xi)|c_n\|\eta_0^{(s+1)}(x/n)\|_{L^2}/(s!n^{3/2+s}) \\ &\qquad + \sum_{1\le s\le 4} |P^{(s)}(\xi)|c_n\|\eta_0^{(s+2)}(x/n)\|_{L^2}/(s!n^{5/2+s}) \to 0, \end{aligned}$$

as $n \to \infty$. Moreover, for any positive integer m, $\|\partial_x^m\eta_n\|_{L^\infty} \to 0$ as $n \to \infty$, so we have

$$\|(3\varphi^2 + 2\gamma_1\varphi)\partial_x^2\eta_n\|_{L^2}^2 \le \|\partial_x^2\eta_n\|_{L^\infty}^2 (3\|\varphi^2\|_{L^2}^2 + |2\gamma_1|\|\varphi\|_{L^2}^2) \to 0,$$
$$\|\partial_x[(3\varphi^2 + 2\gamma_1\varphi)\partial_x^2\eta_n]\|_{L^2}^2 \le \|\partial_x^3\eta_n\|_{L^\infty}^2\|3\varphi^2\|_{L^2}^2 + \|\partial_x^2\eta_n\|_{L^\infty}^2\|6\varphi\varphi'\|_{L^2}^2 \to 0,$$

and

$$\|\partial_x^2[(3\varphi^2 + 2\gamma_1\varphi)\partial_x^2\eta_n]\|_{L^2}^2 \le \|\partial_x^3\eta_n\|_{L^\infty}^2\|12\varphi\varphi' + 4\gamma_1\varphi'\|_{L^2}^2$$
$$+\|\partial_x^2\eta_n\|_{L^\infty}^2\|6\varphi\varphi'' + 6\varphi'^2 + (2\gamma - 1)\varphi''\|_{L^2}^2 + \|\partial_x^4\eta_n\|_{L^\infty}^2\|3\varphi^2 + 2\gamma_1\varphi\|_{L^2}^2 \to 0.$$

Similarly to the above estimates, we have the other estimates, thus

$$\|(3\varphi^2 + 2\gamma_1\varphi)\partial_x^2\eta_n + (12\varphi\varphi' + 4\gamma_1 - \varphi^3 + \varphi + \beta)\partial_x\eta_n + (6\varphi\varphi'' + 6\varphi'^2 + 2\gamma_1\varphi'' - 3\varphi^2\varphi' + \varphi')\eta_n\|_{H^2} \to 0.$$

So from the estimates above,

$$\|(L - \lambda)\eta_n\|_{H^2} \to 0 \quad \text{as } n \to \infty.$$

The proof of Lemma 3.4 is complete.

Therefore all the four conditions of Lemma 3.1 are satisfied by the linearized equation (3.5) and Theorem 3.1 has been proved.

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