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REAL FUNCTIONS

Remarks on Convexity in Dimension $(2, 2)$

by

Wojciech KRYŃSKI

Presented by Wiesław PLEŚNIAK

Summary. We consider different convexity notions for functions $F\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$. We give a new characterisation of polyconvexity and a sufficient condition for quasiconvexity.

1. Introduction. A continuous function $F\colon \mathbb{R}^{n\times m} \to \mathbb{R}$ is *quasiconvex* if

$$F(A) \le \int_{\Omega} F(A + D\varphi(x))\,dx$$

for any matrix $A \in \mathbb{R}^{n\times m}$, and any $\varphi \in C_0^\infty(\Omega, \mathbb{R}^m)$, where $\Omega \subseteq \mathbb{R}^n$ is an open, bounded domain of measure 1. The notion of quasiconvexity was introduced by Morrey [8]. He proved that the lower semicontinuity of the integral functional

$$I(\varphi) = \int_{\Omega} F(D\varphi(x))\,dx$$

defined for sufficiently regular φ is equivalent to the quasiconvexity of F. Unfortunately it is hard to verify if a given function is quasiconvex. The following simpler notions were introduced:

1. F is *rank-one convex* if $F(A) \le \lambda_1 F(A_1) + \lambda_2 F(A_2)$ provided that $\mathrm{rk}(A_1 - A_2) \le 1$ and $A = \lambda_1 A_1 + \lambda_2 A_2$ is a convex combination, i.e. $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 \ge 0$,
2. F is *polyconvex* if $F(A) = G(T(A))$ for a certain convex (in the usual sense) function G, where $T(A)$ is the vector of all determinants of square submatrices of A.

It is well known that a polyconvex function is quasiconvex and a quasiconvex function is rank-one convex (see e.g. [2, 6]). In the present paper we will consider the following notion.

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Definition. A function $F\colon \mathbb{R}^{n\times m}\to\mathbb{R}$ is *k-convex* if for any convex combination $A=\sum_{i=0}^k \lambda_i A_i$ of matrices $A_i\in\mathbb{R}^{n\times m}$ such that $T(A)=\sum_{i=0}^k \lambda_i T(A_i)$, the following holds:

$$F(A)\le \sum_{i=0}^{k}\lambda_i F(A_i).$$

Note that $\operatorname{rk}(A_1-A_2)\le 1$ iff $T(\lambda_1A_1+\lambda_2A_2)=\lambda_1T(A_1)+\lambda_2T(A_2)$ and thus a function F is 1-convex iff F is rank-one convex. Obviously, if F is k-convex, then it is l-convex for any $l<k$.

From now on, we limit ourselves to the case $n=m=2$. It follows from Statement (10) of [4] that F is polyconvex iff it is 5-convex and has a convex lower bound (Theorem 4.4 of [2]). In the present note we will prove that 2-convexity implies k-convexity for any k. In particular, we reduce 5 to 2 in Theorem 4.4 of [2].

It follows from our result that quasiconvexity in dimension 2×2 lies between 2-convexity and 1-convexity. The question whether quasiconvexity is equivalent to rank-one convexity is known as the Morrey conjecture. It is proved to be false in higher dimensions [10] but it is still an open problem in dimension 2×2 [9]. Polyconvexity is known to be essentially stronger than quasiconvexity (see [1, 3, 7, 11]), and so is 2-convexity. Recently, a new necessary condition for quasiconvexity has been found [5]. We also refer to [5] for a list of related topics and further references.

2. Results.

Let $A\in\mathbb{R}^{2\times 2}$. We denote by A^1 and A^2 the first and second columns of A. We write $A=[A^1A^2]$.

Lemma 1. *Let $A=\sum_{i=0}^k\lambda_iA_i$ be a convex combination with $A_i\in\mathbb{R}^{2\times2}$. Then $\sum_{i=0}^k\lambda_i\det A_i=\det A$ if and only if $\sum_{i=0}^k\sum_{j=0}^k\lambda_i\lambda_j\det(A_i-A_j)=0$.*

Proof. If $B,C\in\mathbb{R}^{2\times2}$, then $\det(B-C)=\det B+\det C-\det[B^1C^2]-\det[C^1B^2]$. Hence

$$\begin{aligned}\det A-\sum_{i=0}^k\lambda_i\det A_i&=\det\Big(\sum_{i=0}^k\lambda_iA_i\Big)-\sum_{i=0}^k\lambda_i\det A_i\\&=\sum_{i=0}^k\sum_{j=0}^k\lambda_i\lambda_j\det[A_i^1A_j^2]-\sum_{i=0}^k\sum_{j=0}^k\lambda_i\lambda_j\det A_i\\&=-\frac12\sum_{i=0}^k\sum_{j=0}^k\lambda_i\lambda_j\det(A_i-A_j),\end{aligned}$$

and the lemma follows. ∎

REMARK. Note that, since $\det(A_i - A_j) = \det(A_j - A_i)$, the condition in the lemma is equivalent to $\sum_{i=0}^{k} \sum_{j=i+1}^{k} \lambda_i \lambda_j \det(A_i - A_j) = 0$.

For the sake of convenience we will say that if $A = \sum_{i=0}^{k} \lambda_i A_i$ and $\det A = \sum_{i=0}^{k} \lambda_i \det A_i$ are convex combinations then A is a *geometric convex combination* of the matrices A_i.

THEOREM 1. *If F is 2-convex then F is k-convex for any k.*

Proof. Let $A = \sum_{i=0}^{k} \lambda_i A_i$ and $\sum_{i=0}^{k} \lambda_i \det A_i = \det A$. Assume that there are given geometric convex combinations $B_j = \sum_{i=0}^{k} \lambda_{ij} A_i$, where $j = 0, \ldots, n$. If there exist real numbers $\mu_j \in [0,1]$ such that $\lambda_i = \sum_{j=0}^{n} \mu_j \lambda_{ij}$ for any $i = 0, \ldots, k$, then $A = \sum_{j=0}^{n} \mu_j B_j$ is a geometric convex combination of B_i:

$$\det A = \sum_{i=0}^{k} \lambda_i \det A_i = \sum_{i=0}^{k} \sum_{j=0}^{n} \mu_j \lambda_{ij} \det A_i = \sum_{j=0}^{n} \mu_j \det B_j.$$

Assume that $k > 2$. We will prove that there exists a decomposition $A = \sum_{j=0}^{n} \mu_j B_j$ as above such that n is at most 2, and moreover, for any fixed $j = 0, \ldots, n$ at least one λ_{ij} is zero. In other words, each B_j will be a convex combination of at most k matrices A_i. The assumption will imply $F(A) \le \sum_{j=0}^{n} \mu_j F(B_j)$ (since $n \le 2$) and the inductive procedure will complete the proof.

Set $S_i = \sum_{j=0}^{k} \lambda_i \lambda_j \det(A_i - A_j)$. Then $\sum_{i=0}^{k} S_i = 0$ by Lemma 1. If there exists i such that $S_i = 0$, then the Remark implies that

$$\sum_{j=0, j\neq i}^{k} \sum_{l=j+1, l\neq i}^{k} \lambda_j \lambda_l \det(A_j - A_l) = 0,$$

and one can define $B_0 = (\sum_{j\neq i} \lambda_j)^{-1} \sum_{j\neq i} \lambda_j A_j$ and $B_1 = A_i$. In this way we decompose A into the sum of two matrices $A = (\sum_{j\neq i} \lambda_j) B_0 + \lambda_i B_1$.

If all $S_i \neq 0$ then we may assume that $S_0 < 0$ and $S_k > 0$ (possibly after permutation of indices). This gives

$$\sum_{i=1}^{k} \sum_{j=i+1}^{k} \lambda_i \lambda_j \det(A_i - A_j) > 0 \tag{1}$$

and

$$\sum_{i=0}^{k-1} \sum_{j=i+1}^{k-1} \lambda_i \lambda_j \det(A_i - A_j) < 0. \tag{2}$$

Let us consider the following convex combinations:

$$C_t = \Big((1-t)\lambda_0 + \sum_{i=1}^{k-1} \lambda_i\Big)^{-1} \Big((1-t)\lambda_0 A_0 + \sum_{j=1}^{k-1} \lambda_j A_j\Big)$$

for $t \in [0,1]$ and

$$C_t = \Big((t-1)\lambda_k + \sum_{i=1}^{k-1} \lambda_i\Big)^{-1} \Big((t-1)\lambda_k A_k + \sum_{j=1}^{k-1} \lambda_j A_j\Big)$$

for $t \in [1,2]$. The Darboux theorem, relations (1), (2) and Lemma 1 imply that there exists $t \in [0,2]$ such that C_t is a geometric convex combination of some A_i. Note that there are always at most k different matrices A_i in the sum on the right hand side of the equation which defines C_t. We set $B_0 = C_t$, $B_1 = A_0$ and $B_2 = A_k$. This completes the proof. ∎

Corollary 1. *A function* $F\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ *is polyconvex if and only if it is* 2*-convex and has a convex lower bound.*

Corollary 2. *If* F *is* 2*-convex then* F *is quasiconvex.*

Proof. It is known that one can use continuous, piecewise affine functions instead of smooth functions in the definition of quasiconvexity (cf. [2, p. 354]). For such functions the integral is replaced by a sum of the form $\sum_{i=0}^{k} \lambda_i F(A + A_i)$, where k is sufficiently large. Moreover, one can see that A is a geometric convex combination of $A + A_i$. Thus, if F is k-convex for any k, then F is quasiconvex. The result follows from Theorem 1. ∎

Remark. The following problem arises. Find the smallest number $k = k(m,n)$ such that k-convexity of a function $F\colon \mathbb{R}^{n\times m} \to \mathbb{R}$ implies that F is polyconvex. The natural generalisation of the Morrey conjecture is the question whether quasiconvexity is equivalent to l-convexity for some $l > 1$ (if $m > 2$).

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Wojciech Kryński
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warszawa, Poland
E-mail: krynski@mimuw.edu.pl

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