BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 56, No. 3-4, 2008

FUNCTIONAL ANALYSIS

The Embeddability of c_0 in Spaces of Operators

by

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Summary. Results of Emmanuele and Drewnowski are used to study the containment of c_0 in the space $K_{w^*}(X^*, Y)$, as well as the complementation of the space $K_{w^*}(X^*, Y)$ of w^*-w compact operators in the space $L_{w^*}(X^*, Y)$ of w^*-w operators from X^* to Y.

Definitions and notations. Throughout this paper X and Y will denote real Banach spaces and X^* denotes the continuous linear dual of X. An *operator* $T : X \to Y$ will be a continuous and linear function. By $X \otimes_\lambda Y$ we denote the injective tensor product of X and Y. Notation is consistent with that used in Diestel [5]. Let (e_n) be the Schauder basis of c_0, (e_n^*) be the basis of ℓ_1, and (e_n^2) the unit vector basis of ℓ_2. The set of all continuous linear transformations from X to Y will be denoted by $L(X, Y)$, and the compact (resp. weakly compact) operators will be denoted by $K(X, Y)$ (resp. $W(X, Y)$). The w^*-w continuous (resp. w^*-w continuous compact) maps from X^* to Y will be denoted by $L_{w^*}(X^*, Y)$ (resp. $K_{w^*}(X^*, Y)$).

A bounded subset A of X is called a *limited subset* of X if each w^*-null sequence in X^* tends to 0 uniformly on A. If every limited subset of X is relatively compact, then we say that X has the *Gelfand–Phillips property.* If every weakly compact operator defined on X is completely continuous, then we say that X has the *Dunford–Pettis property* (DPP); see [6] and [1] for inventories of classical results related to the DPP.

Introduction. Numerous authors have studied the containment of c_0 in the spaces of compact operators $K(X, Y)$ and $K_{w^*}(X^*, Y)$. This problem has been studied together with the complementation of the space of com-

2000 *Mathematics Subject Classification*: 46B20, 46B28.
Key words and phrases: spaces of operators, complemented subspaces, compact operators, weakly compact operators, w^*-w compact operators, basic sequence.

DOI: 10.4064/bp56-3-7

pact operators $K_{w^*}(X^*, Y)$ (resp. $K(X, Y)$) in the space $L_{w^*}(X^*, Y)$ (resp. $L(X, Y)$) and the containment of l_∞ in $L_{w^*}(X^*, Y)$ (resp. $L(X, Y)$). See Bator and Lewis [2], Kalton [23], Emmanuele [13], Emmanuele and John [16], Ghenciu [19], Lewis [25], and Tong and Wilken [31] for an indication of the extensive literature that deals with these problems. The survey paper [29] by Ruess is a valuable resource for the structure of the space of operators $K_{w^*}(X^*, Y)$.

Theorem 4 of Kalton [23] states that ℓ_∞ embeds in $K(X, Y)$ if and only if it embeds in X^* or in Y. In [8] Drewnowski generalized Theorem 4 of Kalton and proved that ℓ_∞ embeds in $K_{w^*}(X^*, Y)$ if and only if it embeds in X or in Y. In this paper we use techniques of Emmanuele [11] and Drewnowski's result [8] to obtain results about the complementation of the space $K_{w^*}(X^*, Y)$ of compact w^*-w operators in the space $L_{w^*}(X^*, Y)$ of bounded w^*-w operators. Applications to the complementation of the space $K(X, Y)$ in $W(X, Y)$ are given. We also give sufficient conditions for the containment of c_0 in the space $K_{w^*}(X^*, Y)$, resp. $K(X, Y)$. Results in this paper generalize results in [3], [11], [13], [14], [17], [20], [23], and [25].

Spaces of operators. We recall the following well-known isometries [29]:

1) $L_{w^*}(X^*, Y) \simeq L_{w^*}(Y^*, X)$ and $K_{w^*}(X^*, Y) \simeq K_{w^*}(Y^*, X)$ $(T \mapsto T^*)$,
2) $W(X, Y) \simeq L_{w^*}(X^{**}, Y)$ and $K(X, Y) \simeq K_{w^*}(X^{**}, Y)$ $(T \mapsto T^{**})$.

It is known that if X is infinite-dimensional and $c_0 \hookrightarrow L(X, Y)$, then $\ell_\infty \hookrightarrow L(X, Y)$ (see, e.g., [23] and [25]). Part (i) of the following theorem generalizes this result, as well as Theorem 3 in [3].

Theorem 1.

(i) *Suppose that X and Y are infinite-dimensional and S is a closed linear subspace of $L(X, Y)$ which contains all the rank one operators $x^* \otimes y$, $x^* \in X^*$, $y \in Y$. If $c_0 \hookrightarrow S$ and S is complemented in $L(X, Y)$, then $\ell_\infty \hookrightarrow S$.*
(ii) *Suppose that X and Y fail to have the Schur property, and S is a closed linear subspace of $L_{w^*}(X^*, Y)$ which contains all rank one operators $x \otimes y$, $x \in X$, $y \in Y$. If $c_0 \hookrightarrow S$ and S is complemented in $L_{w^*}(X^*, Y)$, then $\ell_\infty \hookrightarrow S$.*

Proof. (i) Consider the following two cases.

Suppose first that $c_0 \hookrightarrow Y$ and let (y_n) be a copy of (e_n) in Y. Use the Josefson–Nissenzweig theorem and choose a w^*-null normalized sequence (x_n^*) in X^*. Define $J : \ell_\infty \to L(X, Y)$ by

$$J(b)(x) = \sum b_n x_n^*(x) y_n, \qquad x \in X.$$

Then J is an isomorphism, and, if b is finitely supported, $J(b) \in S$.

Now suppose that $c_0 \not\hookrightarrow Y$. Let $B : c_0 \to S$ be an isomorphic embedding. Note that $\sum |\langle B(e_n)(x), y^*\rangle| < \infty$ for all $x \in X$ and $y^* \in Y^*$. Since $c_0 \not\hookrightarrow Y$, $\sum B(e_n)(x)$ is unconditionally convergent in Y for all $x \in X$. Define μ by $\mu(\emptyset) = 0$ and

$$\mu(A) = \sum_{n\in A} B(e_n) \quad \text{(strong operator topology)}$$

for any non-empty subset A of $\mathbb{N}$. Note that μ is bounded, finitely additive and not strongly additive ($\|\mu(\{n\})\| \not\to 0$). Apply the Diestel–Faires theorem to obtain $\ell_\infty \hookrightarrow L(X, Y)$, and observe that if A is a finite subset of $\mathbb{N}$, then $\mu(A) \in S$.

Now suppose that S is complemented in $L(X, Y)$, and let $P : L(X, Y) \to S$ be a projection. Let $\nu(A) = P(\chi_A)$ for $A \subseteq \mathbb{N}$. The first part of the proof shows that $\ell_\infty \hookrightarrow L(X, Y)$, thus ν is well-defined. Then $\nu : \mathcal{P}(\mathbb{N}) \to S$ is bounded and finitely additive. Moreover, $\|\nu(\{n\})\| \not\to 0$. Therefore another application of the Diestel–Faires theorem tells us that $\ell_\infty \hookrightarrow S$.

(ii) Assume first that $c_0 \hookrightarrow Y$. Let (x_n) be a w-null normalized sequence in X and (y_n) be a copy of (e_n) in Y. Define $\phi : \ell_\infty \to L_{w^*}(X^*, Y)$ by

$$\phi(b)(x^*) = \sum b_n x^*(x_n) y_n, \qquad x^* \in X^*.$$

We note that the series converges unconditionally. To show that $\phi(b)$ is a w^*-w operator, we need to prove that $(\phi(x^*_\alpha))$ is w-null for each w^*-null net (x^*_α) in X^*. We can suppose that (x^*_α) is a w^*-null net in B_{X^*} by results about the bounded X topology (or BX topology) for X^* ([10, Chapter V]). Let $\varepsilon > 0$ and $y^* \in B_{Y^*}$. Since $\sum y_n$ is *wuc*, there is an $n \in \mathbb{N}$ such that $\sum_{i=n+1}^{\infty} |y^*(y_i)| < \varepsilon/(2\|b\|_\infty)$. Then

$$\Big| \sum_{i=n+1}^{\infty} b_i x^*_\alpha(x_i) y^*(y_i) \Big| \le \|b\|_\infty \sum_{i=n+1}^{\infty} |y^*(y_i)| < \frac{\varepsilon}{2}.$$

On the other hand, $\lim_\alpha \sum_{i=1}^{n} |b_i x^*_\alpha(x_i) y^*(y_i)| = 0$ since (x^*_α) is a w^*-null net. Therefore, for α large,

$$|\langle \phi(b)(x^*_\alpha), y^*\rangle| \le \Big| \sum_{i=1}^{n} b_i x^*_\alpha(x_i) y^*(y_i) \Big| + \Big| \sum_{i=n+1}^{\infty} b_i x^*_\alpha(x_i) y^*(y_i) \Big| < \varepsilon.$$

Hence $\phi(b)$ is a w^*-w operator. Further, if $b \in \ell_\infty$ is finitely supported, $\phi(b) \in S$. A result in [28] implies that $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$ since $\|\phi(e_n)\| \not\to 0$. Similarly, if $c_0 \hookrightarrow X$ (and Y does not have the Schur property), then $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$.

Without loss of generality assume that $c_0 \not\hookrightarrow X, Y$ and let $B : c_0 \to S$ be an isomorphic embedding. Note that $\sum B(e_n)(x^*)$ is *wuc*, hence unconditionally convergent for each $x^* \in X^*$ (since $c_0 \not\hookrightarrow Y$). Similarly, $\sum B(e_n)^*(y^*)$ is unconditionally convergent in X for each $y^* \in Y^*$. Then

$$\sum B(e_n) \quad \text{(strong operator topology)}$$

is a w^*-w operator from X^* to Y. Define $\mu : \mathcal{P}(\mathbb{N}) \to L_{w^*}(X^*, Y)$ by $\mu(\emptyset) = 0$ and

$$\mu(A) = \sum_{n\in A} B(e_n) \quad \text{(strong operator topology)}$$

if A is a non-empty subset of $\mathbb{N}$. Then μ is bounded (by the Uniform Boundedness Principle) and finitely additive, but $\mu(\{n\}) \not\to 0$. The σ-algebra version of the Diestel–Faires theorem [7] implies that $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$. Observe that if A is a finite subset of $\mathbb{N}$, then $\mu(A) \in S$.

Now suppose that S is complemented in $L_{w^*}(X^*, Y)$, and let $P : L_{w^*}(X^*, Y) \to S$ be a projection. Let $\nu(A) = P(\chi_A)$ for $A \subseteq \mathbb{N}$. Then $\nu : \mathcal{P}(\mathbb{N}) \to S$ is bounded and finitely additive. Moreover, $\|\nu(\{n\})\| \not\to 0$. By another application of the Diestel–Faires theorem we conclude that $\ell_\infty \hookrightarrow S$. ∎

If X is infinite-dimensional and $c_0 \hookrightarrow L_{w^*}(X^*, Y)$, then $L_{w^*}(X^*, Y)$ may fail to contain ℓ_∞. It is not difficult to check that $c_0 \hookrightarrow K_{w^*}(\ell_1, \ell_1)$. In fact, $c_0 \overset{c}{\hookrightarrow} K_{w^*}(\ell_1, \ell_1)$; see the closing remarks in this paper. However, since $K_{w^*}(\ell_1, \ell_1) = L_{w^*}(\ell_1, \ell_1)$, Drewnowski's theorem makes it clear that $\ell_\infty \not\hookrightarrow L_{w^*}(\ell_1, \ell_1)$.

Our first corollary points out that the exclusion of ℓ_∞ is not possible if X and Y do not have the Schur property.

Corollary 2. *Suppose that $c_0 \hookrightarrow L_{w^*}(X^*, Y)$ and X and Y do not have the Schur property. Then $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$.*

Corollary 3 (Ghenciu and Lewis, [20]).

(i) *If X does not have the Schur property and $c_0 \hookrightarrow Y$, then $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$.*
(ii) *If c_0 does not embed in X or Y and $c_0 \hookrightarrow K_{w^*}(X^*, Y)$, then $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$ provided that X and Y do not have the Schur property.*

Proof. Part (i) follows from the proof of Theorem 1, and (ii) is an immediate corollary of the statement of the theorem. ∎

The next theorem is motivated by results in [13].

Theorem 4. *Suppose that X has an unconditional and seminormalized basis (x_i) with biorthogonal coefficients (x_i^*), and $T : X \to Y$ is an operator such that $(T(x_i))$ is a weakly null seminormalized basic sequence in Y. Let $S(X, Y)$ be a closed linear subspace of $L(X, Y)$ which properly contains $K(X, Y)$ such that $\phi(b) \in S(X, Y)$ for all $b \in \ell_\infty$, where $\phi(b)(x) = \sum b_i x_i^*(x) T(x_i)$, $x \in X$. Then $K(X, Y)$ is not complemented in $S(X, Y)$.*

Proof. Let $\delta > 0$ and $(x_{i_j}) = (u_j)$ be a subsequence of (x_i) such that $\|T(u_i) - T(u_j)\| > \delta$ for $i \neq j$. Denote the corresponding subsequence of coefficient functionals by (u_j^*). Note that $\sum b_j u_j^*(x)T(u_j)$ converges unconditionally in Y for each $x \in X$ and $b = (b_i) \in \ell_\infty$.

Let $J : [(T(u_i)] \to \ell_\infty$ be a linear isometry, and let $A : Y \to \ell_\infty$ be a continuous linear extension of J. Now suppose that $K(X,Y)$ is complemented in $S(X,Y)$ and let $P : S(X,Y) \to K(X,Y)$ be a projection. Define $\tau : l_\infty \to L(X,Y)$ by

$$\tau(b)(x) = \sum_j b_j u_j^*(x)T(u_j), \qquad x \in X.$$

Note that $\tau(\ell_\infty) \subseteq S(X,Y)$. Consider the operators $AP\tau : \ell_\infty \to K(X,\ell_\infty)$ and $A\tau : \ell_\infty \to S(X,\ell_\infty)$. Since $\tau(e_j) = u_j^* \otimes T(u_j)$, $\tau(e_j)$ is a rank one operator, thus compact. Then $AP\tau(e_j) = A\tau(e_j)$ for each $j \in \mathbb{N}$. Proposition 5 of Kalton [23] produces an infinite subset M of $\mathbb{N}$ such that

$$AP\tau(b) = A\tau(b), \qquad b \in l_\infty(M).$$

Therefore $A\tau(\chi_M)$ is compact. But $\tau(\chi_M)(u_j) = T(u_j)$, $j \in M$, and $\{T(u_j) : j \in M\}$ is not relatively compact. Therefore $\tau(\chi_M)$ is not compact. However, this is a contradiction since $A_{|_{[T(u_i)]}}$ is an isometry. ∎

Corollary 5 (Emmanuele, [13]). *Let Y be a Banach space without the Schur property. Then $K(\ell_1, Y)$ is not complemented in $W(\ell_1, Y)$.*

Proof. Let (y_n) be a w-null normalized basic sequence in Y, $X = \ell_1$, and $S(\ell_1, Y) = W(\ell_1, Y)$. Define $T : \ell_1 \to Y$ by $T(x) = \sum x_n y_n$, $x = (x_n) \in \ell_1$. If $\phi : \ell_\infty \to L(\ell_1, Y)$ is defined as in the previous theorem, then $\phi(b)(x) = \sum_j b_j x_j y_j$ for $x = (x_n) \in \ell_1$. Since $\phi(b)(e_n^*) = (b_n y_n)$ is w-null, $\phi(b)$ is weakly compact for all $b \in \ell_\infty$. By Theorem 4, $K(\ell_1, Y) \overset{c}{\not\hookrightarrow} W(\ell_1, Y)$. ∎

The next corollary contains principal results of [11], [13] and [16].

Corollary 6.

(i) *If $\ell_\infty \hookrightarrow Y$ and X does not have the Schur property (or $\ell_\infty \hookrightarrow X$ and Y does not have the Schur property), then $K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$.*

(ii) *If $c_0 \hookrightarrow K(X,Y)$ and $K(X,Y) \neq L(X,Y)$, then $K(X,Y)$ is not complemented in $L(X,Y)$.*

(iii) *If $c_0 \hookrightarrow Y$ and X does not have the Schur property (or $c_0 \hookrightarrow X$ and Y does not have the Schur property), then $K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$.*

(iv) *If $c_0 \hookrightarrow K_{w^*}(X^*, Y)$ and X and Y do not have the Schur property, then $K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$.*

Proof. (i) Since $\ell_\infty \hookrightarrow Y$ and ℓ_∞ is injective, ℓ_∞ is complemented in Y. Suppose that $K_{w^*}(X^*, Y) \overset{c}{\hookrightarrow} L_{w^*}(X^*, Y)$. Then $K_{w^*}(X^*, \ell_\infty) \overset{c}{\hookrightarrow} L_{w^*}(X^*, \ell_\infty)$. Let P be a projection of $L_{w^*}(X^*, \ell_\infty)$ onto $K_{w^*}(X^*, \ell_\infty)$. Note that $W(\ell_1, X) \simeq L_{w^*}(X^*, \ell_\infty)$ and $K(\ell_1, X) \simeq K_{w^*}(X^*, \ell_\infty)$. Hence the projection P may be viewed as an operator from $W(\ell_1, X)$ onto $K(\ell_1, X)$. Apply Corollary 5 now.

(ii) Suppose that $K(X, Y) \overset{c}{\hookrightarrow} L(X, Y)$. By Theorem 1, $\ell_\infty \hookrightarrow K(X, Y)$. Apply Theorem 4 of Kalton [23] to conclude that $\ell_\infty \hookrightarrow X^*$ or $\ell_\infty \hookrightarrow Y$. The first case produces a contradiction in view of Lemma 3 of Kalton [23]. If $\ell_\infty \hookrightarrow Y$, then $c_0 \hookrightarrow Y$, and the conclusion follows from Corollary 1 of Feder [17].

(iii) Suppose that $c_0 \hookrightarrow Y$ and X does not have the Schur property. Assume that $K_{w^*}(X^*, Y) \overset{c}{\hookrightarrow} L_{w^*}(X^*, Y)$. Theorem 1 implies that $\ell_\infty \hookrightarrow K_{w^*}(X^*, Y)$. Drewnowski's result [8] implies that $\ell_\infty \hookrightarrow X$ or $\ell_\infty \hookrightarrow Y$. However, this is not possible by part (i).

(iv) The same proof as for (iii). ∎

Our proof of Corollary 6 made use of the following result in [17]:

THEOREM 7 (Feder, [17]). *Suppose T is an operator in $L(X, Y)$ which is not compact and (T_n) is a sequence in $K(X, Y)$ such that for each $x \in X$, the series $\sum T_n(x)$ converges unconditionally to $T(x)$. Then $K(X, Y)$ is not complemented in $L(X, Y)$.*

In [11] Emmanuele proved that the containment of c_0 in $K(X, Y)$ is equivalent to the hypothesis of Feder's theorem. He used this to obtain (ii) of Corollary 6 above. In the next theorem we obtain an analogue of Feder's theorem in $L_{w^*}(X^*, Y)$.

THEOREM 8. *Suppose T is an operator in $L_{w^*}(X^*, Y)$ which is not compact and (T_n) is a sequence in $K_{w^*}(X^*, Y)$ such that for each $x^* \in X^*$, the series $\sum T_n(x^*)$ converges unconditionally to $T(x^*)$. Then $K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$. Furthermore, $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$.*

Proof. Since $L_{w^*}(X^*, Y) \neq K_{w^*}(X^*, Y)$, X and Y do not have the Schur property (if X or Y has the Schur property, $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$). Without loss of generality assume $c_0 \not\hookrightarrow X, Y$ (by Corollary 6(iii)), hence $\ell_\infty \not\hookrightarrow X, Y$. Suppose the operator T and the sequence (T_n) are as in the hypothesis. Since T is not compact, $\sum T_n$ diverges in the norm topology of $K_{w^*}(X^*, Y)$. This divergence and the pointwise unconditional convergence of the series $\sum T_n(x^*)$ allow us to reblock the sum and to assume that $\|T_n\| \not\to 0$.

Now use the Uniform Boundedness Principle, the finite-cofinite algebra of the subsets of $\mathbb{N}$, and the Diestel–Faires theorem to conclude that $c_0 \hookrightarrow K_{w^*}(X^*, Y)$; see the proof of Theorem 1 for details. (Alternatively, note that $\sum T_n$ is weakly unconditionally convergent and not unconditionally convergent.) If $K_{w^*}(X^*, Y)$ were complemented in $L_{w^*}(X^*, Y)$, then Theorem 1 would place ℓ_∞ in $K_{w^*}(X^*, Y)$. Another application of Drewnowski's result [8] would provide the contradiction that ℓ_∞ would embed in either X or Y. To see that ℓ_∞ embeds in $L_{w^*}(X^*, Y)$ simply apply Theorem 1 again. ∎

Remark. The hypothesis of the previous theorem implies that the series $\sum T_n$ is *wuc* (by the Uniform Boundedness Principle) and not unconditionally convergent in $K_{w^*}(X^*, Y)$, hence c_0 embeds in $K_{w^*}(X^*, Y)$. Conversely, if c_0 embeds in $K_{w^*}(X^*, Y)$, but neither in X nor in Y, then there is a sequence (T_n) which satisfies the hypothesis of Theorem 8. In fact, if $c_0 \not\hookrightarrow X, Y$, then $l_\infty \not\hookrightarrow X, Y$ and thus $l_\infty \not\hookrightarrow K_{w^*}(X^*, Y)$ [8]. Let (T_n) be a copy of (e_n) in $K_{w^*}(X^*, Y)$. Define $\phi : \ell_\infty \to L(X^*, Y)$ by

$$\phi(b)(x^*) = \sum b_n T_n(x^*), \qquad x^* \in X^*.$$

This series is unconditionally convergent and $\phi(b)$ is a w^*-w operator. If $\phi(b)$ is compact for each $b \in \ell_\infty$, then $\phi : \ell_\infty \to K_{w^*}(X^*, Y)$ is weakly compact (since $\ell_\infty \not\hookrightarrow K_{w^*}(X^*, Y)$, [28]). Then $\|\phi(e_n)\| = \|T_n\| \to 0$. This is a contradiction. Therefore there is a $b_0 \in \ell_\infty$ such that $\phi(b_0)$ is not compact. The series $\sum b_{0n} T_n$ and the operator $\phi(b_0)$ satisfy the hypothesis of Theorem 8.

We are now in a position to present a concise and straightforward proof of the main result in [13] and to obtain several corollaries concerning the structure of $K(X, Y)$ and $W(X, Y)$.

Theorem 9 ([13, Theorem 4]). *Suppose $c_0 \hookrightarrow K_{w^*}(X^*, Y)$. Then either $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$, or $K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$.*

Furthermore, $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$ if and only if only one of the following is true:

(i) *$c_0 \hookrightarrow Y$ and X has the Schur property*,
(ii) *$c_0 \hookrightarrow X$ and Y has the Schur property.*

Proof. If $c_0 \hookrightarrow K_{w^*}(X^*, Y)$ and $K_{w^*}(X^*, Y) \neq L_{w^*}(X^*, Y)$, then Corollary 6(iv) implies that $K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$.

Now assume that $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$ and c_0 embeds neither in X nor in Y. The proof of Theorem 1 shows that if $c_0 \hookrightarrow K_{w^*}(X^*, Y)$, but $c_0 \not\hookrightarrow X, Y$, then $\ell_\infty \hookrightarrow K_{w^*}(X^*, Y)$. Therefore $\ell_\infty \hookrightarrow X$ or $\ell_\infty \hookrightarrow Y$ [8]. This contradiction shows that either $c_0 \hookrightarrow X$ or $c_0 \hookrightarrow Y$.

If $c_0 \hookrightarrow Y$ and X does not have the Schur property, then $K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$ by Corollary 6(iii). Hence X has the Schur property and (i) must hold. ∎

Corollaries 10–12 make use of the following isometries:

$$W(X,Y) \simeq L_{w^*}(X^{**},Y), \qquad K(X,Y) \simeq K_{w^*}(X^{**},Y).$$

Corollary 10. *Suppose Y is the second Bourgain–Delbaen space which is an $\mathcal{L}_\infty$-space which has the RNP and Y^* is isomorphic to ℓ_1. Then $c_0 \not\hookrightarrow K(Y,Y)$.*

Proof. Since Y^* is a Schur space, it follows that $K(Y,Y) = W(Y,Y)$ and $c_0 \not\hookrightarrow Y^*$. Further, $c_0 \not\hookrightarrow Y$ since Y has the RNP. By Theorem 9, $c_0 \not\hookrightarrow K(Y,Y)$. ∎

Corollary 11. *Suppose $T : X \to Y$ is a weakly compact operator which is not compact and (T_n) is a sequence in $K(X,Y)$ such that for each $x \in X$, the series $\sum T_n(x)$ converges unconditionally to $T(x)$. Then $K(X,Y)$ is not complemented in $W(X,Y)$. Furthermore, $\ell_\infty \hookrightarrow W(X,Y)$.*

Proof. Apply Theorem 8. ∎

Corollary 12.

(i) *If $c_0 \hookrightarrow Y$ and X^* does not have the Schur property, then $K(X,Y)$ is not complemented in $W(X,Y)$ and $\ell_\infty \hookrightarrow W(X,Y)$.*
(ii) *If $c_0 \hookrightarrow K(X,Y)$ and $K(X,Y) \neq W(X,Y)$, then $K(X,Y)$ is not complemented in $W(X,Y)$ and $\ell_\infty \hookrightarrow W(X,Y)$.*

Proof. (i) Apply Corollary 6(iii) to deduce that $K(X,Y) \overset{c}{\not\hookrightarrow} W(X,Y)$. An application of Corollary 2 concludes the proof.

(ii) Apply Theorem 9 to find that $K(X,Y) \overset{c}{\not\hookrightarrow} W(X,Y)$. An application of Corollary 2 concludes the proof. ∎

The next theorem, as well as several subsequent corollaries, show that many familiar spaces of operators contain complemented copies of c_0.

Theorem 13. *Suppose that (x_i) is an unconditional and seminormalized shrinking basis for X and (x_i^*) is the associated sequence of coefficient functionals. Let T be an operator in $L_{w^*}(X^*,Y)$ such that $(T(x_i^*))$ is seminormalized. Then $c_0 \hookrightarrow K_{w^*}(X^*,Y)$, $K_{w^*}(X^*,Y) \overset{c}{\not\hookrightarrow} L_{w^*}(X^*,Y)$, and $\ell_\infty \hookrightarrow L_{w^*}(X^*,Y)$. Moreover, $c_0 \overset{c}{\hookrightarrow} K_{w^*}(X^*,Y)$.*

Proof. Since (x_n) is an unconditional shrinking basis for X, (x_n^*) is an unconditional basis for X^*, and the series $\sum x^*(x_n)x_n^*$ converges unconditionally to x^* for all $x^* \in X^*$ ([32, Thm. 17.7]). Note that $(T(x_i^*))$ is w-null since (x_i^*) is w^*-null. Bessaga–Pełczyński's selection principle allows us to assume that $(T(x_i^*))$ is a w-null basic sequence in Y. If $T_i : X^* \to Y$, $T_i(x^*) = x^*(x_i)T(x_i^*)$, then $T_i \in K_{w^*}(X^*,Y)$ and the series $\sum T_i(x^*)$ converges unconditionally to $T(x^*)$ for all $x^* \in X^*$. Since T is not compact,

$\sum T_n$ is weakly unconditionally convergent and not unconditionally convergent, and thus $c_0 \hookrightarrow K_{w^*}(X^*, Y)$. By Theorem 8, $K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$ and $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$.

Choose $\varepsilon > 0$ and intertwining sequences (m_k), (n_k) of positive integers so that $\|\sum_{i=m_k}^{n_k} T_i\| > \varepsilon$ for each k. Let $L_k = \sum_{i=m_k}^{n_k} T_i$, $k \in \mathbb{N}$. Note that $\sum L_k(x^*)$ converges unconditionally for each $x^* \in X^*$ since $\sum T_i(x^*)$ is unconditionally convergent. Hence $\sum L_k$ is weakly unconditionally convergent in $K_{w^*}(X^*, Y)$. Moreover, $\inf \|L_k\| > 0$. By Lemma 3 on p. 160 of [3], $(L_k) \sim (e_k)$.

Let (y_i^*) in Y^* be a biorthogonal sequence of coefficients of $(T(x_i^*))$. We may suppose that $\|y_i^*\| \le 1$. If $L \in K_{w^*}(X^*, Y)$, then $\langle x_i^* \otimes y_i^*, L\rangle \le \|L(x_i^*)\| \to 0$. Hence $(x_i^* \otimes y_i^*)$ is w^*-null in $(K_{w^*}(X^*, Y))^*$. For each $m_k \le i \le n_k$, $\langle x_i^* \otimes y_i^*, L_k\rangle = \langle x_i^* \otimes y_i^*, T_i\rangle = 1$. Then (L_k) is not limited. By a result on p. 36 of Schlumprecht [30], $c_0 \overset{c}{\hookrightarrow} K_{w^*}(X^*, Y)$. ∎

Theorem 14. *Let X and Y be infinite-dimensional Banach spaces satisfying the following assumption: if T is an operator in $L_{w^*}(X^*, Y)$, then there is a sequence of operators (T_n) in $K_{w^*}(X^*, Y)$ such that for each $x^* \in X^*$, the series $\sum T_n(x^*)$ converges unconditionally to $T(x^*)$. Then the following are equivalent:*

(i) $K_{w^*}(X^*, Y) \ne L_{w^*}(X^*, Y)$.
(ii) *X and Y do not have the Schur property and* $c_0 \hookrightarrow K_{w^*}(X^*, Y)$.
(iii) *X and Y do not have the Schur property and* $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$.
(iv) $K_{w^*}(X^*, Y)$ *is not complemented in* $L_{w^*}(X^*, Y)$.

Proof. (i)⇒(ii). Let $T \in L_{w^*}(X^*, Y)$ be noncompact. Then X and Y do not have the Schur property. Let (T_n) be a sequence as in the hypothesis. By the remark after Theorem 8, $c_0 \hookrightarrow K_{w^*}(X^*, Y)$.

(ii)⇒(iii) by Corollary 3 (or Corollary 2).

(iii)⇒(i). If $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$, then $\ell_\infty \hookrightarrow K_{w^*}(X^*, Y)$. By Drewnowski's result [8], $\ell_\infty \hookrightarrow X$ or $\ell_\infty \hookrightarrow Y$. By Corollary 6(i), $K_{w^*}(X^*, Y) \overset{c}{\not\hookrightarrow} L_{w^*}(X^*, Y)$, a contradiction.

(iv)⇒(i) is trivial, and (ii)⇒(iv) by Corollary 6(iv). ∎

A separable Banach space X has an *unconditional finite-dimensional expansion of the identity* (u.f.d.e.i.) if there is a sequence (A_n) of finite rank operators from X to X such that $\sum A_n(x)$ converges unconditionally to x for all $x \in X$. In this case, (A_n) is called an u.f.d.e.i. of X [18].

Corollary 15. *If either Y or X has an u.f.d.e.i., then the following are equivalent:*

(i) $K_{w^*}(X^*, Y) \ne L_{w^*}(X^*, Y)$.
(ii) *X and Y do not have the Schur property and* $c_0 \hookrightarrow K_{w^*}(X^*, Y)$.

(iii) *X and Y do not have the Schur property and $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$.*
(iv) *$K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$.*
(v) *X and Y do not have the Schur property and $c_0 \overset{c}{\hookrightarrow} K_{w^*}(X^*, Y)$.*

Proof. Suppose Y has an u.f.d.e.i. (A_n). Then $A_n : Y \to Y$ is compact for each n and $y = \sum A_n(y)$ unconditionally for each $y \in Y$. Let $T \in L_{w^*}(X^*, Y)$. Hence $T(x^*) = \sum A_n T(x^*)$ unconditionally for each $x^* \in X^*$ and $A_n T \in K_{w^*}(X^*, Y)$. Apply Theorem 14 to find that the first four statements are equivalent.

Now, if Y has an u.f.d.e.i. then Y must be separable, hence it has the Gelfand–Phillips property [4]. By Theorem 18 in [13], if $c_0 \hookrightarrow K_{w^*}(X^*, Y)$, then $c_0 \overset{c}{\hookrightarrow} K_{w^*}(X^*, Y)$. Hence (ii)⇒(v). (v)⇒(ii) is trivial.

Assume that X has an u.f.d.e.i. (A_n). Then $A_n : X \to X$ is compact and $x = \sum A_n(x)$ unconditionally for each $x \in X$. Let $T \in L_{w^*}(X^*, Y)$. Then $T^*(y^*) = \sum A_n T^*(y^*)$ unconditionally for each $y^* \in Y^*$ and $T_n = A_n T^* \in K_{w^*}(Y^*, X)$. Now apply Theorem 14 and use the isometry $K_{w^*}(X^*, Y) \simeq K_{w^*}(Y^*, X)$. ∎

Corollary 16 ([13, Corollary 9]). *Let X and Y be infinite-dimensional Banach spaces such that X^* or Y has an u.f.d.e.i. Then the following are equivalent*:

(i) *$K(X, Y) \neq W(X, Y)$.*
(ii) *X^* and Y do not have the Schur property and $c_0 \hookrightarrow K(X, Y)$.*
(iii) *X^* and Y do not have the Schur property and $\ell_\infty \hookrightarrow W(X, Y)$.*
(iv) *$K(X, Y)$ is not complemented in $W(X, Y)$.*
(v) *X^* and Y do not have the Schur property and $c_0 \overset{c}{\hookrightarrow} K(X, Y)$.*

Proof. Apply the isometries at the beginning of this section and Corollary 15. ∎

Corollary 17. *Suppose that X^* has an u.f.d.e.i. (A_n) consisting of w^*-w operators. Then the conclusion of Corollary 15 is true.*

Proof. Let (A_n) be an u.f.d.e.i. for X^* consisting of w^*-w operators. Let $T \in L_{w^*}(X^*, Y)$ and $T_n = T A_n$. Then $x^* = \sum A_n(x^*)$ unconditionally for each $x^* \in X^*$, $T^*(Y^*) \subseteq X$, $A_n^*(X^{**}) \subseteq X$, and T_n is compact for each n. We will show that T_n is w^*-w continuous. Let (x_α^*) be a w^*-null net in B_{X^*} and $y^* \in Y^*$. For each $n \in \mathbb{N}$,

$$\langle y^*, T_n(x_\alpha^*)\rangle = \langle A_n^* T^*(y^*), x_\alpha^*\rangle \to 0.$$

Then $T_n \in L_{w^*}(X^*, Y)$, and thus $T_n \in K_{w^*}(X^*, Y)$. Since the series $\sum T_n(x^*)$ converges unconditionally to $T(x^*)$ for each $x^* \in X^*$, an application of Theorem 14, Theorem 18 in [13], and the isometry $K_{w^*}(Y^*, X) \simeq K_{w^*}(X^*, Y)$ concludes the proof. ∎

The following result is motivated by Theorem 1 in [14].

A sequence (X_n) of closed subspaces of a Banach space X is called an *unconditional Schauder decomposition* of X if every $x \in X$ has a unique representation of the form $x = \sum x_n$ with $x_n \in X_n$ for every n, and the series converges unconditionally [26].

COROLLARY 18. *Let X and Y be infinite-dimensional Banach spaces satisfying the following assumptions*:

(a) *Y is complemented in a Banach space Z which has an unconditional Schauder decomposition (Z_n).*
(b) *$L(X^*, Z_n) = K(X^*, Z_n)$ for each n. Then the conclusion of Theorem 14 is true.*

Proof. Let $T \in L_{w^*}(X^*, Y)$, $A_n : Z \to Z_n$, $A_n(\sum z_i) = z_n$, and P the projection of Z onto Y. Define $T_n : X^* \to Y$ by $T_n(x^*) = PA_nT(x^*)$, $x^* \in X^*$, $n \in \mathbb{N}$. Note that T_n is compact since $L(X^*, Z_n) = K(X^*, Z_n)$, and T_n is w^*-w continuous for each n. Since for each $z \in Z$, $z = \sum A_n(z)$ and the convergence is unconditional, $\sum T_n(x^*)$ converges unconditionally to $T(x^*)$ for each $x^* \in X^*$. An application of Theorem 14 gives the conclusion. ■

The hypothesis (b) of the previous theorem is satisfied, for instance, in the following cases:

(1) X is arbitrary and each Z_n is finite-dimensional;
(2) $\ell_1 \not\hookrightarrow X^*$ and each Z_n has the Schur property;
(3) $X = \ell_1$ and each Z_n has the Schur property;
(4) X^{**} has the Schur property and each Z_n has (RDP*).

COROLLARY 19. *If $\ell_1 \not\hookrightarrow X^*$, Y is complemented in a Banach space Z which has an unconditional Schauder decomposition (Z_n), and each Z_n has the Schur property, then the conclusion of Corollary* 15 *is true.*

Proof. Since Z has an unconditional Schauder decomposition (Z_n) and each Z_n has the Schur property, Z, hence Y, has the Gelfand–Phillips property [9]. Apply Corollary 18 and Theorem 18 in [13] to get the conclusion. ■

The following theorem continues a theme of Theorem 13 and gives sufficient conditions for $K_{w^*}(X^*, Y)$ to contain isomorphic (complemented) copies of c_0.

THEOREM 20. *Let X and Y be Banach spaces satisfying the following assumption*: *there exists a Banach space G with an unconditional basis (g_n) and biorthogonal coefficients (g_n^*) and two operators $R : G \to Y$ and $S : G^* \to X$ such that $(R(g_i))$ and $(S(g_i^*))$ are seminormalized sequences and either $(R(g_i))$ or $(S(g_i^*))$ is a basic sequence. Then c_0 embeds in $K_{w^*}(X^*, Y)$ (indeed, in any subspace H of $L(X^*, Y)$ which contains $X \otimes_\lambda Y$).*

Moreover, if $(R(g_i))$ *and* $(S(g_i^*))$ *are basic and* Y *(or* X*) has the Gelfand–Phillips property, then* $K_{w^*}(X^*, Y)$ *contains a complemented copy of* c_0.

Proof. Suppose that $p \le \|R(g_i)\| \le q$ and $p \le \|S(g_i^*)\| \le q$ for all i. Let $S(g_i^*) \otimes R(g_i) \in K_{w^*}(X^*, Y)$, $\langle S(g_i^*) \otimes R(g_i), x^*\rangle = x^*(S(g_i^*))R(g_i)$, $x^* \in X^*$.

Assume without loss of generality that $(R(g_i))$ is a basic sequence. Choose $C_1 > 0$ so that for all real numbers (b_i) and all positive integers $m \le n$,

$$\Big\|\sum_{i=1}^{m} b_i R(g_i)\Big\| \le C_1 \Big\|\sum_{i=1}^{n} b_i R(g_i)\Big\|.$$

Then $\|b_i R(g_i)\| \le 2C_1 \|\sum_{j=1}^{n} b_i R(g_i)\|$ for each $1 \le i \le n$.

We have, for any sequence (a_n) of real numbers,

$$\begin{aligned}\Big\|\sum_{i=1}^{n} a_i[S(g_i^*) \otimes R(g_i)]\Big\|_\lambda &= \sup\Big\{\Big\|\sum_{i=1}^{n} a_i x^*(S(g_i^*))R(g_i)\Big\| : x^* \in B_{X^*}\Big\}\\ &\ge \sup\Big\{\frac{1}{2C_1}\,\|a_i x^*(S(g_i^*))R(g_i)\| : x^* \in B_{X^*}\Big\}\\ &\ge \frac{1}{2C_1}\,p|a_i|\,\|S(g_i^*)\| \ge \frac{1}{2C_1}\,p^2|a_i|\end{aligned}$$

for each $1 \le i \le n$. Hence

$$\Big\|\sum_{i=1}^{n} a_i[S(g_i^*) \otimes R(g_i)]\Big\|_\lambda \ge \frac{1}{2C_1} p^2 (\max_{i=1}^{n} |a_i|).$$

On the other hand, S and R induce an operator $S \otimes_\lambda R : G^* \otimes_\lambda G \to X \otimes_\lambda Y$, which maps $(g_n^* \otimes g_n)$ into $(S(g_n^*) \otimes R(g_n))$ ([7, Chapter VIII]). So we have

$$\Big\|\sum_{i=1}^{n} a_i[S(g_i^*) \otimes R(g_i)]\Big\|_\lambda \le \|S \otimes_\lambda R\| \Big\|\sum_{i=1}^{n} a_i(g_i^* \otimes g_i)\Big\|_\lambda.$$

Let $\varepsilon(\{g_i^*(g)g_i\}) = \sup\{\sum |g^*(g_i^*(g)g_i)| : g^* \in B_{G^*}\}$ for $g \in G$ and let M be the unconditional basis constant of the unconditional basis (g_n).

If $g \in G$ and $g^* \in B_{G^*}$, then $g = \sum g_i^*(g)g_i$ unconditionally, $\sum |g^*(g_i^*(g)g_i)| \le 2M\|g\|$, and $\sup\{\varepsilon(\{g_i^*(g)g_i\}) : g \in B_G\} \le 2M$. Consequently,

$$\begin{aligned}\Big\|\sum_{i=1}^{n} a_i(g_i^* \otimes g_i)\Big\|_\lambda &\le \sup\Big\{\sum_{i=1}^{n} |a_i g_i^*(g) g^*(g_i)| : g \in B_G,\ g^* \in B_{G^*}\Big\}\\ &\le 2M(\max_{i=1}^{n} |a_i|),\end{aligned}$$

and therefore

$$\Big\|\sum_{i=1}^{n} a_i[S(g_i^*)\otimes R(g_i)]\Big\|_\lambda \le 2M\|S\otimes_\lambda R\| \max_{i=1}^{n}|a_i|.$$

Hence $(S(g_n^*)\otimes R(g_n)) \sim (e_n)$ and thus $c_0 \hookrightarrow K_{w^*}(X^*,Y)$.

To prove the last part of the theorem, suppose that Y has the Gelfand–Phillips property and both $(R(g_n))$ and $(S(g_n^*))$ are basic. If $(R(g_n))$ is limited, then $R(g_n)\to 0$ since $(R(g_n))$ is relatively compact and the only weak limit of a basic sequence is zero [5, p. 42]. Therefore $(R(g_n))$ is not limited. By a result of Schlumprecht [30], we can choose a w^*-null sequence (y_n^*) in Y^* such that $\langle y_n^*, R(g_m)\rangle = \delta_{nm}$. Let (x_n^*) be a bounded sequence in X^* such that $\langle x_n^*, S(g_m^*)\rangle = \delta_{nm}$. We may assume that $\|x_n^*\|\le 1$. Then $(x_n^*\otimes y_n^*)$ is a w^*-null sequence in $(K_{w^*}(X^*,Y))^*$ since for each $T\in K_{w^*}(X^*,Y)$,

$$\langle x_n^*\otimes y_n^*, T\rangle = \langle T(x_n^*), y_n^*\rangle \le \|T^*(y_n^*)\| \to 0.$$

Also, $\langle x_n^*\otimes y_n^*, S(g_m^*)\otimes R(g_m)\rangle = \delta_{nm}$, thus $(S(g_m^*)\otimes R(g_m))$ is not limited. By Theorem 1.3.2 in [30], $c_0 \overset{c}{\hookrightarrow} K_{w^*}(X^*,Y)$. ∎

Remark. From Theorem 20 and the first example at the end of the paper it follows that c_0 embeds in $K_{w^*}(X^*,Y)$ when ℓ_2 embeds in both X and Y. In fact, $c_0 \overset{c}{\hookrightarrow} K_{w^*}(X^*,Y)$.

Corollary 21 ([11, Theorem 3]). *Let X and Y be Banach spaces satisfying the following assumption: there exists a Banach space G with an unconditional basis (g_n) and biorthogonal coefficients (g_n^*) and two operators $R: G\to Y$ and $S: G^*\to X^*$ such that $(R(g_i))$ and $(S(g_i^*))$ are normalized basic sequences. Then $c_0\hookrightarrow K(X,Y)$.*

Moreover, if Y (or X^*) has the Gelfand–Phillips property, then $K(X,Y)$ contains a complemented copy of c_0.

Proof. Apply Theorem 20 and the isometry $K_{w^*}(X^{**},Y)\simeq K(X,Y)$. ∎

Recall that a basis (x_n) for X is said to be *perfectly homogeneous* if it is seminormalized and every seminormalized block basic sequence with respect to (x_n) is equivalent to (x_n) [32]. A perfectly homogeneous basis is unconditional. The unit vector bases of c_0 and ℓ_p, $1\le p<\infty$, are, up to equivalence, the only perfectly homogeneous bases (Zippin) [32, p. 609].

Theorem 22. *Suppose that (x_n^*) is a perfectly homogeneous basic sequence in X^*, $[x_n^*]^*\hookrightarrow X$ and $T:[x_n^*]\to Y$ is a non-completely continuous operator. Then $c_0\hookrightarrow K_{w^*}(X^*,Y)$, $\ell_\infty\hookrightarrow L_{w^*}(X^*,Y)$, and $K_{w^*}(X^*,Y)$ is not complemented in $L_{w^*}(X^*,Y)$.*

Proof. Suppose that (x_n^*) is a perfectly homogeneous basic sequence, $[x_n^*]^*\hookrightarrow X$, and $T:[x_n^*]\to Y$ is an operator which is not completely continuous. Let $U=[x_n^*]$, and let (u_n^*) be a weakly null sequence in U so that

$(T(u_n^*)) \nrightarrow 0$. Without loss of generality, suppose that $\varepsilon > 0$ and $\|T(u_n^*)\| > \varepsilon$ for each n. Apply the Bessaga–Pełczyński selection principle [5] and let (v_n^*) be a subsequence of (u_n^*) so that (v_n^*) is equivalent to a block basic sequence of (x_n^*). In fact, an inspection of the Bessaga–Pełczyński theorem shows that we may assume that (v_n^*) is seminormalized. Therefore $(v_n^*) \sim (x_n^*)$. Since (x_n^*) is unconditional, $c_0 \hookrightarrow K_{w^*}(X^*, Y)$ by Theorem 20. Apply Corollary 2 and Corollary 6(iv) to conclude the argument. ■

Corollary 23.

(a) *Assume that* $\ell_2 \hookrightarrow X$ *and there is an operator* $T : \ell_2 \to Y$ *such that the sequence* $(T(e_n^2))$ *is seminormalized. Then the four statements in the conclusion of Theorem* 14 *hold.*
(b) *Assume that* $\ell_2 \hookrightarrow Y$ *and there is an operator* $T : \ell_2 \to X$ *such that the sequence* $(T(e_n^2))$ *is seminormalized. Then the four statements in the conclusion of Theorem* 14 *hold.*

Proof. We prove (a); the case (b) is similar. An application of Theorem 22 (or 20) gives $c_0 \hookrightarrow K_{w^*}(X^*, Y)$. We note that X and Y are not Schur spaces by hypothesis. Thus, (ii) holds. The proof of Theorem 14 shows that (ii)⇒(i), (ii)⇒(iii), and (ii)⇒(iv). ■

Remark. A similar proof shows that if $\ell_2 \hookrightarrow X$ and $\ell_p \hookrightarrow Y$ for some $p \geq 2$, then the four statements in the conclusion of Theorem 14 hold.

Corollary 24. *Assume that* $\ell_2 \hookrightarrow X^*$ *and there is an operator* $T : \ell_2 \to Y$ *such that the sequence* $(T(e_n^2))$ *is seminormalized. Then the first four statements in the conclusion of Corollary* 16 *are true.*

Proof. Apply Corollary 23. ■

In [17] Feder proved that $K(C(S), L^1)$ is not complemented in $L(C(S), L^1)$ when S is not dispersed. See also [12]. The following corollary improves Feder's result.

Corollary 25. *Assume that* S *is a Hausdorff compact space which is not dispersed. Then* $K(C(S), L^1)$ *is not complemented in* $W(C(S), L^1)$.

Proof. Since S is not dispersed, $\ell_1 \hookrightarrow C(S)$ [24]. Then $L^1 \hookrightarrow C(S)^*$ [27]. Also, the Rademacher functions span ℓ_2 inside of L^1, and thus $\ell_2 \hookrightarrow C(S)^*$. Corollary 21 implies that $c_0 \hookrightarrow K(C(S), L^1)$. By Corollary 24, $K(C(S), L^1)$ is not complemented in $W(C(S), L^1)$. ■

See the last section of this paper for a generalization of Corollary 25.

Corollary 26 ([13, Corollary 12]). *Assume that* X *has the DPP and there is an operator* $T : \ell_2 \to Y$ *such that the sequence* $(T(e_n^2))$ *is seminormalized. Then the first four statements in the conclusion of Corollary* 16 *are equivalent.*

Proof. We only have to show that (i)⇒(ii). Since $K(X,Y) \neq W(X,Y)$, X^* and Y do not have the Schur property. Since X has the DPP and X^* is not a Schur space, $\ell_1 \hookrightarrow X$ [21], [6].

Then $L^1 \hookrightarrow X^*$ (by a result in [27]), hence $\ell_2 \hookrightarrow X^*$ [5]. By Theorem 20, $c_0 \hookrightarrow K(X,Y)$. The rest follows from Corollary 24. ∎

In [22] the authors proved that if X and Y are weakly sequentially complete and $K_{w^*}(X^*,Y) = L_{w^*}(X^*,Y)$, then $K_{w^*}(X^*,Y)$ is weakly sequentially complete. Now we give a partial converse.

COROLLARY 27. *If Y (or X) has an u.f.d.e.i. and $K_{w^*}(X^*,Y)$ is weakly sequentially complete, then $K_{w^*}(X^*,Y) = L_{w^*}(X^*,Y)$.*

Proof. By Corollary 15, if $K_{w^*}(X^*,Y) \neq L_{w^*}(X^*,Y)$, then $c_0 \hookrightarrow K_{w^*}(X^*,Y)$, a contradiction. ∎

Closing remarks. Emmanuele made the following two observations on p. 334 of [11]:

(a) If $\ell_1 \hookrightarrow X$ and $\ell_p \hookrightarrow Y$ for some $p \geq 2$, then $c_0 \hookrightarrow K(X,Y)$ and $K(X,Y) \overset{c}{\not\hookrightarrow} L(X,Y)$.
(b) If $1/p + 1/p' = 1$ and $1 < p' \leq q < \infty$, then $c_0 \overset{c}{\hookrightarrow} \ell_p \otimes_\varepsilon \ell_q$.

In case (a) we can actually show that $K(X,Y) \overset{c}{\not\hookrightarrow} W(X,Y)$. Suppose that $\ell_1 \hookrightarrow X$ and $\ell_p \hookrightarrow Y$, $p \geq 2$. Then $L_1 \hookrightarrow X^*$, and thus $\ell_2 \hookrightarrow X^*$. By Theorem 20, $c_0 \hookrightarrow K_{w^*}(X^{**},Y)$. By Corollary 6, $K_{w^*}(X^{**},Y)$ is not complemented in $L_{w^*}(X^{**},Y)$. Now use the natural isometries at the beginning of the previous section to conclude that $K(X,Y)$ is not complemented in $W(X,Y)$.

Since $K(\ell_p,\ell_q) = K_{w^*}(\ell_p,\ell_q) \neq L(\ell_p,\ell_q) = L_{w^*}(\ell_p,\ell_q)$, Theorem 13 allows us to see that $c_0 \overset{c}{\hookrightarrow} K(\ell_p,\ell_q)$, $\ell_\infty \hookrightarrow L(\ell_p,\ell_q)$, and $K(\ell_p,\ell_q) \overset{c}{\not\hookrightarrow} L(\ell_p,\ell_q)$ whenever $1 < p \leq q < \infty$.

Since $X \hookrightarrow K_{w^*}(X^*,Y)$, obviously $c_0 \hookrightarrow K_{w^*}(\ell_1,Y)$ for every Banach space Y. By Theorem 18 in Emmanuele [13] , $c_0 \overset{c}{\hookrightarrow} K_{w^*}(\ell_1,Y)$ whenever Y has the Gelfand–Phillips property. Thus c_0 is complemented in $K_{w^*}(\ell_1,\ell_1)$. Further, Theorem 13, as well as the Emmanuele result just cited, show $c_0 \overset{c}{\hookrightarrow} K_{w^*}(\ell_1,\ell_p)$, $1 < p < \infty$. In fact, we can conclude more. Suppose that Z contains an infinite-dimensional subspace Y which has a shrinking and semi-normalized basis (y_n). Let (y_n^*) be the associated sequence of coefficient functionals. Define $L : \ell_1 \to Y$ by $L(\lambda) = \sum_{i=1}^{\infty} \lambda_i y_i$. Then $L^*(y_k^*) = e_k \in c_0$ for each k. Since (y_n) is shrinking, L is a w^*-w continuous operator and satisfies the hypotheses of Theorem 13. (Theorems 14 and 20 also apply to this setting.) In fact, if one defines $\widehat{L} : \ell_1 \to Z$ by $\widehat{L}(\lambda) = L(\lambda)$, then $\widehat{L} \in L_{w^*}(\ell_1,Z)$. Thus $c_0 \overset{c}{\hookrightarrow} K_{w^*}(\ell_1,Z)$, $\ell_\infty \hookrightarrow L_{w^*}(\ell_1,Z)$, and $K_{w^*}(\ell_1,Z) \overset{c}{\not\hookrightarrow} L_{w^*}(\ell_1,Z)$.

We note that $K_{w^*}(\ell_1, Z)$ may also contain copies of c_0 which fail to be complemented in this space of operators as well as copies of c_0 which are complemented. For example, ℓ_∞ contains all spaces with shrinking bases, and thus $c_0 \overset{c}{\hookrightarrow} K_{w^*}(\ell_1, \ell_\infty)$. However, ℓ_∞ naturally (and isometrically) embeds in $K_{w^*}(\ell_1, \ell_\infty)$, and thus the canonical copy of c_0 contained in ℓ_∞ cannot be complemented in this space of operators.

Similar arguments show that if $1 < p \le q < \infty$ and $\ell_q \hookrightarrow Z$, then $c_0 \overset{c}{\hookrightarrow} K_{w^*}(\ell_p, Z) = K(\ell_p, Z)$, $\ell_\infty \hookrightarrow L_{w^*}(\ell_p, Z) = L(\ell_p, Z)$ and $K(\ell_p, Z) \overset{c}{\not\hookrightarrow} L(\ell_p, Z)$. Note also that $K(\ell_p, \ell_\infty)$ contains both complemented and uncomplemented copies of c_0.

Examples. The first example shows that there are Banach spaces X and Y such that $c_0 \not\hookrightarrow X, Y$, $c_0 \hookrightarrow K_{w^*}(X^*, Y)$, but $K_{w^*}(X^*, Y) \ne L_{w^*}(X^*, Y)$. Clearly c_0 does not embed in ℓ_2. A direct application of Theorem 20 shows that $c_0 \hookrightarrow K_{w^*}(\ell_2, \ell_2)$ and the identity operator from ℓ_2 to ℓ_2 shows that $K_{w^*}(\ell_2, \ell_2) \ne L_{w^*}(\ell_2, \ell_2)$.

The next example [15] shows that we can find Banach spaces X and Y such that $c_0 \not\hookrightarrow K_{w^*}(X^*, Y)$, but $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$. Let $E = F$ be the Bourgain–Delbaen space which is an $\mathcal{L}_\infty$ space with RNP and such that E^* is a Schur space even though $c_0 \not\hookrightarrow E$. Assume that $c_0 \hookrightarrow K_{w^*}(E^{**}, E)$ and let (T_n) be a copy of c_0 in $K_{w^*}(E^{**}, E)$. Define $\phi : \ell_\infty \to L(E^{**}, E)$ by $\phi(b)(x^{**}) = \sum b_n T_n(x^{**})$. Since $c_0 \not\hookrightarrow E^*$, the series $\sum b_n T_n^*(y^*)$ converges unconditionally for each $y^* \in E^*$, hence $\phi(b) \in L_{w^*}(E^{**}, E)$. Note that $\|\phi(e_n)\| = \|T_n\| \not\to 0$. A result of Rosenthal [28] implies that $\ell_\infty \hookrightarrow L_{w^*}(E^{**}, E)$. On the other hand, $K_{w^*}(E^{**}, E) = L_{w^*}(E^{**}, E)$ since E^* is a Schur space. By Drewnowski's result, $\ell_\infty \hookrightarrow E$ or $\ell_\infty \hookrightarrow E^*$, a contradiction. Hence $c_0 \not\hookrightarrow K_{w^*}(E^{**}, E)$. Thus the spaces $X = E^*$ and $Y = E$ are as desired.

Alternatively, for $1 \le q < p$, $L(\ell_p, \ell_q) = K(\ell_p, \ell_q)$ (Pitt). Kalton showed that for $1 \le q < p$, $L(\ell_p, \ell_q)$ is reflexive [23]. Thus $c_0 \not\hookrightarrow K(\ell_p, \ell_q) \simeq K_{w^*}(\ell_p^{**}, \ell_q)$, and the spaces $X = \ell_p^*$ and $Y = \ell_q$ are as desired.

We conclude the paper by asking the following question.

Question. Are there Banach spaces X, Y such that $K_{w^*}(X^*, Y) \ne L_{w^*}(X^*, Y)$ and $c_0 \not\hookrightarrow K_{w^*}(X^*, Y)$?

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Received November 14, 2008 (7689)