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PROBABILITY THEORY AND STOCHASTIC PROCESSES

Limiting Behaviour of Dirichlet Forms for Stable Processes on Metric Spaces

by

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Summary. Supposing that the metric space in question supports a fractional diffusion, we prove that after introducing an appropriate multiplicative factor, the Gagliardo seminorms $\|f\|_{W^{\sigma,2}}$ of a function $f \in L^2(E,\mu)$ have the property

$$\frac{1}{C}\,\mathcal{E}(f,f) \le \liminf_{\sigma\nearrow 1}\,(1-\sigma)\|f\|_{W^{\sigma,2}} \le \limsup_{\sigma\nearrow 1}\,(1-\sigma)\|f\|_{W^{\sigma,2}} \le C\mathcal{E}(f,f),$$

where $\mathcal{E}$ is the Dirichlet form relative to the fractional diffusion.

1. Inroduction. For $f \in L^p(\mathbb{R}^d)$, $0<\sigma<1$, $p>1$, consider the so-called *Gagliardo seminorm* of f:

$$\|f\|_{W^{\sigma,p}(\Omega)} = \bigg(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+\sigma p}}\,dx\,dy\bigg)^{1/p}, \tag{1.1}$$

where Ω is a connected open subset of $\mathbb{R}^d$. The restriction to $\sigma<1$ is mandatory: when $\sigma \ge 1$, then the finiteness of (1.1) results in f being a constant function (see e.g. [8]). The seminorm (1.1) is the intrinsic seminorm in the fractional Sobolev space $W^{\sigma,p}(\Omega)$ (see [1, par. 7.43]). We are interested in the behaviour of (1.1) as σ approaches the critical value $\sigma=1$. Bourgain, Brézis and Mironescu in [6], and further in [7], established the relation

$$\lim_{\sigma\nearrow 1}\,(1-\sigma)\|f\|^p_{W^{\sigma,p}(\Omega)} = C_p\int_\Omega |\nabla f|^p\,dx = C_p\|f\|^p_{W^{1,p}(\Omega)} \tag{1.2}$$

(Ω is a smooth bounded domain in $\mathbb{R}^d$, $f\in W^{1,p}(\Omega)$, $p>1$). Note that the

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meaning of $\|\cdot\|_{W^{\sigma,p}(\Omega)}$ is different for $\sigma<1$ and for $\sigma=1$, which is annoying but consistent with traditional notation. For the special case $p=2$, $\Omega=\mathbb{R}^d$, (1.2) follows from the previous work of Maz'ya and Nagel [18].

From another perspective, in this case ($\Omega=\mathbb{R}^d$, $p=2$, $\alpha<1$) the expression

$$\mathcal{E}^{(\alpha)}(f,f)=C_\alpha\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(f(x)-f(y))^2}{|x-y|^{d+2\alpha}}\,dx\,dy \qquad (=\|f\|^2_{W^{\alpha,2}})$$

(with domain $\mathcal{D}(\mathcal{E}^{(\alpha)})=W^{\alpha,2}(\mathbb{R}^d)$) is the Dirichlet form of the subordinated symmetric 2α-stable process on $\mathbb{R}^d$, while the Dirichlet integral

$$\mathcal{E}(f,f)=\int_{\mathbb{R}^d}|\nabla f|^2\,dx \quad (=\|f\|^2_{W^{1,2}}),$$

with domain $W^{1,2}$, is the Dirichlet form of the Brownian motion, and therefore the relation (1.2) asserts that the Dirichlet form of the Brownian motion on $\mathbb{R}^d$ can be recovered from the Dirichlet forms of stable processes.

In this note, we are concerned with a similar phenomenon arising for Brownian-like diffusions (*fractional diffusions*, see [2] for the definition) and related stable processes on metric measure spaces. Namely, suppose that $\mathcal{E}(f,f)$ is the Dirichlet form of the diffusion on a metric space (E,ϱ) equipped with an Ahlfors d-regular measure μ, and for $\alpha\in(0,1)$, $\mathcal{E}^{(\alpha)}$ is the Dirichlet form of the subordinated 2α-stable process. Then a similar statement holds: for any $f\in\mathcal{D}(\mathcal{E})$,

$$\lim_{\alpha\nearrow 1}\mathcal{E}^{(\alpha)}(f,f)=\mathcal{E}(f,f). \tag{1.3}$$

Similarly to the classical case, we can consider the Gagliardo seminorms

$$\|f\|_{W^{\sigma,2}}=\bigg(\int_E\int_E\frac{(f(x)-f(y))^2}{\varrho(x,y)^{d+2\sigma}}\,d\mu(x)\,d\mu(y)\bigg)^{1/2}, \tag{1.4}$$

which are now nontrivial up to $\sigma=d_w/2$, d_w being the *walk dimension* of (E,ϱ,μ) (in [20] it was proved that the finiteness of (1.4) with $\sigma\ge d_w/2$ implies $f\equiv\text{const}$). Since it is known (see Stós [21]) that the Dirichlet form of the 2α-stable process compares to $\|f\|^2_{W^{\alpha d_w,2}}$, our statement (1.3) obliges $\|f\|_{W^{\sigma,2}}$ to tend to ∞ when $\sigma\nearrow d_w/2$, and also, for $f\in\mathcal{D}(\mathcal{E})$,

$$\begin{aligned}\frac{1}{C}\,\mathcal{E}(f,f)&\le\liminf_{\alpha\nearrow 1}\,(1-\alpha)\|f\|_{W^{\alpha d_w/2,2}}\\&\le\limsup_{\alpha\nearrow 1}\,(1-\alpha)\|f\|_{W^{\alpha d_w/2,2}}\le C\mathcal{E}(f,f).\end{aligned}$$

Since satisfactory differential techniques are unavailable in the setting of general metric spaces, our proof requires a different approach than the original one in [6, 7].

2. Diffusion processes and their Dirichlet forms. Suppose that (E,ϱ) is a separable, locally compact metric space and that μ is a Radon measure on E such that

$$C_1 r^d \le \mu(B(x,r)) \le C_2 r^d \tag{2.1}$$

for some $d \ge 1$ and all $x \in E$, $r > 0$ (i.e. the measure μ is *Ahlfors d-regular*). We require E to satisfy the *chain condition*:

(C) for any $x, y \in E$ and $n \ge 1$ there exists a "chain" $x = x_0, x_1, \ldots, x_n = y$ such that $\varrho(x_i, x_{i+1}) \le (C/n)\varrho(x,y)$ (C a universal constant).

Further, assume that (E,ϱ,μ) supports a *Markovian kernel* $\{p(t,x,y)\}$, i.e. a family of measurable functions $p(t,\cdot,\cdot) : E \times E \to \mathbb{R}_+$, $t > 0$, which satisfies:

(M1) $p(t,x,y) = p(t,y,x)$ for all $t > 0$ and $x, y \in E$ (symmetry),
(M2) $p(s+t,x,y) = \int_E p(s,x,z)p(t,z,y)d\mu(z)$ for all $s,t > 0$ and $x,y \in E$ (the Chapman–Kolmogorov identity, or the Markov property),
(M3) $\int_E p(t,x,y)\,d\mu(y) = 1$ for all $t > 0$ and $x \in E$ (normalization),
(M4) $p(t,x,y) > 0$ for all $t > 0$ and $x, y \in E$ (irreducibility).

Our further assumption is that the Markovian kernel $p(t,x,y)$ satisfies the following estimate for all $t > 0$ and $x, y \in E$:

$$\begin{aligned} \frac{c_{1.1}}{t^{d/\beta}} \exp\left\{ -c_{1.2}\left(\frac{\varrho(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right\} &\le p(t,x,y) \\ &\le \frac{c_{1.3}}{t^{d/\beta}} \exp\left\{ -c_{1.4}\left(\frac{\varrho(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right\}. \end{aligned} \tag{2.2}$$

Examples of such spaces are the nested fractals ([2]) and other post-critically finite self-similar sets ([15]), Sierpiński carpets ([3]), and spaces that support the 2-Poincaré inequality ([22]). It is known that the parameter β does not depend on the particular kernel $p(\cdot,\cdot,\cdot)$, and is one of the characteristic constants of (E,ϱ,μ), called the *walk dimension* of E and denoted by $d_w(E)$ (or just d_w). Under the chain condition **(C)**, it is known (see [2], [13]) that $2 \le d_w \le d+1$. For the Euclidean space $\mathbb{R}^d$, as well as other spaces supporting the 2-Poincaré inequality, the walk dimension is equal to 2, regardless of d. The walk dimension of the Sierpiński gasket in $\mathbb{R}^d$ is equal to $\log(d+3)/\log 2 > 2$. The exact value of the walk dimension for the Sierpiński carpet is unknown.

It has been proven lately in [14] that if we require the basic estimate (2.2) to be of the form $(c/t^{d/\beta})\Phi(\varrho(x,y)/t^{1/\beta})$ with $\Phi : [0,\infty) \to [0,\infty)$ decreasing, then either Φ is an exponential function $\exp(-cs^{\beta/(\beta-1)})$, $\beta \ge 2$, and the corresponding Markov process is a diffusion, or it is equal to $1/(1+s)^{d+\beta}$ and the process is not diffusive.

Denote by $(P_t)_{t\geq 0}$ the semigroup of selfadjoint contraction operators on $L^2(E,\mu)$ associated with $\{p(t,x,y)\}$, given by

$$L^2(E,\mu) \ni f(x) \mapsto P_t f(x) = \int_E p(t,x,y) f(y)\, d\mu(y).$$

We require the semigroup to be continuous at zero, i.e.

(M4) $\lim_{t\to 0^+} P_t f = f$ for all $f \in L^2(E,\mu)$, the limit taken in $L^2(E,\mu)$.

Such a strongly continuous semigroup on $L^2(E,\mu)$ gives rise to a Dirichlet form $\mathcal{E}$. There are several ways of defining it; the most convenient for our setting is the following (see [9], [11]). For $f \in L^2(E,\mu)$ set

$$\mathcal{E}_t(f,f) = \frac{1}{t}\,\langle (f - P_t f), f\rangle_{L^2(E,\mu)}. \tag{2.3}$$

Because of **(M1)** and **(M3)**, we have

$$\mathcal{E}_t(f,f) = \frac{1}{2t} \int_E \int_E (f(x) - f(y))^2 p(t,x,y)\, d\mu(x)\, d\mu(y). \tag{2.4}$$

By an easy application of the spectral theorem, for any given $f \in L^2$, the mapping $t \mapsto \mathcal{E}_t(f,f)$ is decreasing. Therefore we can set

$$\begin{aligned} \mathcal{D}(\mathcal{E}) &= \{f \in L^2(E,\mu) : \sup_{t>0} \mathcal{E}_t(f,f) < \infty\},\\ \mathcal{E}(f,f) &= \lim_{t\downarrow 0} \mathcal{E}_t(f,f). \end{aligned} \tag{2.5}$$

Assuming **(M1)–(M5)** and (2.2), it has been shown (see [16], [19], [13]) that the domain of this Dirichlet form, $\mathcal{D}(\mathcal{E})$, is actually equal to the space

$$\mathcal{L} = \mathrm{Lip}(d_w/2, 2, \infty)(E).$$

The definition of this space is the following. For $f \in L^2(E,\mu)$ and $n = 1, 2, \ldots$, let

$$a_n(f) = \iint_{2^{-(n+1)} < \varrho(x,y) \leq 2^{-n}} (f(x) - f(y))^2\, d\mu(x)\, d\mu(y).$$

Then

$$f \in \mathcal{L} \;\Leftrightarrow\; \sup_{n\geq 0}\, [2^{n(d+d_w)} a_n(f)] < \infty. \tag{2.6}$$

The norm in $\mathcal{L}$ is

$$\|f\|_{\mathcal{L}}^2 = \|f\|_2^2 + \sup_{n\geq 0}\, [2^{n(d+d_w)} a_n(f)],$$

and turns $\mathcal{L}$ into a Banach space. Also, there exists a universal constant C such that for $f \in \mathcal{L}$ one has

$$\frac{1}{C}\,\mathcal{E}(f,f) \leq \sup_{n\geq 0}\, [2^{n(d+d_w)} a_n(f)] \leq C\mathcal{E}(f,f).$$

Consider now the expression

$$E^{(\alpha)}(f,f) = \int_E\int_E \frac{(f(x)-f(y))^2}{\varrho(x,y)^{d+\alpha d_w}}\, d\mu(x)\, d\mu(y). \tag{2.7}$$

and introduce the following spaces $\Lambda_\alpha^{2,2}(E)$:

$$f \in \Lambda_\alpha^{2,2}(E) \Leftrightarrow E^{(\alpha)}(f,f) < \infty,$$

with the norm

$$\|f\|^2_{\Lambda_\alpha^{2,2}} = \|f\|_2^2 + E^{(\alpha)}(f,f).$$

In the fractal setting, they were first considered in [21], and subsequently appeared in several articles. For a detailed account, see [12] or [17]. Note that these spaces have been known by the name of Besov–Slobodetskiĭ spaces. In [20], we have proven that for $\alpha \ge 1$, the finiteness of $E^{(\alpha)}(f,f)$ for a function $f \in L^2(E,\mu)$ results in $f \equiv$ const, and so the choice of the smoothness parameter α is restricted to $(0,1)$.

The following lemma is similar to its classical counterpart, nevertheless we give its proof for completeness.

Lemma 2.1. *We have* $\mathrm{Lip}(d_w/2, 2, \infty)(E) \subset \Lambda_\alpha^{2,2}(E)$. *Moreover, this embedding is continuous*,

$$\|f\|_{\Lambda_\alpha^{2,2}} \le C\|f\|_{\mathcal{L}}.$$

Proof. Take $f \in \mathcal{L}$. Split the integral defining $E^{(\alpha)}(f,f)$ into two parts: the first over the region $\varrho(x,y) > 1$, the other over $\varrho(x,y) \le 1$.

The first one is finite for any $f \in L^2$: indeed,

$$\begin{aligned}
&\iint\limits_{\varrho(x,y)>1} \frac{(f(x)-f(y))^2}{\varrho(x,y)^{d+\alpha d_w}}\, d\mu(x)\, d\mu(y) \\
&\quad = \sum_{n=0}^{\infty} \iint\limits_{2^n<\varrho(x,y)\le 2^{n+1}} \frac{(f(x)-f(y))^2}{\varrho(x,y)^{d+\alpha d_w}}\, d\mu(x)\, d\mu(y) \\
&\quad \le \sum_n \frac{1}{2^{n(d+\alpha d_w)}} \iint\limits_{2^n<\varrho(x,y)\le 2^{n+1}} (f(x)-f(y))^2\, d\mu(x)\, d\mu(y) \\
&\quad \le 2\sum_n \frac{1}{2^{n(d+\alpha d_w)}} \iint\limits_{2^n<\varrho(x,y)\le 2^{n+1}} (f(x)^2+f(y)^2)\, d\mu(x)\, d\mu(y) \\
&\quad = 4\sum_n \frac{1}{2^{n(d+\alpha d_w)}} \iint\limits_{2^n<\varrho(x,y)\le 2^{n+1}} f(x)^2\, d\mu(x)\, d\mu(y) \qquad \text{(by symmetry)} \\
&\quad = 4\sum_n \frac{1}{2^{n(d+\alpha d_w)}} \int_E f(x)^2 \mu(\{y : 2^n < \varrho(x,y) \le 2^{n+1}\})\, d\mu(x) \le C\|f\|_2^2
\end{aligned}$$

(in the last equality we used (2.1)).

As to the remaining integral, we write

$$\iint_{\varrho(x,y)\le 1} \frac{(f(x)-f(y))^2}{\varrho(x,y)^{d+\alpha d_w}}\,d\mu(x)\,d\mu(y)$$
$$\le \sum_{n=0}^{\infty} 2^{n(d+\alpha d_w)} \iint_{2^{-(n+1)}<\varrho(x,y)\le 2^{-n}} (f(x)-f(y))^2\,d\mu(x)\,d\mu(y)$$
$$\le 2^{d+\alpha d_w}\sum_{n=0}^{\infty}[2^{n(d+d_w)}a_n(f)]\,\frac{1}{2^{nd_w(1-\alpha)}} \le \frac{2^{d+\alpha d_w}2^{d_w(1-\alpha)}}{2^{d_w(1-\alpha)}-1}\,\|f\|_{\mathcal{L}}$$

and the continuity of the embedding is proven. ■

In particular, we see that all the spaces $\Lambda_\alpha^{2,2}(E)$, $\alpha<1$, are dense in $L^2(E,\mu)$. This is so because $\mathcal{L}$, being the domain of the Dirichlet form of a Markov process, is in particular dense in $L^2(E)$.

3. Stable processes and their Dirichlet forms

3.1. *Preliminaries on stable processes on metric spaces.* The following definition of a stable process on a metric space supporting a fractional diffusion is taken from [5].

For a fixed parameter $\alpha\in(0,1)$, let $(\xi_t)_{t\ge 0}$ be the *α-stable subordinator*, i.e. the process whose Laplace transform is given by $\mathbb{E}\exp(-u\xi_t)=\exp(-tu^\alpha)$. Let $\eta_t(u)$, $t>0, u\ge 0$, be its one-dimensional distribution density. For $t>0$ and $x,y\in X$ define

$$p^\alpha(t,x,y)=\int_0^\infty p(u,x,y)\eta_t(u)\,du.$$

It is classical (see e.g. [4, p. 18]) that $p^\alpha(t,x,y)$ is the transition density of a Markov process, which we denote by X^α and call the *symmetric 2α-stable process on E*. For further properties of this process and its transition density we refer the reader to [5].

From the property (see Th. 37.1 of [10])

$$\lim_{u\to\infty}\eta_1(u)u^{1+\alpha/2}=\frac{\alpha}{2\Gamma(1-\alpha/2)}$$

and the scaling relation

$$\eta_t(u)=t^{-2/\alpha}\eta_1(t^{-2/\alpha}u),\qquad t,u>0,$$

one deduces:

(P1) $\lim_{t\to 0} t^{-1}\eta_t(u)=(\alpha/\Gamma(1-\alpha))u^{-1-\alpha}$ for $u>0$,

(P2) (formula (9) of [5]) $\eta_t(u)\le ctu^{-1-\alpha}$ for $t,u>0$,

(P3) (formula (10) of [5]) $\eta_t(u) \geq ctu^{-1-\alpha}$ for $t > 0$ and $u > u_0 t^{1/\alpha}$, where $u_0 = u_0(\alpha)$.

The Dirichlet form of the 2α-stable process on E is defined by (2.3)–(2.5) and will be denoted by $\mathcal{E}^{(\alpha)}(f,f)$. In [21] it was proven that $\mathcal{D}(\mathcal{E}^{(\alpha)}) = \Lambda_\alpha^{2,2}(E)$, and that there exists a universal constant $D = D(\alpha)$ such that

$$\frac{1}{D} E^{(\alpha)}(f,f) \leq \mathcal{E}^{(\alpha)}(f,f) \leq D E^{(\alpha)}(f,f). \tag{3.1}$$

3.2. *The main theorem.* First, we prove the following theorem.

THEOREM 3.1. *Suppose* $f \in \mathcal{D}(\mathcal{E})$ $(= \mathrm{Lip}(d_w/2, 2, \infty)(E))$. *Then*

$$\lim_{\alpha \nearrow 1} \mathcal{E}^{(\alpha)}(f,f) = \mathcal{E}(f,f). \tag{3.2}$$

Proof. In view of Lemma 2.1 and the characterization of the domain $\mathcal{D}(\mathcal{E}^{(\alpha)})$, $\mathcal{E}^{(\alpha)}(f,f)$ is well-defined. The explicit formula for $\mathcal{E}^{(\alpha)}$ is

$$\begin{aligned}
\mathcal{E}^{(\alpha)}(f,f) &= \lim_{t\to 0} \frac{1}{2t} \int_E \int_E p^\alpha(t,x,y)(f(x)-f(y))^2 \, d\mu(x)\, d\mu(y) \\
&= \lim_{t\to 0} \int_0^\infty \frac{1}{2t} \Big(\int_E \int_E p(u,x,y)(f(x)-f(y))^2 \, d\mu(x)\, d\mu(y) \Big) \eta_t(u)\, du \qquad (3.3) \\
&= \frac{1}{2} \int_0^\infty \Big(\int_E \int_E p(u,x,y)(f(x)-f(y))^2 \, d\mu(x)\, d\mu(y) \Big) \lim_{t\to 0} \frac{\eta_t(u)}{t}\, du.
\end{aligned}$$

To justify the last step (of putting the limit under the integral sign) we use the Lebesgue dominated convergence theorem: since for all $u, t > 0$ we have $\eta_t(u) \leq ctu^{-1-\alpha}$ (property **(P2)**), the integrand in (3.3), being equal to $\frac{1}{2t}\langle f - P_u f, f\rangle \frac{\eta_t(u)}{u}$, can be estimated by

$$\frac{c}{u^{1+\alpha}} \langle f - P_u f, f\rangle. \tag{3.4}$$

For large u the contraction property of the semigroup yields

$$(3.4) \leq \frac{c\|f\|_2^2}{u^{1+\alpha}},$$

which is integrable for large u, and for small u write

$$(3.4) = \frac{2c\,\mathcal{E}_u(f,f)}{u^\alpha} \leq \frac{2c\,\mathcal{E}(f,f)}{u^\alpha},$$

which in turn is integrable in the vicinity of 0 as long as $\alpha < 1$.

Next, by **(P1)**, one has

$$\lim_{t\to 0} \frac{\eta_t(u)}{t} = \frac{\alpha}{\Gamma(1-\alpha)} \frac{1}{u^{1+\alpha}},$$

and so by Fubini,

$$\begin{aligned}\mathcal{E}^{(\alpha)}(f,f) &= \frac{\alpha}{2\Gamma(1-\alpha)} \int_E\int_E \Big(\int_0^\infty \frac{p(u,x,y)}{u^{1+\alpha}}\,du\Big)(f(x)-f(y))^2\,d\mu(x)\,d\mu(y)\\ &= \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{1}{u^\alpha}\,\mathcal{E}_u(f,f)\,du. \qquad (3.5)\end{aligned}$$

It follows that for any fixed number $a>0$, any $\alpha\in(0,1)$, and any $f\in\mathcal{D}(\mathcal{E})$, as a result of the monotonicity of $\mathcal{E}_u(f,f)$,

$$\begin{aligned}\mathcal{E}^{(\alpha)}(f,f) &\ge \frac{\alpha}{\Gamma(1-\alpha)}\int_0^a \frac{1}{u^\alpha}\,\mathcal{E}_u(f,f)\,du \ge \frac{\alpha}{\Gamma(1-\alpha)}\,\mathcal{E}_a(f,f)\int_0^a\frac{1}{u^\alpha}\,du\\ &= \frac{\alpha}{\Gamma(1-\alpha)}\,\frac{a^{1-\alpha}}{1-\alpha}\,\mathcal{E}_a(f,f).\end{aligned}$$

In particular, we can choose $a=1-\alpha$, which yields the estimate

$$\mathcal{E}_a(f,f) \ge \frac{\alpha}{\Gamma(1-\alpha)}\,\frac{(1-\alpha)^{1-\alpha}}{1-\alpha}\,\mathcal{E}_{1-\alpha}(f,f).$$

Since $\lim_{t\to 0^+} t^t = 1$ and $\lim_{t\to 0^+}\mathcal{E}_t(f,f)=\mathcal{E}(f,f)$, and $\lim_{t\to 0^+} t\,\Gamma(t)=1$, we obtain

$$\liminf_{\alpha\nearrow 1}\mathcal{E}^{(\alpha)}(f,f)\ge \mathcal{E}(f,f). \qquad (3.6)$$

The matching upper bound is simpler: this time around, write the integral (3.5) as

$$\begin{aligned}\mathcal{E}^{(\alpha)}(f,f) &= \frac{\alpha}{\Gamma(1-\alpha)}\Big(\int_0^1\frac{1}{u^\alpha}\,\mathcal{E}_u(f,f)\,du + \int_1^\infty \frac{1}{u^\alpha}\,\mathcal{E}_u(f,f)\,du\Big)\\ &=: \frac{\alpha}{\Gamma(1-\alpha)}\,(I_1+I_2).\end{aligned}$$

These integrals are dealt with separately: I_1 will give the proper asymptotics, and I_2 will be negligible.

More precisely, since $\mathcal{E}_u(f,f)\to\mathcal{E}(f,f)$ and the limit is increasing, one has

$$I_1 \le \mathcal{E}(f,f)\int_0^1 \frac{du}{u^\alpha} = \frac{\mathcal{E}(f,f)}{1-\alpha}.$$

The integral I_2 can be rewritten as

$$I_2 = \int_1^\infty \frac{1}{u^{1+\alpha}}\,\langle f-P_uf,f\rangle\,du,$$

and since the P_u's are contractions, $|\langle f - P_u f, f\rangle| \le 2\|f\|_2^2$. Therefore

$$|I_2| \le 2\|f\|_2^2 \int_1^\infty \frac{du}{u^{1+\alpha}} = \frac{2}{\alpha}\,\|f\|_2^2,$$

and so

$$\mathcal{E}^{(\alpha)}(f,f) \le \frac{\alpha}{\Gamma(1-\alpha)(1-\alpha)}\left(\mathcal{E}(f,f) + \frac{2}{\alpha}\,(1-\alpha)\|f\|_2^2\right),$$

giving

$$\limsup_{\alpha\nearrow 1} \mathcal{E}^{(\alpha)}(f,f) \le \mathcal{E}(f,f). \ \blacksquare$$

As a corollary, we obtain

THEOREM 3.2. *Suppose that $\mathcal{E}$ is the Dirichlet form associated with a fractional diffusion on (E,ϱ,μ) and let $E^{(\alpha)}(f,f)$, $\alpha\in(0,1)$, be defined by* (2.7). *Then for any $f \in \mathrm{Lip}(d_w/2,2,\infty)(E)$ we have*

$$\text{(3.7)}\qquad \frac{1}{C}\,\mathcal{E}(f,f) \le \liminf_{\alpha\nearrow 1}\,(1-\alpha)E^{(\alpha)}(f,f) \le \limsup_{\alpha\nearrow 1}\,(1-\alpha)E^{(\alpha)}(f,f) \le C\,\mathcal{E}(f,f)$$

where the constant C does not depend on f.

Proof. This follows from the representation (3.5) and the transition density estimate (2.2). ■

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