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FUNCTIONS OF A COMPLEX VARIABLE

On Some Correspondence between Holomorphic Functions in the Unit Disc and Holomorphic Functions in the Left Halfplane

by

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Summary. We study a correspondence L between some classes of functions holomorphic in the unit disc and functions holomorphic in the left halfplane. This correspondence is such that for every f and $w \in \mathbb{H}$, $\exp(L(f)(w)) = f(\exp w)$.

In particular, we prove that the famous class S of univalent functions on the unit disc is homeomorphic via L to the class $S(\mathbb{H})$ of all univalent functions g on $\mathbb{H}$ for which $g(w + 2\pi i) = g(w) + 2\pi i$ and $\lim_{\operatorname{Re} z \to -\infty}(g(w) - w) = 0$.

1. Introduction and preliminaries. A usual way to establish a correspondence between holomorphic functions in the left halfplane and those in the unit disc is to take the composition with the fractional linear map $\varphi(z) = \frac{z+1}{z-1}$. Note that $\varphi \circ \varphi = \mathrm{Id}$.

In this note we shall use the fact that the exponential function $\exp(z) = e^z$ is a covering map from the left halfplane onto the punctured unit disc to define another correspondence between some classes of holomorphic functions.

Let us introduce some notations. $\mathbb{D}$ will denote the unit disc and $\mathbb{H}$ will denote the left halfplane $\{z : \operatorname{Re} z < 0\}$. Let ln stand for the branch of the logarithm such that $-\pi < \operatorname{Im} \ln z \leq \pi$ and $\ln 1 = 0$.

We now define some classes of functions holomorphic on $\mathbb{D}$ or $\mathbb{H}$.

1) $S_0 = \{f \in H(\mathbb{D}) : f(0) = 0 \text{ and } f(z) \neq 0 \text{ for } z \neq 0\}$.

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2) S_k consists of the functions from S_0 which have a zero of order k at zero.

Each function from S_k can be written as

$$f(z) = cz^k f_1(z), \quad \text{where } c \neq 0,\ f_1(0) = 1$$

and $f_1(z)$ does not vanish on $\mathbb{D}$. We have $S_0 = \bigcup_{k=1}^{\infty} S_k$.

3) S_k^1 consists of the functions from S_k for which $c = 1$.
4) S contains all functions from S_1^1 which are *univalent* on $\mathbb{D}$. In other words, it is the class of functions f univalent on $\mathbb{D}$ and such that $f(0) = 0$ and $f'(0) = 1$. This is the most important of all the classes considered. Note that $f(z) = z + z^2$ belongs to S_1^1 but not to S.
5) $S_0(\mathbb{H})$ consists of the functions h holomorphic on $\mathbb{H}$ for which there exist $k \in \mathbb{N}$ and $a \in \mathbb{C}$ with $-\pi < \operatorname{Im} a \leq \pi$ such that

 (i) $h(w) - kw$ is a $2\pi i$-periodic function,
 (ii) if $\{w_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{H}$ for which $\lim_{n\to\infty} \operatorname{Re} w_n = -\infty$ then

$$\lim_{n\to\infty} (h(w_n) - kw_n) = a.$$

6) $S_k(\mathbb{H})$ consists of all functions from $S_0(\mathbb{H})$ for which (i) and (ii) hold with the given $k \in \mathbb{N}$.
7) $S_k^0(\mathbb{H})$ consists of all functions from $S_k(\mathbb{H})$ for which $a = 0$.
8) $S(\mathbb{H})$ is the class of *univalent* functions from $S_1^0(\mathbb{H})$.

If $h \in S_k(\mathbb{H})$ then $h(w + 2\pi i) = h(w) + 2k\pi i$.

In particular, if $h \in S_1(\mathbb{H})$ then

$$h(z + 2\pi i) = h(z) + 2\pi i.$$

Note that all sets $S_k(\mathbb{H})$ and $S_k^0(\mathbb{H})$ are *convex*.

2. The correspondence. Let $f \in S_0$. Then $f \in S_k$ for some $k \in \mathbb{N}$, and f can be written as $f(z) = cz^k f_1(z)$, $c \neq 0$, $f_1(0) = 1$. By a monodromy argument there exists g holomorphic on $\mathbb{D}$ with $g(0) = 0$ such that $f_1(z) = e^{g(z)}$. For $w \in \mathbb{H}$ put

$$L(f)(w) := \ln c + kw + g(e^w).$$

It is easy to check that $L(f)(w) \in S_0(\mathbb{H})$. Since e^w is $2\pi i$-periodic, (i) holds. Condition (ii) is satisfied because if $\operatorname{Re} w_n \to -\infty$ then $e^{w_n} \to 0$ and $g(e^{w_n}) \to 0$. We have

$$(*) \qquad e^{L(f)(w)} = f(e^w) \quad \text{for each } w \in \mathbb{H}.$$

THEOREM 1. *For each $k \in \mathbb{N}$ the mapping L is one-to-one and maps S_k* **onto** *$S_k(\mathbb{H})$. It also maps S_k^1 onto $S_k^0(\mathbb{H})$.*

Proof. The fact that L is one-to-one follows directly from $(*)$. If $f_1, f_2 \in S_k$ and $L(f_1) = L(f_2)$ then $f_1(e^w) = f_2(e^w)$ for each $w \in \mathbb{H}$, and so $f_1 = f_2$.

Now let $h \in S_k(\mathbb{H})$. Put $g_1(w) = h(w) - kw - a$. If $z \in \mathbb{D}$, $z \neq 0$, then there exists $w \in \mathbb{H}$ such that $e^w = z$. Define $g(z) = g_1(w)$. Since g_1 is $2\pi i$-periodic, g is well defined on $\mathbb{D} \setminus \{0\}$. We have $g(z) = g_1(\ln z)$ for each branch of $\ln z$. Since $\exp(z) = e^z$ is a covering map, $g(z)$ is holomorphic on $\mathbb{D} \setminus \{0\}$. Condition (ii) implies that $g_1(w) \to 0$ if $\operatorname{Re} w \to -\infty$. Hence $g(z) \to 0$ for $z \to 0$. Thus $g(z)$ extends to a function holomorphic on $\mathbb{D}$ by setting $g(0) = 0$. Now define $f(z) = e^a z^k e^{g(z)}$. Then $f \in S_k$ and $L(f) = h$.

It also follows from the above proof that L maps S_k^1 onto $S_k^0(\mathbb{H})$.

Let now $H(\mathbb{D})$ and $H(\mathbb{H})$ be the spaces of holomorphic functions on $\mathbb{D}$ and $\mathbb{H}$ respectively, endowed with the compact-open topology. We shall consider S_k^1 and $S_k^0(\mathbb{H})$ as topological subspaces of $H(\mathbb{D})$ and $H(\mathbb{H})$ respectively.

We have

Theorem 2. *The mapping L is a homeomorphism between S_k^1 and $S_k^0(\mathbb{H})$ for each $k \in \mathbb{N}$.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}} \subset S_k^1$ converge almost uniformly to $f_0 \in S_k^1$. We have $f_n(z) = z^k e^{g_n(z)}$, $g_n(0) = 0$ for each $n \in \mathbb{N}$ and $f_0(z) = z^k e^{g_0(z)}$, $g_0(0) = 0$. The almost uniform convergence of f_n implies that e^{g_n} converges almost uniformly to e^{g_0}. Since $g_n' = (e^{g_n})'/e^{g_n}$ and $g_0' = (e^{g_0})'/e^{g_0}$ we see that g_n' converges almost uniformly to g_0'. Since $g_n(0) = g_0(0) = 0$ for all n, we have $g_n(z) = \int_0^z g_n'(\xi)d\xi$. Thus g_n converges almost uniformly to g_0. It follows that $L(f_n)(w) = kw + g_n(e^w)$ converges almost uniformly on $\mathbb{H}$ to $L(f_0)(w) = kw + g_0(e^w)$.

Conversely, if $L(f_n)$ converges almost uniformly to $L(f_0)$ on $\mathbb{H}$ then g_n converges almost uniformly to g_0 and hence f_n converges almost uniformly to f_0.

Warning. The mapping L is **not** continuous between S_k and $S_k(\mathbb{H})$ (with compact-open topology) (because of the term $a = \ln c$ in the definition of L).

We can also prove

Proposition 1. *The class S_k^1 is a closed subset of $H(\mathbb{D})$, and $S_k^0(\mathbb{H})$ is a closed subset of $H(\mathbb{H})$.*

Proof. The Hurwitz theorem implies that S_k^1 is a closed subset of $H(\mathbb{D})$. If the sequence $h_n(w) = L(f_n)(w)$ converges almost uniformly on $\mathbb{H}$ then the sequence $g_n(e^w)$ converges almost uniformly on $\mathbb{H}$ since for $f_n = z^k e^{g_n(z)}$ with $g_n(0) = 0$ we have $h_n(w) = kw + g_n(e^w)$. Then g_n converges almost uniformly on $\mathbb{D} \setminus \{0\}$ and therefore on $\mathbb{D}$. It follows that f_n converges on $\mathbb{D}$ to $f_0 \in S_k^1$. We have $L(f_0) = \lim_{n\to\infty} h_n \in S_k^0(\mathbb{H})$.

Remark 1. The assumptions $g(0) = 0$ and $-\pi < \operatorname{Im} a \leq \pi$ were introduced to ensure that L is a one-to-one correspondence between S_0

and $S_0(\mathbb{H})$. If we omit them we obtain a 1-∞ correspondence. For every $f \in S_0$ we shall have a countable family of functions $\{L_m(f)\}_{m\in\mathbb{Z}}$, $L_m(f) = L(f) + 2m\pi i$.

Remark 2. Let $f \in S_k$ and let

$$f_m := \sqrt[m]{f(z^m)} = c^{1/m} \cdot z^k \cdot e^{g(z^m)/m} \qquad \text{for } m \in \mathbb{N}.$$

Then

$$L(f_m)(w) = \frac{\ln c}{m} + kw + \frac{1}{m}\, g(e^{mw}).$$

Remark 3. The correspondence L can be used to construct other classes of holomorphic functions. Let $\varphi(z) = \frac{z+1}{z-1}$. Let $f \in S_1$. We have $L(f)(w + 2\pi i) = L(f)(w) + 2\pi i$. Put $\Lambda(f) = \varphi \circ L(f) \circ \varphi$. The function $\Lambda(f)$ maps $\mathbb{D}$ into the Riemann sphere $\widehat{\mathbb{C}}$ and has the following properties.

1) The nontangential limit of $\Lambda(f)$ at 1 is equal to 1.
2) We have

$$\forall_{k\in\mathbb{Z}} \quad u_k \circ \Lambda(f) = \Lambda(f) \circ u_k, \qquad u_0 = \mathrm{Id},$$
$$u_k(z) = \frac{\overline{a}_k}{a_k} \cdot \frac{z - a_k}{1 - \overline{a}_k z}, \qquad a_k = \frac{k\pi i}{1 + k\pi i} \qquad \text{if } k \neq 0.$$

For $f(z) = ze^{g(z)}$ with $g(0) = 0$,

$$\Lambda(f)(z) = \frac{2z + g(e^{\frac{z+1}{z-1}}) \cdot (z-1)}{2 + g(e^{\frac{z+1}{z-1}}) \cdot (z-1)}.$$

3. The case of univalent functions. We start from

Theorem 3. *Let $f \in S_1$. The function f is univalent iff $L(f)$ is univalent.*

Proof. Let $f = cze^{g(z)} \in S_1$. Assume that f is univalent. Let $L(f)(w_1) = L(f)(w_2)$. Since f is univalent and $e^{L(f)(w)} = f(e^w)$, we see that $e^{w_1} = e^{w_2}$ and $w_1 = w_2 + 2m\pi i$ for some $m \in \mathbb{Z}$.

We have $L(f)(w) = \ln c + w + g(e^w)$. Hence $L(f)(w_1) = L(f)(w_2)$ and $w_1 = w_2 + 2m\pi i$ imply that $m = 0$ and $w_1 = w_2$.

Assume now that $L(f)$ is univalent. Since $f \in S_1$ we have $L(f)(w + 2\pi i) = L(w) + 2\pi i$. Assume that $f(z_1) = f(z_2)$. If it is equal to zero then $z_1 = z_2 = 0$ by the definition of S_1. Hence we can assume that there exist $w_1, w_2 \in \mathbb{H}$ such that $z_1 = e^{w_1}$ and $z_2 = e^{w_2}$. This implies, as before, that $e^{L(f)(w_1)} = e^{L(f)(w_2)}$, so there exists $m \in \mathbb{Z}$ for which $L(f)(w_1) = L(f)(w_2) + 2m\pi i = L(f)(w_2 + 2m\pi i)$. Thus $w_1 = w_2 + 2m\pi i$ and $e^{w_1} = e^{w_2}$. Hence $z_1 = z_2$.

Theorem 3 is not true for S_k with $k > 1$. The function $f(z) = z^k$, $k > 1$, is not univalent but $L(f)(w) = kw$ is univalent.

Proposition 2. *For every $f \in S_k$ there exists $f_1 \in S_1$ such that $f = f_1^k$.*

Proof. For $f(z) = cz^k e^{g(z)}$ take $f_1(z) = c^{1/k} z e^{g(z)/k}$.

Theorem 3 and Proposition 2 yield

THEOREM 3′. *Let $f \in S_k$. The function $L(f)$ is univalent iff $f = f_1^k$ where $f_1 \in S_1$ is univalent.*

Proof. There exists $m \in \mathbb{Z}$ such that

$$L(f) = L(f_1^k) = k \cdot L(f_1) + 2m\pi i.$$

EXAMPLES.

1. Let $f(z) = z + z^2/2$. Then $f \in S_1$ is univalent and hence $L(f)(w) = w + \ln(1 + e^w/2)$ is univalent on $\mathbb{H}$.
2. If $f(z) = z + z^2$ then $L(f)(w) = w + \ln(1 + e^w)$. The function $L(f)$ is not univalent because f is not.

In the rest of this note we shall study the famous class S of univalent functions from S_1^1.

Let us consider S as a subset of $H(\mathbb{D})$ with compact-open topology and $S(\mathbb{H})$ as a subset of $H(\mathbb{H})$ with compact-open topology. Recall that $S(\mathbb{H})$ is the set of univalent functions from $S_1^0(\mathbb{H})$.

The Hurwitz theorem implies that S is closed in $H(\mathbb{D})$. Proposition 1 together with Theorems 1, 2 and 3 shows that $S(\mathbb{H})$ is closed in $H(\mathbb{H})$.

As an immediate consequence of Theorem 3 we have

THEOREM 4. *The mapping L is a homeomorphism from S onto $S(\mathbb{H})$.*

COROLLARY. *The class $S(\mathbb{H})$ is compact in $H(\mathbb{H})$. More generally, for each $k \in \mathbb{N}$ the class $\widetilde{S}_k^0(\mathbb{H})$ consisting of all univalent functions from $S_k^0(\mathbb{H})$ is compact in $H(\mathbb{H})$.*

Proof. The class S is compact since for each $f \in S$ and $z \in \mathbb{D}$, $|f(z)| \leq \sum_{n=1}^{\infty} n|z|^n = |z|/(1-|z|)^2$ (De Branges Theorem). Thus $S(\mathbb{H})$ must also be compact. Moreover, by Theorem 3′, $\widetilde{S}_k^0(\mathbb{H})$ is the continuous image of S under the mapping $f \mapsto L(f^k)$. Hence $\widetilde{S}_k^0(\mathbb{H})$ is compact.

REMARK 4. The class $S_1^0(\mathbb{H})$ is **not** compact since S_1^1 is **not** compact. It contains all functions $f_c(z) = ze^{cz}$, $c \in \mathbb{C}$. The set of values $f_c''(0) = 2c$ is not bounded.

We now consider two important definitions:

DEFINITION 1. A univalent function $f \in H(\mathbb{D})$ with $f(0) = 0$ is *starlike* iff $f(\mathbb{D})$ is a domain starlike with respect to zero.

DEFINITION 2. A univalent function $f \in H(\mathbb{D})$ with $f(0) = 0$ is *convex* iff $f(\mathbb{D})$ is a convex domain.

The following facts are well known and can be found in [D] or [P].

THEOREM A. *A function f is starlike iff* $\operatorname{Re}\{zf'(z)/f(z)\} > 0$.

THEOREM B. *A function f is convex iff the function $f_1(z) = zf'(z)$ is starlike.* (One assumes here that $f \in H(\mathbb{D})$ and $f(0) = 0$.)

THEOREM 5.

(1) *$f \in S$ is starlike iff* $\operatorname{Re}(L(f))'(w) > 0$ *for all $w \in \mathbb{H}$.*
(2) *$f \in S$ is convex iff*

$$\forall_{w\in\mathbb{H}} \quad \operatorname{Re}\biggl((L(f))'(w) + \frac{(L(f))''(w)}{(L(f))'(w)}\biggr) > 0.$$

Proof. (1) Let $f(z) = ze^{g(z)}$. We have

$$\frac{zf'(z)}{f(z)} = \frac{z(e^{g(z)} + ze^{g(z)}g'(z))}{ze^{g(z)}} = 1 + zg'(z),$$

$L(f)(w) = w + g(e^w)$ and $(L(f))'(w) = 1 + g'(e^w)e^w = 1 + zg'(z)$ for $z = e^w$. If $z = 0$ then $zf'(z)/f(z) = 1$. Hence and by Theorem A, f is starlike iff $\operatorname{Re}(L(f))'(w) > 0$.

(2) By Theorem B, $f(z) = ze^{g(z)}$ is convex iff $f_1(z) = zf'(z) = z(e^{g(z)} + ze^{g(z)}g'(z)) = ze^{g(z)+\ln(1+zg'(z))}$ is starlike. Note that $\operatorname{Re}(1 + zg'(z)) > 0$ for $z \in \mathbb{D}$.

Hence and by the first part of Theorem 5, f is convex iff $\operatorname{Re}(L(f_1))'(w) > 0$ for each $w \in \mathbb{H}$. We have

$$L(f_1) = L(f) + \ln((L(f))'(w))$$

since $L(f_1)(w) = w + g(e^w) + \ln(1 + e^w g'(e^w))$. We obtain

$$(L(f_1))' = (L(f))' + \frac{(L(f))''}{(L(f))'}.$$

Thus f is convex iff

$$\forall_{w\in\mathbb{H}} \quad \operatorname{Re}(L(f_1))'(w) = \operatorname{Re}\biggl((L(f))'(w) + \frac{(L(f))''(w)}{(L(f))'(w)}\biggr) > 0. \ \blacksquare$$

REMARK. We thank the referee for pointing out that the operator very similar to our L was used in Krzyż's paper [K, proof of Theorem 1] to study quasiconformal automorphisms of the unit disc. Very recently Chéritat [Ch] used Krzyż's operator to construct a holomorphic function with a strange Siegel disc.

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