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POTENTIAL THEORY

On Polynomially Bounded Harmonic Functions on the $\mathbb{Z}^d$ Lattice

by

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Summary. We prove that if $f : \mathbb{Z}^d \to \mathbb{R}$ is harmonic and there exists a polynomial $W : \mathbb{Z}^d \to \mathbb{R}$ such that $f + W$ is nonnegative, then f is a polynomial.

1. Introduction. Harmonic functions on the integer lattice are closely related to lattice random walks and have been studied by many authors; an introduction and detailed references can be found in a modern monograph by Woess [8]. Many different methods have been successfully applied, including the extreme point theory [2] and martingale approach [4]. The present paper grew out of the author's bachelor thesis [7] which extended results and methods of Darkiewicz [3]. A similar result for sublinear functions on compactly generated groups of polynomial growth has been obtained by Hebisch and Saloff-Coste [6, Theorem 6.1] by using Gaussian estimates for iterated kernels of random walks.

2. Preliminaries and main results. Let $d \in \mathbb{N}$ and let $(e_i)_{i=1}^d$ be the standard orthonormal basis for $\mathbb{R}^d$. A function $f : \mathbb{Z}^d \to \mathbb{R}$ is called *harmonic* if it has the mean value property,

$$f(x) = \frac{1}{2d} \sum_{i=1}^{d} [f(x + e_i) + f(x - e_i)] \quad \text{for all } x \in \mathbb{Z}^d.$$

We say that $f : \mathbb{Z}^d \to \mathbb{R}$ is a *polynomial* if there exists a polynomial $F : \mathbb{R}^d \to \mathbb{R}$ such that $f = F|_{\mathbb{Z}^d}$.

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For $t \geq 0$ let $Y_1^{(t)}, \ldots, Y_d^{(t)}, Z_1^{(t)}, \ldots, Z_d^{(t)}$ be independent Poisson random variables with mean t.

We will use the following notation:

- $\|x\|_p = (\sum_{i=1}^d |x_i|^p)^{1/p}$ for $p \in [1, \infty)$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$,
- $X_i^{(t)} = Y_i^{(t)} - Z_i^{(t)}$ for $i = 1, \ldots, d$, $X^{(t)} = \sum_{i=1}^d X_i^{(t)} e_i$,
- $g_t(l) = \mathbb{P}(Y_1^{(t)} - Z_1^{(t)} = l)$ for $l \in \mathbb{Z}$,
- $G_t(k) = \prod_{i=1}^m g_t(k_i)$ for $k = (k_1, \ldots, k_m) \in \mathbb{Z}^m$,
- $q_t(l) = \mathbb{P}(Y_1^{(t)} = l) = e^{-t} t^l / l!$ for $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Note that if $t \in \mathbb{N}$ then $q_t(0) \leq q_t(1) \leq \cdots \leq q_t(t-1) = q_t(t) \geq q_t(t+1) \geq q_t(t+2) \geq \cdots$.

We consider the space of all exponentially bounded functions,

$$\mathcal{L} = \{f : \mathbb{Z}^d \to \mathbb{R} \mid \exists_{c_1, c_2 > 0} \ |f(x)| \leq c_1 e^{c_2 \|x\|_1} \text{ for all } x \in \mathbb{Z}^d\},$$

and define a family of operators $(\mathcal{P}_t)_{t \geq 0}$, $\mathcal{P}_t : \mathcal{L} \to \mathcal{L}$, by

$$\mathcal{P}_t(f)(x) = \mathbb{E} f(x + X^{(t)}).$$

THEOREM 2.1. *The family $(\mathcal{P}_t)_{t \geq 0}$ is a well-defined semigroup of operators. Moreover, harmonic functions belonging to $\mathcal{L}$ lie in the domain $\mathcal{D}_A$ of the infinitesimal generator A of the semigroup $(\mathcal{P}_t)_{t \geq 0}$, and for $f \in \mathcal{D}_A$ we have*

$$(Af)(x) = \frac{d}{dt} \mathcal{P}_t(f)(x) \bigg|_{t=0} = \sum_{k \in \mathbb{Z}^d : \|k\|_1 = 1} f(x + k) - 2d f(x).$$

In particular, if $f \in \mathcal{L}$ is harmonic, then $(Af)(x) = 0$ for all $x \in \mathbb{Z}^d$, and so

$$\mathcal{P}_t(f)(x) = \sum_{k \in \mathbb{Z}^d} G_t(k) f(x + k) = f(x) \quad \text{for all } x \in \mathbb{Z}^d.$$

Proof. If $f \in \mathcal{L}$, then there exist $c_1, c_2, \tilde{c}_1(t) > 0$ such that

$$|\mathbb{E} f(x + X^{(t)})| \leq c_1 \mathbb{E} e^{c_2 \|x + X^{(t)}\|_1} \leq c_1 e^{c_2 \|x\|_1} (\mathbb{E} e^{c_2 |X_1^{(t)}|})^d = \tilde{c}_1(t) e^{c_2 \|x\|_1},$$

so $\mathcal{P}_t(f) \in \mathcal{L}$. Observe that $\mathcal{P}_0(f) = f$. If $s, t \geq 0$ and $\tilde{X}^{(s)}$ is a copy of $X^{(s)}$ independent of $X^{(t)}$, then $X^{(t)} + \tilde{X}^{(s)} \sim X^{(t+s)}$, so one can easily check that $(\mathcal{P}_t)_{t \geq 0}$ is a semigroup. The last part is a simple calculation. ∎

LEMMA 2.2. *If $(r_i)_{i \in \mathbb{N}}$ are independent ± 1 symmetric Bernoulli random variables and M is a Poisson variable with mean $4t$, such that M and $(r_i)_{i \in \mathbb{N}}$ are independent, then*

$$X_1^{(t)} \sim \frac{1}{2} (r_1 + \cdots + r_{2M}).$$

Moreover, for $l \in \mathbb{N}_0$,

$$g_t(l) = g_t(-l) = \sum_{n=0}^{\infty} e^{-4t} \frac{t^n}{n!} \binom{2n}{n+l},$$

so if $l_1, l_2 \in \mathbb{Z}$ *and* $0 \le l_1 \le l_2$, *then*

$$g_t(l_1) \ge g_t(l_2).$$

Proof. To prove the first statement, it is enough to show that the characteristic functions of both random variables are equal. We have

$$\begin{aligned}\phi_{X_1^{(t)}}(x) &= \phi_{Y_1^{(t)}}(x)\phi_{Z_1^{(t)}}(-x) = e^{t(e^{ix}-1)}e^{t(e^{-ix}-1)} = e^{t(2\cos x - 2)} \\ &= e^{-4t\sin^2(x/2)}\end{aligned}$$

and

$$\begin{aligned}\phi_{(r_1+\dots+r_{2M})/2}(x) &= \sum_{n=0}^{\infty} \mathbb{P}(M=n)\phi_{(r_1+\dots+r_{2n})/2}(x) \\ &= \sum_{n=0}^{\infty} e^{-4t} \frac{(4t)^n}{n!} (\phi_{r_1/2}(x))^{2n} = e^{-4t}e^{4t(\phi_{r_1/2}(x))^2} \\ &= e^{4t(-1+\cos^2(x/2))} = e^{-4t\sin^2(x/2)},\end{aligned}$$

as

$$\phi_{r_1/2}(x) = \phi_{r_1}(x/2) = \frac{1}{2}(e^{-ix/2} + e^{ix/2}) = \cos(x/2).$$

To finish the proof observe that for $l \in \mathbb{N}_0$ we have

$$\begin{aligned}g_t(l) &= \mathbb{P}\bigg(\frac{1}{2}(r_1 + \dots + r_{2M}) = l\bigg) = \sum_{n=0}^{\infty} \mathbb{P}(M=n)\mathbb{P}(r_1 + \dots + r_{2n} = 2l) \\ &= \sum_{n=0}^{\infty} e^{-4t} \frac{(4t)^n}{n!} \frac{1}{2^{2n}} \binom{2n}{n+l} = \sum_{n=0}^{\infty} e^{-4t} \frac{t^n}{n!} \binom{2n}{n+l}\end{aligned}$$

and $\binom{2n}{n+l_1} \ge \binom{2n}{n+l_2}$ for $0 \le l_1 \le l_2$. ■

LEMMA 2.3. *For every* $\varepsilon > 0$ *and* $d \in \mathbb{N}$ *we can find* $0 < s < t$ *such that*

$$g_t(k) \ge (1-\varepsilon)g_s(k-1) \quad \text{for } k \in \mathbb{Z}$$

and

$$G_t(k) \ge (1-\varepsilon)G_s(k-e_1) \quad \text{for } k \in \mathbb{Z}^d.$$

Proof. If the first inequality holds for $k = 1, 2, \dots, m$ then it holds for $k = 0, -1, \dots, -m$. Indeed, for $k = -1, -2, \dots, -m$ we have (see Lemma 2.2)

$$\begin{aligned}\mathbb{P}(X_1^{(t)} = k) &= \mathbb{P}(X_1^{(t)} = -k) \geq (1-\varepsilon)\mathbb{P}(X_1^{(s)} = -k-1) \\ &= (1-\varepsilon)\mathbb{P}(X_1^{(s)} = k+1) \geq (1-\varepsilon)\mathbb{P}(X_1^{(s)} = k-1)\end{aligned}$$

and

$$\mathbb{P}(X_1^{(t)} = 0) \geq \mathbb{P}(X_1^{(t)} = 1) \geq (1-\varepsilon)\mathbb{P}(X_1^{(s)} = 0) \geq (1-\varepsilon)\mathbb{P}(X_1^{(s)} = -1).$$

For $k \geq 1$ we have

$$\mathbb{P}(X_t = k) = \sum_{l=0}^{\infty} \mathbb{P}(Y_t = l+k)\mathbb{P}(Z_t = l) = \sum_{l=0}^{\infty} e^{-2t} \frac{t^{2l+k}}{l!(l+k)!},$$

$$\mathbb{P}(X_s = k-1) = \sum_{l=0}^{\infty} e^{-2s} \frac{s^{2l+k-1}}{l!(l+k-1)!}.$$

Let $s > 1$ be such that $\sqrt{s} \in \mathbb{N}$ and set $t = s + \sqrt{s}$. We then have

$$\mathbb{P}(X_t = k) \geq \sum_{l=\sqrt{s}}^{\infty} e^{-2t} \frac{t^{2l+k}}{l!(l+k)!} = \sum_{l=0}^{\infty} e^{-2t} \frac{t^{2(l+\sqrt{s})+k}}{(l+\sqrt{s})!(l+\sqrt{s}+k)!}.$$

It is enough to prove that

$$\inf_{k\geq 1,\, l\geq 0} \left(e^{-2t} \frac{t^{2(l+\sqrt{s})+k}}{(l+\sqrt{s})!(l+\sqrt{s}+k)!} \Big/ e^{-2s} \frac{s^{2l+k-1}}{l!(l+k-1)!} \right) \xrightarrow[s\to\infty]{} 1.$$

We consider the expression

$$p_{l,k}(s) := e^{2(s-t)} s t^{2\sqrt{s}} \left(\frac{t}{s}\right)^{l+k} \frac{(l+k-1)!}{(l+\sqrt{s}+k)!} \left(\frac{t}{s}\right)^{l} \frac{l!}{(l+\sqrt{s})!}.$$

The function $\mathbb{N} \ni n \mapsto (t/s)^n (n-1)!/(n+\sqrt{s})!$ has its minimum at $n = s(1+\sqrt{s})/(t-s) = t$. Similarly, the function $\mathbb{N}_0 \ni n \mapsto (t/s)^n n!/(n+\sqrt{s})!$ has its minimum at $n = s\sqrt{s}/(t-s) = s$. Therefore

$$\begin{aligned}p_{l,k}(s) \geq p_{s,t-s}(s) &= e^{2(s-t)} s t^{2\sqrt{s}} \left(\frac{t}{s}\right)^{t+s} \frac{(t-1)!}{(t+\sqrt{s})!} \frac{s!}{t!} \\ &= e^{-2\sqrt{s}} s (s+\sqrt{s})^{2\sqrt{s}} \left(\frac{s+\sqrt{s}}{s}\right)^{2s+\sqrt{s}} \frac{s!}{(s+2\sqrt{s})!} \frac{1}{s+\sqrt{s}}.\end{aligned}$$

Using Stirling's formula we get $s!/(s+2\sqrt{s})! \approx e^{2\sqrt{s}} s^s/(s+2\sqrt{s})^{s+2\sqrt{s}}$ as $s \to \infty$, hence we arrive at

$$\begin{aligned}&\inf_{k\geq 1,\, l\geq 0} p_{l,k}(s) \approx s^{-s-\sqrt{s}+1} (s+\sqrt{s})^{2s+3\sqrt{s}-1} (s+2\sqrt{s})^{-s-2\sqrt{s}} \\ &= \sqrt{s}^{-2s-2\sqrt{s}+2+2s+3\sqrt{s}-1} (1+\sqrt{s})^{-\sqrt{s}-1} (1+\sqrt{s})^{2s+4\sqrt{s}} (s+2\sqrt{s})^{-s-2\sqrt{s}} \\ &= \left(\frac{\sqrt{s}}{1+\sqrt{s}}\right)^{\sqrt{s}+1} \left(\frac{s+2\sqrt{s}+1}{s+2\sqrt{s}}\right)^{s+2\sqrt{s}} \xrightarrow[s\to\infty]{} e^{-1}e = 1.\end{aligned}$$

To prove the second part observe that the first inequality yields

$$\begin{aligned} G_t(k) &= g_t(k_1)\dots g_t(k_d) \ge (1-\varepsilon)g_s(k_1-1)g_t(k_2)\dots g_t(k_d) \\ &\ge (1-\varepsilon)^d G_s(k-e_1), \end{aligned}$$

since

$$g_t(l) = g_t(|l|) \ge g_t(|l|+1) \ge (1-\varepsilon)g_s(|l|) = (1-\varepsilon)g_s(l). \ ■$$

A sequence $(x_i)_{i=0}^n \subset \mathbb{Z}^d$ is called a *path* in $\mathbb{Z}^d$ between x_0 and x_n if $\|x_i - x_{i+1}\|_1 = 1$ for $i = 0, \dots, n-1$. For $k \in \mathbb{Z}^d$ let $L_n(k)$ denote the number of paths in $\mathbb{Z}^d$ between 0 and k.

LEMMA 2.4. *Let $f : \mathbb{Z}^d \to \mathbb{R}$ be harmonic. Suppose there exists a polynomial $W : \mathbb{Z}^d \to \mathbb{R}$ such that $f(x) \ge -W(x)$. Then $f \in \mathcal{L}$.*

Proof. Using simple induction we can prove that for f harmonic and $n \in \mathbb{N}$ we have

$$f(0) = \frac{1}{(2d)^n} \sum_{k \in \mathbb{Z}^d} f(k) L_n(k).$$

Let $l \in \mathbb{Z}^d$. Then $L_{\|l\|_1}(l) \ge 1$ and

$$\begin{aligned} f(0)(2d)^{\|l\|_1} &= \sum_{k\in\mathbb{Z}^d} (f(k) + W(k)) L_{\|l\|_1}(k) - \sum_{k \in \mathbb{Z}^d} W(k) L_{\|l\|_1}(k) \\ &\ge (f(l) + W(l)) - \max_{k:\, \|k\|_1 \le \|l\|_1} |W(k)| \cdot (2d)^{\|l\|_1}, \end{aligned}$$

hence

$$f(l) \le f(0)(2d)^{\|l\|_1} + (2d)^{\|l\|_1} \max_{k:\, \|k\|_1 \le \|l\|_1} |W(k)| - W(l) \le c_1 e^{c_2 \|l\|_1}$$

for some $c_1, c_2 > 0$ which depend only on f and W but not on l. Since f is polynomially bounded from below we have $f \in \mathcal{L}$. ■

Now we may recover the classical strong Liouvillle property of harmonic functions on $\mathbb{Z}^d$. Woess [8] traces back its weak form to Blackwell [1]; see also [2] and [5].

THEOREM 2.5. *If $f : \mathbb{Z}^d \to \mathbb{R}$ is harmonic and $f \ge 0$ then f is constant.*

Proof. By Lemma 2.4 we have $f \in \mathcal{L}$. Let $x \in \mathbb{Z}^d$. Lemma 2.3 implies that there exist $t > s > 0$ such that

$$\begin{aligned}
f(x)-f(x+e_1) &= P_t(f)(x)-P_s(f)(x+e_1)\\
&= \sum_{k\in\mathbb{Z}^d} f(x+k)G_t(k) - \sum_{k\in\mathbb{Z}^d} f(x+k+e_1)G_s(k)\\
&= \sum_{k\in\mathbb{Z}^d} f(x+k)(G_t(k)-G_s(k-e_1))\\
&\ge -\varepsilon \sum_{k\in\mathbb{Z}^d} f(x+k)G_s(k-e_1) = -\varepsilon f(x+e_1).
\end{aligned}$$

By letting $\varepsilon \to 0$ we get $f(x) \ge f(x+e_1)$. Applying this inequality to the harmonic function $x \mapsto g(x) = f(-x)$ we get $f(x) = f(x+e_1)$ and similarly $f(x) = f(x+e_i)$ for $i = 1, \ldots, d$. ■

We will now prove some auxiliary lemmas.

LEMMA 2.6. *Let $n \in \mathbb{N}$ and let $k \in \mathbb{Z}$ satisfy $|k| \le n$. Then*

$$\frac{1}{2\sqrt{n}}\left(1-\frac{k^2}{n}\right) \le \frac{1}{2^{2n}}\binom{2n}{n+k} \le \frac{1}{\sqrt{2n+1}}\, e^{-\frac{k^2}{2n}} \le \frac{1}{\sqrt{n+1}}\, e^{-\frac{k^2}{2n}}.$$

Proof. We can assume $k \ge 0$. By multiplying the obvious inequalities $(2j-1)^2 \ge 2j(2j-2)$ for $j = 2, 3, \ldots, n$ and $(2j)^2 \ge (2j-1)(2j+1)$ for $j = 1, 2, \ldots, n$ we arrive at $((2n-1)!!)^2 \ge \frac{1}{2}(2n)!!(2n-2)!!$ and $((2n)!!)^2 \ge (2n-1)!!(2n+1)!!$, so that

$$\frac{1}{4n} \le \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \le \frac{1}{2n+1}.$$

To finish the proof it suffices to observe that

$$\frac{1}{2^{2n}}\binom{2n}{n+k} = \frac{(2n-1)!!}{(2n)!!} \cdot \prod_{j=1}^{k}\left(1-\frac{k}{n+j}\right)$$

and

$$1-\frac{k^2}{n} \le \left(1-\frac{k}{n}\right)^k \le \prod_{j=1}^{k}\left(1-\frac{k}{n+j}\right) \le \left(1-\frac{k}{2n}\right)^k \le e^{-\frac{k^2}{2n}}. \ ■$$

LEMMA 2.7. *There exists a constant $C > 0$ such that for $k \in \mathbb{Z}^d \setminus \{0\}$,*

$$G_{\|k\|_1^2}(k) \ge C^d \|k\|_1^{-2d}.$$

Proof. Let $t > 0$ and $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$. We have (see Lemma 2.2)

$$g_t(k_i) \ge e^{-4t}\,\frac{t^n}{n!}\binom{2n}{n+k_i} \ge e^{-4t}\,\frac{t^n}{n!}\binom{2n}{n+\|k\|_1} \qquad (i = 1, \ldots, d,\ n \in \mathbb{N}).$$

We set $t = \|k\|_1^2$ and $n = 4t$. Then $e^{-4t}t^n = e^{-n}n^n/4^n$, so that

$$g_t(k_i) \ge q_n(n) \cdot \frac{1}{2^{2n}}\binom{2n}{n+\|k\|_1} \ge q_n(n) \cdot \frac{1}{2\sqrt{n}}\left(1-\frac{\|k\|_1^2}{n}\right) = \frac{3}{16}\, q_n(n)/\|k\|_1,$$

where we have used Lemma 2.6. Note that by Chebyshev's inequality,

$$\mathbb{P}(|Y_1^{(n)} - n| \geq 2\sqrt{n}) = \mathbb{P}(|Y_1^{(n)} - \mathbb{E}Y_1^{(n)}| \geq 2\sqrt{n}) \leq \frac{D^2 Y_1^{(n)}}{4n} = 1/4,$$

so that

$$\begin{aligned} 3/4 \leq \mathbb{P}(|Y_1^{(n)} - n| < 2\sqrt{n}) &= \sum_{m\in\mathbb{N}_0:\, |m-n|<2\sqrt{n}} q_n(m) \\ &\leq \operatorname{card}\{m \in \mathbb{N}_0 : |m-n| < 2\sqrt{n}\} \cdot q_n(n) \leq 8\|k\|_1 \cdot q_n(n). \end{aligned}$$

Hence

$$g_t(k_i) \geq \frac{3}{32\|k\|_1} \cdot \frac{3}{16\|k\|_1} = \frac{C}{\|k\|_1^2}$$

and therefore

$$G_{\|k\|_1^2}(k) = \prod_{i=1}^{d} g_t(k_i) \geq C^d \|k\|_1^{-2d}. \blacksquare$$

LEMMA 2.8. *Let $W : \mathbb{R}^d \to \mathbb{R}$ be a polynomial. Define $H_W : \mathbb{R} \to \mathbb{R}$ by*

$$H_W(t) = \mathcal{P}_t(W)(0) = \sum_{k\in\mathbb{Z}^d} G_t(k)W(k).$$

Then H_W is a polynomial.

Proof. H_W is well defined since $W|_{\mathbb{Z}^d} \in \mathcal{L}$. Because of the product structure of G_t it is enough to consider the case $d = 1$ and $W(z) = z^l$ for $l \in \mathbb{N}$. The characteristic function

$$\phi_{X_1^{(t)}}(z) = e^{-4t\sin^2(z/2)}$$

is smooth, so that

$$H_W(t) = \mathbb{E}[(X_1^{(t)})^l] = (-i)^l \frac{d^l \phi_{X_1^{(t)}}}{dz^l}(0),$$

which is clearly a polynomial in t. $\blacksquare$

LEMMA 2.9. *Let $f : \mathbb{Z}^d \to \mathbb{R}$ be harmonic. Suppose there exists a polynomial $W : \mathbb{Z}^d \to \mathbb{R}$ such that $f \geq -W$. Then $|f| \leq R$ for some polynomial $R : \mathbb{Z}^d \to \mathbb{R}$.*

Proof. We have $f \in \mathcal{L}$ (see Lemma 2.4). Proposition 2.1 yields

$$f(0) = \sum_{k\in\mathbb{Z}^d} G_t(k)f(k),$$

hence for all $l \in \mathbb{Z}^d$,

$$\begin{aligned} f(0) &= \sum_{k\in\mathbb{Z}^d} G_t(k)(f(k) + W(k)) - \sum_{k\in\mathbb{Z}^d} G_t(k)W(k) \\ &\geq G_t(l)(f(l) + W(l)) - H_W(t). \end{aligned}$$

Therefore

$$f(0) + H_W(t) \geq G_t(l)(f(l) + W(l)).$$

There exists a constant $c = c(d) > 0$ such that (see Lemma 2.7) for all $l \neq 0$,

$$G_{\|l\|_1^2}(l) \geq c\|l\|_1^{-2d}.$$

Hence for $l \neq 0$,

$$f(0) + H_W(\|l\|_1^2) \geq c(f(l) + W(l))\|l\|_1^{-2d}$$

and therefore

$$f(l) \leq c^{-1}\|l\|_1^{2d}\left(f(0) + H_W(\|l\|_1^2)\right) - W(l).$$

Since the right-hand side is polynomially bounded from above in l, we have $f(l) \leq P(l)$ for some polynomial $P : \mathbb{R}^d \to \mathbb{R}$ and all $l \in \mathbb{Z}^d$. One can easily check that $|f(l)| \leq 1 + [P(l)]^2 + [W(l)]^2$. ∎

Lemma 2.10. *For all $x \in \mathbb{Z}$, $n \in \mathbb{N}$, $a, b \in \mathbb{R}$ and $p \geq 0$ we have*

$$|a + b|^p \leq 2^p(|a|^p + |b|^p)$$

and

$$\left||x|^n - |x+1|^n\right| \leq 1 + 2^n|x|^{n-1}.$$

Proof. Without loss of generality we may assume that $|a| \leq |b|$. Then

$$|a + b|^p \leq (2|b|)^p \leq 2^p(|a|^p + |b|^p).$$

To prove the second inequality note that

$$\left||(x+1)^n| - |x^n|\right| \leq |(x+1)^n - x^n| = \left|\sum_{k=0}^{n-1}\binom{n}{k}x^k\right| \leq 1 + \sum_{k=1}^{n-1}\binom{n}{k}|x|^{n-1}$$
$$\leq 1 + 2^n|x|^{n-1}. \blacksquare$$

Lemma 2.11. *If $t > 0$ then*

$$g_t(0) \leq \frac{1}{2\sqrt{t}}$$

and

$$\mathbb{E}|X_1^{(t)}|^m \leq b(m)t^{m/2} + c(m)$$

for some constants $b(m), c(m) > 0$ and $m \in \mathbb{N}$.

Proof. Let M be the Poisson variable with mean $4t$. By Lemma 2.2, Lemma 2.6 and Jensen's inequality we have

$$g_t(0) = \sum_{n=0}^{\infty} e^{-4t}\frac{(4t)^n}{n!}\frac{1}{2^{2n}}\binom{2n}{n} \leq \sum_{n=0}^{\infty} e^{-4t}\frac{(4t)^n}{n!}\frac{1}{\sqrt{n+1}} = \mathbb{E}\,\frac{1}{\sqrt{M+1}}$$
$$\leq \left(\mathbb{E}\,\frac{1}{M+1}\right)^{1/2}$$

and

$$\mathbb{E}\,\frac{1}{M+1} = \sum_{n=0}^{\infty} e^{-4t}\,\frac{(4t)^n}{(n+1)!} = \frac{1}{4t}\sum_{n=0}^{\infty} e^{-4t}\,\frac{(4t)^{n+1}}{(n+1)!} \le \frac{1}{4t}.$$

To prove the second part, let M, r_1, $r_2, \ldots$ be as in Lemma 2.2. For fixed $k \in \mathbb{N}$ and all $i \le k$ we have $\mathbb{E}e^{r_i/\sqrt{k}} = 1 + \sum_{s=1}^{\infty} k^{-s}/(2s)! \le 1 + ek^{-1} \le e^{e/k}$, so that

$$\frac{1}{m!}\,\mathbb{E}\biggl(\frac{r_1+\cdots+r_k}{\sqrt{k}}\biggr)_+^m \le \mathbb{E}\exp\biggl(\frac{r_1+\cdots+r_k}{\sqrt{k}}\biggr) = \prod_{i=1}^{k}\mathbb{E}e^{r_i/\sqrt{k}} \le e^e.$$

Hence

$$\mathbb{E}|r_1+\cdots+r_k|^m = 2\mathbb{E}(r_1+\cdots+r_k)_+^m \le 2e^e m!\cdot k^{m/2}$$

and therefore, by Lemma 2.2,

$$\mathbb{E}|X_1^{(t)}|^m \le 2e^e m!\cdot 2^{-m}\cdot \mathbb{E}(2M)^{m/2} \le 2e^e m!\cdot (\mathbb{E}M^m)^{1/2}.$$

Now,

$$\begin{aligned}\mathbb{E}M^m &= \mathbb{E}M^m I_{M<m} + \mathbb{E}M^m I_{M\ge m} \le m^m + m^m\mathbb{E}(M-m+1)^m\\ &\le m^m\biggl(1+\sum_{k=m}^{\infty} e^{-4t}\,\frac{(4t)^k}{k!}\,k(k-1)\ldots(k-m+1)\biggr)\\ &= m^m(1+(4t)^m)\end{aligned}$$

and it is obvious (see Lemma 2.10) that

$$\mathbb{E}|X_1^{(t)}|^m \le b(m)t^{m/2} + c(m)$$

for some constants $b(m), c(m) > 0$. ■

Now we state the key lemma of this paper. Similar estimates for sublinear harmonic functions have been obtained in a more general setting in [6, Theorem 6.1].

Lemma 2.12. *Let $n \in \mathbb{N}$ and let $f : \mathbb{Z}^d \to \mathbb{R}$ be harmonic. Suppose that there exists a constant a_n such that*

$$|f(x)| \le a_n(1+\|x\|_n^n)$$

for all $x \in \mathbb{Z}^d$. Then there exists a constant a_{n-1} such that for all $x \in \mathbb{Z}^d$,

$$|f(x+e_1)-f(x)| \le a_{n-1}(1+\|x\|_{n-1}^{n-1}).$$

Proof. For $x \in \mathbb{Z}^d$ and any $t > 0$ we have

$$f(x) = \sum_{k\in\mathbb{Z}^d} G_t(k)f(x+k)$$

and

$$f(x+e_1) = \sum_{k\in\mathbb{Z}^d} G_t(k)f(x+e_1+k) = \sum_{k\in\mathbb{Z}^d} G_t(k-e_1)f(x+k),$$

hence

$$
\begin{aligned}
|f(x+e_1)-f(x)| &\leq \sum_{k\in\mathbb{Z}^d} |G_t(k-e_1)-G_t(k)|\,|f(x+k)| \\
&\leq \sum_{k\in\mathbb{Z}^d} |G_t(k-e_1)-G_t(k)| a_n(1+\|x+k\|_n^n) \\
&= \sum_{k\in\mathbb{Z}^d\,:\,k_1\leq 0} (G_t(k)-G_t(k-e_1)) a_n(1+\|x+k\|_n^n) \\
&\quad + \sum_{k\in\mathbb{Z}^d\,:\,k_1>0} (G_t(k-e_1)-G_t(k)) a_n(1+\|x+k\|_n^n) \\
&= \sum_{k\in\mathbb{Z}^d\,:\,k_1\leq -1} G_t(k) a_n(|x_1+k_1|^n-|x_1+k_1+1|^n) \\
&\quad + \sum_{k\in\mathbb{Z}^d\,:\,k_1\geq 1} G_t(k) a_n(|x_1+k_1+1|^n-|x_1+k_1|^n) \\
&\quad + \sum_{k\in\{0\}\times\mathbb{Z}^{d-1}} G_t(k) a_n(1+\|x+k\|_n^n) \\
&\quad + \sum_{k\in\{0\}\times\mathbb{Z}^{d-1}} G_t(k) a_n(1+\|x+k+e_1\|_n^n).
\end{aligned}
$$

We have used the product structure of G_t and Lemma 2.2. By using Lemma 2.10 we get

$$
\begin{aligned}
&\sum_{k_1\leq -1,\,k\in\mathbb{Z}^d} G_t(k)(|x_1+k_1|^n-|x_1+k_1+1|^n) \\
&\qquad\qquad + \sum_{k_1\geq 1, k\in\mathbb{Z}^d} G_t(k)(|x_1+k_1+1|^n-|x_1+k_1|^n) \\
&\quad \leq \sum_{k\in\mathbb{Z}^d} G_t(k)(2^n|x_1+k_1|^{n-1}+1) = 1+2^n\sum_{k_1\in\mathbb{Z}} g_t(k_1)|x_1+k_1|^{n-1} \\
&\quad \leq 1+2^{2n-1}\sum_{k_1\in\mathbb{Z}} g_t(k_1)(|x_1|^{n-1}+|k_1|^{n-1}) \\
&\quad = 1+2^{2n-1}(|x_1|^{n-1}+\mathbb{E}|X_1^{(t)}|^{n-1}).
\end{aligned}
$$

We also have, again by using Lemma 2.10 several times,

$$
\begin{aligned}
&\sum_{k\in\{0\}\times\mathbb{Z}^{d-1}} G_t(k)(1+\|x+k\|_n^n) + \sum_{k\in\{0\}\times\mathbb{Z}^{d-1}} G_t(k)(1+\|x+k+e_1\|_n^n) \\
&\qquad \leq \sum_{k\in\{0\}\times\mathbb{Z}^{d-1}} G_t(k)(2+2^n\|x\|_n^n+2^n\|x+e_1\|_n^n+2^{n+1}\|k\|_n^n) \\
&\qquad \leq g_t(0)(2+2^n\|x\|_n^n+2^n\|x+e_1\|_n^n+d\,2^{n+1}\mathbb{E}|X_1^{(t)}|^n) \\
&\qquad \leq 4^{n+1}g_t(0)(1+\|x\|_n^n+d\,\mathbb{E}|X_1^{(t)}|^n),
\end{aligned}
$$

so we arrive at

$$\begin{aligned}
&|f(x+e_1)-f(x)| \\
&\quad \le a_n[1+2^{2n-1}(|x_1|^{n-1}+\mathbb{E}|X_1^{(t)}|^{n-1})+4^{n+1}g_t(0)(1+\|x\|_n^n+d\,\mathbb{E}|X_1^{(t)}|^n)] \\
&\quad \le 4^{n+2}a_n d[(1+\|x\|_{n-1}^{n-1}+\mathbb{E}|X_1^{(t)}|^{n-1})+g_t(0)(\|x\|_n^n+\mathbb{E}|X_1^{(t)}|^n)].
\end{aligned}$$

From Lemma 2.11 we infer that there exists a constant $C=C(n,d)$ such that for every $t>0$ and every $x\in\mathbb{Z}^d$,

$$|f(x+e_1)-f(x)|\le Ca_n[1+\|x\|_{n-1}^{n-1}+t^{(n-1)/2}+t^{-1/2}(\|x\|_n^n+t^{n/2})].$$

By setting $t=(1+\|x\|_1)^2$ we complete the proof. ■

LEMMA 2.13. *Let $f:\mathbb{Z}^d\to\mathbb{R}$ be such that $f_i(x)=f(x+e_i)-f(x)$ are polynomials for $i=1,\dots,d$. Then f is a polynomial.*

Proof. First we consider the case $d=1$. Note that $f(x)-f(0)$ is determined by values of f_1. Define a sequence of polynomials $(W_k)_{k=0}^{\infty}$ by

$$x^m=\sum_{k=0}^{m-1}\binom{m}{k}W_k(x),\qquad m=1,2,\dots.$$

A simple induction yields $W_k(x+1)-W_k(x)=x^k$ and $W_k(0)=0$. It follows that if $f_1(x)=\sum_{i=0}^{l}a_ix^i$ then $f(x)=f(0)+\sum_{i=0}^{l}a_iW_i(x)$. If $d>1$ then

$$\begin{aligned}
f(x_1,\dots,x_d)&=f(x_1,x_2,\dots,x_d)-f(0,x_2,\dots,x_d)\\
&\quad+f(0,x_2,\dots,x_d)-f(0,0,x_3,\dots,x_d)\\
&\quad+\dots+f(0,\dots,0,x_1)-f(0,\dots,0)+f(0).
\end{aligned}$$

By using the same argument as in the case $d=1$ we see that

$$f(0,\dots,x_i,\dots,x_d)-f(0,\dots,x_{i+1},\dots,x_d)\qquad(i=1,\dots,d)$$

are polynomials. ■

MAIN THEOREM 2.14. *Let $f:\mathbb{Z}^d\to\mathbb{R}$ be harmonic. Suppose there exists a polynomial $W:\mathbb{Z}^d\to\mathbb{R}$ such that $f(k)\ge -W(k)$ for $k\in\mathbb{Z}^d$. Then f is a polynomial.*

Proof. There exists $n\in\mathbb{N}$ such that $|f(x)|\le a_n(1+\|x\|_n^n)$ (see Lemma 2.9). We claim that together with the harmonicity of f this already implies that f is a polynomial. We prove this by induction on n. For $n=0$ the claim is a consequence of Proposition 2.5. For $n>1$ let $f_i(x)=f_i(x+e_1)-f(x)$. Note that f_i, $i=1,\dots,d$, are also harmonic. By Lemma 2.12 and induction hypothesis, f_i are polynomials, hence by Lemma 2.13 we conclude that f is a polynomial as well. ■

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