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PARTIAL DIFFERENTIAL EQUATIONS

Property C for ODE and Applications to an Inverse Problem for a Heat Equation

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Summary. Let $\ell_j := -d^2/dx^2 + k^2 q_j(x)$, $k = \text{const} > 0$, $j = 1, 2$, $0 < \operatorname{ess\,inf} q_j(x) \leq \operatorname{ess\,sup} q_j(x) < \infty$. Suppose that $(*)$ $\int_0^1 p(x) u_1(x,k) u_2(x,k)\, dx = 0$ for all $k > 0$, where p is an arbitrary fixed bounded piecewise-analytic function on $[0,1]$, which changes sign finitely many times, and u_j solves the problem $\ell_j u_j = 0$, $0 \leq x \leq 1$, $u_j'(0,k) = 0$, $u_j(0,k) = 1$. It is proved that $(*)$ implies $p = 0$. This result is applied to an inverse problem for a heat equation.

1. Introduction. Property C stands for completeness of the set of products of solutions to homogeneous equations. This notion was introduced by the author in [3], [4], [8], and used widely as a powerful tool for proving uniqueness theorems for many inverse problems ([5]–[12]). In [8] Property C was proved for the pair of operators

$$\left\{ \frac{d^2}{dx^2} + k^2 - q_1(x), \frac{d^2}{dx^2} + k^2 - q_2(x) \right\},$$

where

$$q_1, q_2 \in L_{1,1} := \left\{ q : q = \overline{q}, \int_0^\infty (1+x)|q(x)|\, dx < \infty \right\}.$$

The novel point in our paper is the proof of Property C for a pair of differential operators with a different dependence on the spectral parameter. This new version of Property C turns out to be basic, for example, in the proof of uniqueness for an inverse problem for a heat equation with a discontinuous thermal conductivity.

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The aim of this paper is to prove Property C for the pair $\{\ell_1, \ell_2\}$, where

$$\ell_j := -\frac{d^2}{dx^2} + k^2 q_j(x), \qquad j = 1, 2,$$
$$0 < \operatorname{ess\,inf}\, q_j(x) \le \operatorname{ess\,sup}\, q_j(x) < \infty,$$

and k is a positive number. These standing assumptions about q_j are not repeated in the formulation of the theorems below.

Denote by M_0 the set of $L^1(0,1)$ functions which change sign at most finitely many times in the interval $[0,1]$.

We say that a function p has finitely many changes of sign in $[0,1]$ if there is a partition of this interval into finitely many subintervals on which the function does not change sign, that is, p is either non-negative or non-positive on each of the subintervals of this partition.

Definition 1. Let

$$\ell_j u_j(x,k) = 0, \quad u_j'(0,k) = 0, \quad u_j(0,k) = 1, \quad 0 \le x \le 1, \tag{1}$$

and let $W \subset L^1(0,1)$ be some set. If, for all $h \in W$, the orthogonality relation

$$\int_0^1 h(x) u_1(x,k) u_2(x,k)\, dx = 0 \quad \forall k > 0, \tag{2}$$

implies $h = 0$, then we say that the pair $\{\ell_1, \ell_2\}$ has *Property C* for the set W.

Our first result is:

Theorem 1. *The pair $\{\ell_1, \ell_2\}$ has Property C for the set M_0.*

Remark. The set M_0 is dense in $L^2(0,1)$. Therefore one might expect that the result of Theorem 1 implies that the pair $\{\ell_1, \ell_2\}$ has Property C in $L^2(0,1)$. However, such a conclusion does not follow from Theorem 1: one can construct an example of a set W, dense in $L^2(0,1)$, and a set $\{\phi_j\}_{j=1}^\infty$, such that the conditions $f \in W$ and $(f, \phi_j) = 0$ for all j imply $f = 0$, but there exists an $h \in L^2(0,1)$, $h \ne 0$, such that $(h, \phi_j) = 0$ for all j. See an example in [13, pp. 164–165]. Here is another example. Let $W = C(0,1)$. Then W is dense in $L^2(0,1)$. Choose $h \notin W$, $h \in L^2(0,1)$, $h \ne 0$. Let V be the orthogonal complement to h in $L^2(0,1)$, and $\{\phi_j\}_{j=1}^\infty$ be an orthonormal basis of V. If $f \in C(0,1)$ and $(f, \phi_j) = 0$ for all j, then $f = ch$, where $c = \text{const}$. Since $f \in C(0,1)$ on the one hand, and $f = ch \notin C(0,1)$, unless $c = 0$, on the other hand, we conclude that $c = 0$ and $f = 0$. However, $h \ne 0$ and $(h, \phi_j) = 0$ for all j. Therefore the set $\{\phi_j\}_{j=1}^\infty$ is not complete in $L^2(0,1)$, but there is no $f \in C(0,1)$, $f \ne 0$, such that $(f, \phi_j) = 0$ for all j.

It is an open problem whether the pair $\{\ell_1, \ell_2\}$ has Property C in $L^2(0,1)$.

Let us give an example of applications of Property C.

Denote by M a set of real-valued integrable functions such that if $q_j \in M$, $j = 1, 2$, are arbitrary members of M, then the function $p(x) := q_2(x) - q_1(x)$ is in M_0.

Let us mention some examples of such sets M:

1) M_1 is the set of piecewise-constant functions on $[0, 1]$ with finitely many discontinuity points (see [2]),
2) M_2 is the set of piecewise-analytic real-valued functions on $[0, 1]$ with finitely many discontinuity points,
3) M_3 is the set of functions of the form $q_0 + q$, where q_0 is a fixed real-valued integrable function, and $q \in M_2$.

Consider the problem

$$U_t = (a(x)U')', \quad 0 \le x \le 1, \quad t > 0; \quad U' := \frac{\partial U}{\partial x}, \tag{3}$$

$$U(x, 0) = 0, \quad U(0, t) = 0, \quad U(1, t) = F(t), \tag{4}$$

$$a(1)U'(1, t) = G(t). \tag{5}$$

Denote by M_2' the subset of functions in M_2 which satisfy the inequalities

$$0 < \inf a(x) \le \sup a(x) < \infty. \tag{6}$$

The function F satisfies $F \not\equiv 0$, $F \ge 0$, $F(t) = 0$ if $t > T$, where $T > 0$ is an arbitrary fixed number, and $F \in L^1([0, T])$.

Problem (3)–(4) has a unique solution.

The above assumptions about a and F will not be repeated in the formulation of Theorem 2.

The function G is an (extra) measured datum, which is the heat flux at the point $x = 1$.

The extra datum G cannot be given arbitrarily: it is determined by F since problem (3)–(4) is uniquely solvable. If G is an arbitrary function, no matter how smooth it is, problem (3)–(5) may have no solutions.

In this paper we are concerned with the uniqueness of solution to the inverse problem, stated below, and do not discuss the problem of existence of solutions, that is, conditions on the data F and G under which the data are compatible.

The inverse problem is:

IP_1: *Given* $\{F(t), G(t)\}_{t>0}$, *find* $a \in M_2'$.

The function $a \in M_2'$ has finitely many discontinuity points, and the solution to (3)–(4) is understood in the weak sense.

Let us formulate IP_1 in an equivalent but different form. Let

$$
\begin{aligned}
v(x,\lambda) &:= \int_0^\infty U(x,t)e^{-\lambda t}\,dt, \qquad f(\lambda) := \int_0^\infty F(t)e^{-\lambda t}\,dt,\\
g(\lambda) &:= \int_0^\infty G(t)e^{-\lambda t}\,dt.
\end{aligned}
\tag{7}
$$

Take the Laplace transform of (3)–(5) to get

$$-(a(x)v')' + \lambda v = 0, \quad 0 \le x \le 1, \quad \lambda > 0, \tag{8}$$

$$v(0,\lambda) = 0, \quad v(1,\lambda) = f(\lambda), \quad a(1)v'(1,\lambda) = g(\lambda). \tag{9}$$

Let $u(x,\lambda) := a(x)v'(x,\lambda)$. Differentiate (8) to get

$$-u'' + \lambda a^{-1}(x)u = 0, \quad 0 \le x \le 1, \quad u'(0,\lambda) = 0, \tag{10}$$

$$u(1,\lambda) = g(\lambda), \quad u'(1,\lambda) = \lambda f(\lambda). \tag{11}$$

Let

$$\lambda = k^2, \quad k > 0; \quad a^{-1}(x) := q(x).$$

Then $q(x) \ge c_0 > 0$, $c_0 = \text{const}$,

$$-u'' + k^2 q(x)u = 0, \quad 0 \le x \le 1, \quad u'(0,k^2) = 0, \quad u(1,k^2) = g(k^2), \tag{12}$$

and the data are $\{g(k^2), k^2 f(k^2)\}_{k>0}$.

Note that if $a \in M_2'$, then $q \in M_2'$.

Now, IP_1 can be reformulated as follows:

IP_2: *Given the data $\{g(k^2), k^2 f(k^2)\}_{k>0}$, find $q \in M_2'$.*

Our second result is:

THEOREM 2. *IP_2 has at most one solution.*

Theorem 2 implies the uniqueness of solution to IP_1 in the class of piecewise-analytic strictly positive functions a with finitely many discontinuity points in $[0,1]$.

In the literature, IP_1 has been considered earlier (see, e.g., [9], [10] and references therein) in the case when $a \in H^2([0,1])$, where H^2 is the Sobolev space. For piecewise-constant thermal conductivity coefficients $a(x)$ with finitely many discontinuity points, IP_1 was studied recently in [2]. An inverse problem for equation (3) with different extra data, namely $U(\xi_n,t)$ for $t > 0$, $\xi_n \in [0,1]$, $1 \le n \le N$, $\min_{1\le n\le N}|\xi_n - \xi_{n+1}| \ge \sigma > 0$, where σ is a fixed number, $N = 3\nu$, and ν is the number of discontinuity points of a, was studied in [1].

In IP_1 the extra data are collected at just one point $x = 1$.

Our arguments prove the uniqueness result of Theorem 2 in the case when the data in IP_1 are the values $\{F(t), G(t)\}_{t\in[0,T+\epsilon]}$ for an arbitrarily small $\epsilon > 0$. These data determine a uniquely because the solution $U(x,t)$

is an analytic function of t in a neighborhood of the set (T, ∞), so the knowledge of $U(x,t)$ on $[0, T+\epsilon]$ determines $U(x,t)$ uniquely for all $t > 0$.

2. Proofs

Proof of Theorem 1. The solution to (1) solves the equation

$$u_j(x,k) = 1 + k^2 \int_0^x (x-s) q_j(s) u_j(s,k)\, ds, \qquad x \geq 0, \qquad j = 1,2. \tag{13}$$

This is a Volterra equation. It has a unique solution $u_j(x,k)$. This solution has the following properties:

$$u_j(x,k) \geq 1, \quad u_j'(x,k) \geq 0, \quad u_j''(x,k) > 0, \quad 0 \leq x \leq 1, \tag{14}$$

$$\frac{\partial^i u_j}{(\partial k^2)^i} \geq 0, \quad i = 1, 2, \ldots, \tag{15}$$

$$\lim_{k\to\infty} \frac{u_j(y,k)}{u_j(x,k)} = 0, \qquad 0 \leq y < x \leq 1. \tag{16}$$

Properties (14)–(15) are immediate consequences of (13). Let us prove (16). One has

$$u_j(x,k) = u_j(y,k) + \int_y^x u_j'(s,k)\, ds. \tag{17}$$

From (13) and (14) one obtains

$$u_j'(x,k) = k^2 \int_0^x q_j(s) u_j(s,k)\, ds \geq k^2 \int_0^x q_j(s)\, ds. \tag{18}$$

From (17) and (18) one gets

$$\begin{aligned} \frac{u_j(x,k)}{u_j(y,k)} &= 1 + \int_y^x \frac{u_j'(s,k)}{u_j(y,k)}\, ds = 1 + k^2 \int_y^x \frac{\int_0^s q_j(z) u_j(z,k)\, dz}{u_j(y,k)}\, ds \\ &\geq 1 + k^2 \int_y^x ds \int_y^s q_j(z)\, dz \geq 1 + \frac{1}{2} k^2 c_0 (x-y)^2 \to \infty \end{aligned} \tag{19}$$

as $k \to \infty$, where $q_j(x) \geq c_0 = \text{const} > 0$.

Thus, (16) is proved.

Since $h \in M_0$, the interval $[0,1]$ is a union of finitely many intervals without common interior points on each of which the function h keeps constant sign. Let $[z,1]$ be such an interval. We want to prove that $h = 0$ on this interval. If this is done then similarly, in a finite number of steps, one proves that $h = 0$ on the whole interval $[0,1]$, and thus the proof of Theorem 1 is complete.

Let us rewrite relation (2) as

$$\int_z^1 h(x)u_1(x,k)u_2(x,k)\,dx = -\int_0^z h(x)u_1(x,k)u_2(x,k)\,dx \le u_1(z,k)u_2(z,k)\int_0^z |h(x)|\,dx, \tag{20}$$

where the monotonicity and positivity of u_j were used (see (14)). Without loss of generality assume that $h(x) > 0$ on $[z,1]$ and fix an arbitrary $y \in (z,1)$. Then

$$\int_z^1 h(x)u_1(x,k)u_2(x,k)\,dx \ge \int_y^1 h(x)\,dx\,u_1(y,k)u_2(y,k). \tag{21}$$

From (20) and (21) one gets

$$\int_y^1 h(x)\,dx \le \frac{u_1(z,k)u_2(z,k)}{u_1(y,k)u_2(y,k)}\int_0^z |h(x)|\,dx, \qquad y > z. \tag{22}$$

Let $k \to \infty$ in (22) and use (16) to get $\int_y^1 h(x)\,dx = 0$. Since $h(x) \ge 0$ on $[z,1]$, it follows that $h = 0$ on $[y,1]$. Since the point $y \in (z,1)$ is arbitrary, it follows that $h = 0$ on $[z,1]$. Theorem 1 is proved. ∎

Proof of Theorem 2. Assume there are pairs of functions $\{\psi_1, q_1\}$ and $\{\psi_2, q_2\}$ which solve (12) and (11) with $\lambda = k^2$. Let $w = \psi_1 - \psi_2$. Then

$$w'(0,k) = w(1,k) = w'(1,k) = 0, \tag{23}$$

and

$$-w'' + k^2 q_1(x)w = k^2 p(x)\psi_2, \qquad p(x) := q_2(x) - q_1(x). \tag{24}$$

Multiply (24) by $u_1(x,k)$, integrate over $[0,1]$, and then integrate by parts using (23) to obtain

$$k^2\int_0^1 p(x)u_1(x,k)\psi_2(x,k)\,dx = 0 \quad \forall k > 0. \tag{25}$$

Since $\psi_2(x) = c(k)u_2(x,k)$, where $c(k) = \text{const} \ne 0$, and $p \in M_0$, it follows from (25) and Theorem 1 that $p = 0$, so $q_1 = q_2$.

Theorem 2 is proved. ∎

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