BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 62, No. 2, 2014

COMBINATORICS

Combinatorics of Dyadic Intervals: Consistent Colourings

by

Anna KAMONT and Paul F. X. MÜLLER

Presented by Gilles PISIER

Summary. We study the problem of consistent and homogeneous colourings for increasing families of dyadic intervals. We determine when this problem can be solved and when it cannot.

1. Introduction. Combinatorics of coloured dyadic intervals refers to a set of techniques created for the study of operators defined through their action on the Haar system. We refer to the treatment of averaging projections by P. W. Jones [5], the proof of the vector-valued $T(1)$ theorem by T. Figiel [1, 2], the use of the stripe operators in J. Lee, P. F. X. Müller and S. Müller [7], and the study of rearrangement operators on L^p spaces by P. F. X. Müller [9], K. Smela [11] and A. Kamont and P. F. X. Müller [6].

Here we study a very natural colouring problem on dyadic trees. We start out with a coloured collection $\mathcal{C}$ of dyadic intervals, where we assume that the colours are distributed homogeneously over $\mathcal{C}$. Given any collection $\mathcal{H}$ containing $\mathcal{C}$ we ask if there exists an equally homogeneous colouring of $\mathcal{H}$ that preserves the colours of $\mathcal{C}$ (a consistent colouring of $\mathcal{H}$). The nature of this problem depends very much on what we agree to call a homogeneous distribution of colours. Our choice of homogeneity is very restrictive, and consequently in working on the problem of consistent colouring we encountered delicate combinatorial questions.

Let $\mathcal{D}$ denote the collection of dyadic intervals in the unit interval $[0,1]$, and let

$$\mathcal{D}_j = \{I \in \mathcal{D} : |I| = 2^{-j}\}.$$

2010 *Mathematics Subject Classification*: 05A18, 91A05.
Key words and phrases: coloured dyadic intervals, consistent and homogeneous colouring, 2-person games.

DOI: 10.4064/ba62-2-1

We consider a large collection $\mathcal{C} \subset \mathcal{D}_j$. We assume that the intervals in $\mathcal{C}$ are painted with d distinct colours, giving rise to a decomposition

$$\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_d.$$

It is intuitively clear what it means that the colours $\{1, \ldots, d\}$ are homogeneously distributed among the intervals of $\mathcal{C}$. For instance, we would demand that there exists $\eta > 0$ so that

$$\eta \max_{1\le i\le d} |\mathcal{C}_i| \le \min_{1\le i\le d} |\mathcal{C}_i|, \tag{1.1}$$

where $|\mathcal{C}_i|$ denotes the cardinality of the collection $\mathcal{C}_i$. A much stronger measure of homogeneity arises when we ask for (1.1) to hold over the prespecified collection of testing intervals

$$\mathcal{T} = \{J \in \mathcal{D} : |J| > 2^{-j}\}.$$

Specifically, if $|\mathcal{C} \cap L| > d$, we would demand that there exists $\eta > 0$ such that

$$\eta \max_{1\le i\le d} |\mathcal{C}_i \cap L| \le \min_{1\le i\le d} |\mathcal{C}_i \cap L| \quad \text{for each } L \in \mathcal{T}, \tag{1.2}$$

where

$$\mathcal{C}_i \cap L = \{I \in \mathcal{C}_i : I \subset L\}.$$

We use an additional rule to express homogeneity with respect to testing intervals that satisfy $|\mathcal{C} \cap L| \le d$. The necessity of such a rule arises from the fact that the cardinalities $|\mathcal{C}_i \cap L|$ take values in $\mathbb{N} \cup \{0\}$, hence if $|\mathcal{C} \cap L| < d$, then (1.2) has to fail. Thus, there are two regimes: high cardinality and low cardinality of $\mathcal{C} \cap L$, with transition at $|\mathcal{C} \cap L| = d$. The following definition contains the homogeneity conditions for both regimes, and it addresses the discrete nature of our gauge functions

$$\mathcal{C}_i \mapsto |\mathcal{C}_i \cap L|, \quad L \in \mathcal{T}.$$

Definition 1.1. Let $\mathcal{C} \subset \mathcal{D}_j$, and fix $d \in \mathbb{N}$, $0 < \eta \le 1/2$. Let $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_d$ be some decomposition of $\mathcal{C}$. This decomposition is called an (η, d)-*homogeneous colouring* of $\mathcal{C}$ if for each $L \in \mathcal{D}$ with $|L| \ge 1/2^j$ one of the following holds:

Either $|\mathcal{C} \cap L| > d$, and then

$$\eta \max_{1\le i\le d} |\mathcal{C}_i \cap L| \le \min_{1\le i\le d} |\mathcal{C}_i \cap L|, \tag{1.3}$$

or else $|\mathcal{C} \cap L| \le d$, and then

$$|\mathcal{C}_i \cap L| \le 1 \quad \text{for each } 1 \le i \le d, \tag{1.4}$$

Remark. We remark that for each (uncoloured) $\mathcal{C} \subset \mathcal{D}_j$, $d \in \mathbb{N}$ and $\eta = 1/2$ there is always an (η, d)-homogeneous colouring that can be obtained as follows: Enumerate the intervals in $\mathcal{C}$ from left to right, and simply put

$$\mathcal{C}_r = \{\Gamma_l \in \mathcal{C} : l = r \bmod d\}, \qquad 1 \leq r \leq d. \tag{1.5}$$

Later, we refer to this colouring as the *colouring modulo d*.

The use of this colouring rule—applied to intervals of equal length—appeared in a context similar to ours in [4, p. 200] (see also [3, p. 359] and [10, p. 199]).

The problem of consistent colouring. The problem we treat in this paper is the following. We are given two disjoint collections $\mathcal{C}, \mathcal{U} \subset \mathcal{D}_j$. Assume that the collection $\mathcal{C}$ is coloured, that is, it is given an (η, d)-homogeneous colouring

$$\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_d.$$

The collection $\mathcal{U}$ consists of uncoloured intervals. We would like to colour the intervals in $\mathcal{U}$ using the same colours $\{1, \ldots, d\}$, that is, to decompose $\mathcal{U}$ as

$$\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_d$$

in such a way that the union $\mathcal{H} = \mathcal{C} \cup \mathcal{U}$ has an (η, d)-homogeneous colouring given by

$$\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_d, \quad \text{where} \quad \mathcal{H}_i = \mathcal{C}_i \cup \mathcal{U}_i \quad \text{for } 1 \leq i \leq d.$$

That is, we want to obtain an (η, d)-homogeneous colouring of $\mathcal{H} \supset \mathcal{C}$ keeping the pre-existing (η, d)-homogeneous colouring of $\mathcal{C}$.

We refer to this question as *the problem of finding a colouring of $\mathcal{H}$ consistent with an existing colouring of $\mathcal{C}$.* Our treatment of this problem is as follows:

1. We isolate a condition on $\mathcal{U}$ and $\mathcal{C}$ (previsibility; see Definition 2.1) implying that the problem of consistent colouring for $\mathcal{H} = \mathcal{C} \cup \mathcal{U}$ has a solution (see Theorem 2.2).
2. We give examples where the problem of consistent colouring for $\mathcal{H} = \mathcal{C} \cup \mathcal{U}$ has just one solution. Moreover, we give examples (of $\mathcal{C}$, its decomposition $\{\mathcal{C}_i\}$ and $\mathcal{U}$) for which the problem of consistent colouring for $\mathcal{H} = \mathcal{C} \cup \mathcal{U}$ does not have a solution (see Proposition 3.1).
3. In Section 4 we reformulate the problem of consistent colouring as a two-person game. Our results—Theorem 2.2 and Proposition 3.1—translate into winning strategies for the respective players.

For the appearance of successive colourings of dyadic intervals in the context of averaging projections see [5, pp. 871–875]. In [6] we constructed supporting trees for some rearrangement operators and thereby proved their boundedness on vector valued L_p spaces. Initially, our approach to defining the supporting trees was via inclusion-exclusion principles and consistent colourings as studied in the present paper.

2. Constructing a consistent colouring. In the following we isolate a criterion which guarantees the existence of a consistent colouring. To formulate this criterion, we use a dyadic interval $L \in \mathcal{D}$ together with its immediate dyadic successors L', L'', i.e. intervals $L', L'' \in \mathcal{D}$ such that $L = L' \cup L''$ and $|L'| = |L''| = \frac{1}{2}|L|$. The problem of consistent colouring leads us to the following condition: We are given disjoint collections $\mathcal{C}, \mathcal{U} \subset \mathcal{D}_j$ and $d \in \mathbb{N}$. We say that the pair $(\mathcal{C}, \mathcal{U})$ is *d-previsible* if with $\mathcal{H} = \mathcal{C} \cup \mathcal{U}$, the conditions

$$|\mathcal{H} \cap L''| \geq d, \quad \mathcal{C} \cap L'' \neq \emptyset, \quad \mathcal{U} \cap L'' \neq \emptyset$$

imply

$$|\mathcal{H} \cap L'| \geq d.$$

For case of reference in the argument below, we give the following—equivalent—definition.

Definition 2.1. Let $\mathcal{C}, \mathcal{U} \subset \mathcal{D}_j$, $\mathcal{C} \cap \mathcal{U} = \emptyset$. Let $d \in \mathbb{N}$. The pair $(\mathcal{C}, \mathcal{U})$ of collections is called *d-previsible* if for every $L \in \mathcal{D}$ with $|L| \geq 1/2^{j-1}$ and its dyadic succesors L', L'', the following holds:

$$|(\mathcal{U} \cup \mathcal{C}) \cap L'| < d \quad \text{and} \quad |(\mathcal{U} \cup \mathcal{C}) \cap L''| \geq d$$

implies

$$\mathcal{U} \cap L'' = \emptyset \quad \text{or} \quad \mathcal{C} \cap L'' = \emptyset.$$

Now, the following theorem gives a sufficient condition for existence of consistent colourings.

Theorem 2.2. *Fix $d \in \mathbb{N}$ and $0 < \eta \leq 1/2$. Let $\mathcal{C} \subset \mathcal{D}_j$, and let $\{\mathcal{C}_i : 1 \leq i \leq d\}$ be a fixed (η, d)-homogeneous colouring of $\mathcal{C}$. Let $\mathcal{U} \subset \mathcal{D}_j$ be such that the pair $(\mathcal{C}, \mathcal{U})$ is d-previsible. Then there is a colouring $\{\mathcal{U}_i : 1 \leq i \leq d\}$ of $\mathcal{U}$ such that $\{\mathcal{H}_i = \mathcal{C}_i \cup \mathcal{U}_i : 1 \leq i \leq d\}$ is an (η, d)-homogeneous colouring of $\mathcal{H} = \mathcal{C} \cup \mathcal{U}$.*

Remark. A first attempt to prove Theorem 2.2 by an inductive argument would be the following: Find first an (η, d)-homogeneous colouring of $\mathcal{H} \cap K$ for $K \in \mathcal{D}_{j-\alpha}$, where $2^\alpha \leq d < 2^{\alpha+1}$. Then carry over these colourings—inductively and backwards in time—to larger collections $\mathcal{H} \cap L$, $L \in \mathcal{D}_s$ with $s > j - \alpha$, as follows: Assume that for a dyadic interval L with successors L', L'', separate (η, d)-homogeneous colourings of $\mathcal{H} \cap L'$ and $\mathcal{H} \cap L''$ are fixed. Then check that the union of these colourings gives an (η, d)-homogeneous colouring of the union $\mathcal{H} \cap L = (\mathcal{H} \cap L') \cup (\mathcal{H} \cap L'')$. If this procedure worked, at stage s we would have produced an (η, d)-homogeneous colouring of $\mathcal{H} \cap K$ for each $K \in \mathcal{D}$ with $|K| \leq 2^{-s}$. However, such a deterministic approach cannot work, as the following example shows.

Example. Take $d = 3r$, $r \in \mathbb{N}$, and fix a dyadic interval L with successors L', L''. Assume that collections $\mathcal{C}, \mathcal{U}$ are such that $|\mathcal{C} \cap L'| = |\mathcal{C} \cap L''| = r$

and $|\mathcal{U}\cap L'| = |\mathcal{U}\cap L''| = r$. Take an (η,d)-homogeneous colouring of $\mathcal{C}\cap L$ such that

$$\begin{aligned} |\mathcal{C}_i\cap L'| &= 1 \quad \text{for } 1\le i\le r, \qquad |\mathcal{C}_i\cap L'| = 0 \quad \text{for } r+1\le i\le 3r,\\ |\mathcal{C}_i\cap L''| &= 1 \quad \text{for } r+1\le i\le 2r,\\ |\mathcal{C}_i\cap L''| &= 0 \quad \text{for } 1\le i\le r \text{ and } 2r+1\le i\le 3r. \end{aligned}$$

Note that then

$$|\mathcal{C}_i\cap L| = 1 \quad \text{for } 1\le i\le 2r, \qquad |\mathcal{C}_i\cap L| = 0 \quad \text{for } 2r+1\le i\le 3r.$$

Next, we can choose colourings of $\mathcal{U}\cap L'$ and $\mathcal{U}\cap L''$ such that

$$\begin{aligned} |\mathcal{U}_i\cap L'| &= 1 \quad \text{for } r+1\le i\le 2r,\\ |\mathcal{U}_i\cap L'| &= 0 \quad \text{for } 1\le i\le r \text{ and } 2r+1\le i\le 3r,\\ |\mathcal{U}_i\cap L''| &= 1 \quad \text{for } 1\le i\le r, \qquad |\mathcal{U}_i\cap L''| = 0 \quad \text{for } r+1\le i\le 3r. \end{aligned}$$

Then, knowing the colouring of $\mathcal{C}\cap K$ and $\mathcal{U}\cap K$ for $K = L', L''$ with $\mathcal{H} = \mathcal{C}\cup\mathcal{U}$ we have

$$|\mathcal{H}_i\cap K| = 1 \quad \text{for } 1\le i\le 2r, \qquad |\mathcal{H}_i\cap K| = 0 \quad \text{for } 2r+1\le i\le 3r,$$

so we have separate (η,d)-homogeneous colourings of $\mathcal{H}\cap L'$ and $\mathcal{H}\cap L''$. However, by taking their union we get a colouring of $\mathcal{H}\cap L$ such that

$$|\mathcal{H}_i\cap L| = 2 \quad \text{for } 1\le i\le 2r, \qquad |\mathcal{H}_i\cap L| = 0 \quad \text{for } 2r+1\le i\le 3r,$$

which is not (η,d)-homogeneous.

Note however that the above example does not contradict the assertion of Theorem 2.2. In fact, given $\mathcal{C}$ and its colouring as above, it is quite easy to obtain a colouring of $\mathcal{U}$ so that the conclusion of Theorem 2.2 holds. For $K = L', L''$, we put

$$|\mathcal{U}_i\cap K| = 0 \quad \text{for } 1\le i\le 2r, \qquad |\mathcal{U}_i\cap K| = 1 \quad \text{for } 2r+1\le i\le 3r.$$

Taking the union with the colouring of $\mathcal{C}\cap K$, we find that $0\le|\mathcal{H}_i\cap K|\le 1$, so we have an (η,d)-homogeneous colouring of $\mathcal{H}\cap K$, $K = L', L''$. Finally,

$$|\mathcal{H}_i\cap L| = 1 \quad \text{for } 1\le i\le 2r, \qquad |\mathcal{H}_i\cap L| = 2 \quad \text{for } 2r+1\le i\le 3r,$$

so we have a $(1/2,d)$-homogeneous colouring of $\mathcal{H}\cap L$. ■

In response to these examples, we will introduce a stopping time argument—running backwards in time—that produces the (η,d)-homogeneous colouring of Theorem 2.2. At stage s of our inductive argument, we will produce consistent (η,d)-homogeneous colourings of collections $\mathcal{H}\cap K$ for $K\in\mathcal{D}$ with $K\le 2^{-s}$ provided that K satisfies

$$|\mathcal{H}\cap K|\ge d \quad\text{and}\quad \mathcal{C}\cap K\ne\emptyset.$$

Proof of Theorem 2.2. We are going to define a colouring of $\mathcal{U}$ by an inductive argument. Let α be such that $2^\alpha\le d<2^{\alpha+1}$. Let us observe that if $1/2^j \le |L| \le 1/2^{j-\alpha}$, then $|\mathcal{H} \cap L| \le 2^\alpha \le d$. Thus, if the homogeneity conditions (1.4) respectively (1.3) are satisfied for $L \in \mathcal{D}$ with $|L| \ge 1/2^{j-\alpha}$, then they are satisfied for each $L \in \mathcal{D}$ with $|L| \ge 1/2^j$. Therefore, in our procedure of colouring $\mathcal{U}$ we consider only $L \in \mathcal{D}_k$ with $k \le j - \alpha$.

The inductive argument is used to prove the following statement at each stage s with $j - \alpha \ge s \ge 0$:

Inductive hypothesis at stage s: Let $K \in \mathcal{D}_s$. If $|\mathcal{H} \cap K| < d$ or $\mathcal{C} \cap K = \emptyset$, then intervals in $\mathcal{U} \cap K$ are still uncoloured. If $|\mathcal{H} \cap K| \ge d$ and $\mathcal{C} \cap K \ne \emptyset$, then all intervals in $\mathcal{U} \cap K$ are coloured, and the colouring of $\mathcal{H} \cap K$ is (η, d)-homogeneous; as $|\mathcal{H} \cap K| \ge d$, this means that $|\mathcal{H}_i \cap K| \ge 1$ and

$$\eta \max_{1 \le i \le d} |\mathcal{H}_i \cap K| \le \min_{1 \le i \le d} |\mathcal{H}_i \cap K|. \tag{2.1}$$

I. *Start of the induction.* Let $L \in \mathcal{D}_{j-\alpha}$. Then either $|\mathcal{H} \cap L| < d$ or $|\mathcal{H} \cap L| = d$.

I.1. If $|\mathcal{H} \cap L| < d$ or $\mathcal{C} \cap L = \emptyset$, then all intervals in $\mathcal{U} \cap L$ are left uncoloured.

I.2. If $|\mathcal{H} \cap L| = d$ and $\mathcal{C} \cap L \ne \emptyset$, then also $|\mathcal{C} \cap L| \le d$, which implies that $|\mathcal{C}_i \cap L| \le 1$ for $1 \le i \le d$. In that case it is possible to colour all intervals in $\mathcal{U} \cap L$ so that $|\mathcal{H}_i \cap L| = 1$ for $1 \le i \le d$.

II. *Inductive step.* Let $\nu < j - \alpha$. The inductive assumption states that there is a colouring at stage $\nu + 1$. We need to prove that there is a colouring at stage ν. For this take $L \in \mathcal{D}_\nu$. Then $L = L' \cup L''$ with $L', L'' \in \mathcal{D}_{\nu+1}$, $\nu + 1 \le j - \alpha$. Each interval in $\mathcal{C} \cap L$ or $\mathcal{U} \cap L$ is included in L' or L'', so we have

$$\begin{aligned} |\mathcal{C} \cap L| &= |\mathcal{C} \cap L'| + |\mathcal{C} \cap L''|, \\ |\mathcal{U} \cap L| &= |\mathcal{U} \cap L'| + |\mathcal{U} \cap L''|, \\ |\mathcal{H} \cap L| &= |\mathcal{H} \cap L'| + |\mathcal{H} \cap L''|. \end{aligned}$$

Then we have two main cases:

II.1. $|\mathcal{H} \cap L| < d$ or $\mathcal{C} \cap L = \emptyset$. If $|\mathcal{H} \cap L| < d$ then also $|\mathcal{H} \cap L'|, |\mathcal{H} \cap L''| < d$. If $\mathcal{C} \cap L = \emptyset$, then also $\mathcal{C} \cap L' = \emptyset$ and $\mathcal{C} \cap L'' = \emptyset$. In both cases, by the induction hypothesis, all intervals in both $\mathcal{U} \cap L'$ and $\mathcal{U} \cap L''$ are uncoloured.

If $\nu > 0$, then leave the intervals in $\mathcal{U} \cap L$ uncoloured.

If $\nu = 0$, then $L = [0, 1]$, and the induction ends. This means that $|\mathcal{H}| < d$ or $\mathcal{C} = \emptyset$. If $|\mathcal{H}| < d$, then it is enough to assign to elements of $\mathcal{U}$ colours different from colours of elements of $\mathcal{C}$. If $\mathcal{C} = \emptyset$, then it is enough to colour $\mathcal{H}$ by the modulo d method (see (1.5)).

II.2. $|\mathcal{H} \cap L| \ge d$ and $\mathcal{C} \cap L \ne \emptyset$. It follows that at least one of $\mathcal{C} \cap L', \mathcal{C} \cap L''$ must be nonempty. Now we isolate next two subcases:

II.2.A. $|\mathcal{H} \cap L| \ge d$, $\mathcal{C} \cap L \ne \emptyset$ and both $\mathcal{C} \cap L' \ne \emptyset$, $\mathcal{C} \cap L'' \ne \emptyset$.

II.2.B. $|\mathcal{H} \cap L| \geq d$, $\mathcal{C} \cap L \neq \emptyset$, $\mathcal{C} \cap L' = \emptyset$, but $\mathcal{C} \cap L'' \neq \emptyset$. (The case $\mathcal{C} \cap L' \neq \emptyset$ and $\mathcal{C} \cap L'' = \emptyset$ is symmetric, and there is no need to treat it separately.)

We first deal with II.2.A, and then with II.2.B. Each of these cases splits into subcases. Case II.2.B is treated by reducing its subcases to appropriate subcases of II.2.A.

II.2.A.1. $|\mathcal{H} \cap L'| \geq d$ and $|\mathcal{H} \cap L''| \geq d$. Recall that $\mathcal{C} \cap L' \neq \emptyset$ and $\mathcal{C} \cap L'' \neq \emptyset$. Then by induction hypothesis all intervals in $\mathcal{U} \cap L'$ and in $\mathcal{U} \cap L''$ are already coloured, i.e. all intervals in $\mathcal{U} \cap L$ are coloured. Moreover, by (2.1), for $1 \leq i, k \leq d$,

$$\eta|\mathcal{H}_i \cap L| = \eta|\mathcal{H}_i \cap L'| + \eta|\mathcal{H}_i \cap L''| \leq |\mathcal{H}_k \cap L'| + |\mathcal{H}_k \cap L''| = |\mathcal{H}_k \cap L|.$$

Of course, we also have $|\mathcal{H}_i \cap L| \geq 1$.

II.2.A.2. $|\mathcal{H} \cap L'| < d$ and $|\mathcal{H} \cap L''| < d$. Then by induction hypothesis all intervals in $\mathcal{U} \cap L'$ and in $\mathcal{U} \cap L''$ are uncoloured, but the intervals in $\mathcal{C} \cap L$ are coloured.

Now, we need to colour all intervals in $\mathcal{U} \cap L = (\mathcal{U} \cap L') \cup (\mathcal{U} \cap L'')$. To simplify notation, let

$$m = |\mathcal{C} \cap L'|, \quad n = |\mathcal{C} \cap L''|, \quad x = |\mathcal{U} \cap L'|, \quad y = |\mathcal{U} \cap L''|.$$

We have

$$0 \leq m, n \leq d - 1, \quad 0 \leq m + x, n + y \leq d - 1,$$
$$d \leq |\mathcal{H} \cap L| = m + x + n + y \leq 2(d - 1).$$

First consider the case $m + n < d$. Then

$$0 \leq |\mathcal{C}_i \cap L| \leq 1 \quad \text{for } 1 \leq i \leq d.$$

For simplicity, assume that the intervals in $\mathcal{C} \cap L'$ have colours $1, \ldots, m$, and the intervals in $\mathcal{C} \cap L''$ have colours $m + 1, \ldots, m + n$. Now, we colour the intervals in $\mathcal{U} \cap L$. First, colour the intervals in $\mathcal{U} \cap L'$ using colours $m + n + 1, \ldots, d$, continuing if necessary (i.e. if $x > d - (m + n)$) with $x - (d - (m + n))$ colours from among $m + 1, \ldots, m + n$; since $m + x < d$, in this way we assign colours to all intervals in $\mathcal{U} \cap L'$. Next, we assign colours to intervals in $\mathcal{U} \cap L''$. If $m + n + x < d$, then first assign colours $m+n+x+1, \ldots, d$, then continue with $1, \ldots, m$, and finish if necessary with $m+n+1, \ldots, m+n+x$. If $m+n+x \geq d$, then just choose y different colours from $1, \ldots, m$ and $m + n + 1, \ldots, d$. With such a colouring of intervals in $\mathcal{U} \cap L'$ and in $\mathcal{U} \cap L''$ we find that both

$$|\mathcal{H}_i \cap L'| \leq 1 \quad \text{and} \quad |\mathcal{H}_i \cap L''| \leq 1, \quad 1 \leq i \leq d.$$

This implies that if $K \subset L'$ or $K \subset L''$ then $|\mathcal{H} \cap K| < d$ and $|\mathcal{H}_i \cap K| \leq 1$.

Moreover, we get $1 \le |\mathcal{H}_i \cap L| \le 2$, which implies

$$\eta \max_{1\le i\le d} |\mathcal{H}_i \cap L| \le \frac{1}{2} \max_{1\le i\le d} |\mathcal{H}_i \cap L| \le \min_{1\le i\le d} |\mathcal{H}_i \cap L|.$$

It remains to consider the case $m+n \ge d$. Then the homogeneity assumption on the decomposition of $\mathcal{C}$ ((1.4) for L', L'' and (1.3) for L) implies

$$1 \le |\mathcal{C}_i \cap L| = |\mathcal{C}_i \cap L'| + |\mathcal{C}_i \cap L''| \le 2, \qquad 1 \le i \le d.$$

For simplicity, assume that the intervals in $\mathcal{C} \cap L'$ have colours $1, \dots, m$ and the intervals in $\mathcal{C} \cap L''$ have colours $m+1, \dots, d$ and $1, \dots, m+n-d$ (note that $m+n-d < m$, since by assumption $n < d$). To colour the intervals in $\mathcal{U} \cap L'$ choose x colours from $m+1, \dots, d$. To colour the intervals in $\mathcal{U} \cap L''$ choose y colours from $m+n-d+1, \dots, m$. This is possible since $m+x < d$ and $n+y < d$. Observe that in this way we get

$$0 \le |\mathcal{H}_i \cap L'|, |\mathcal{H}_i \cap L''| \le 1 \quad \text{and} \quad 1 \le |\mathcal{H}_i \cap L| \le 2, \qquad 1 \le i \le d.$$

Therefore, if $K \subset L'$ or $K \subset L''$ then $|\mathcal{H}_i \cap K| \le 1$, while for L we have

$$\eta \max_{1\le i\le d} |\mathcal{H}_i \cap L| \le \frac{1}{2} \max_{1\le i\le d} |\mathcal{H}_i \cap L| \le \min_{1\le i\le d} |\mathcal{H}_i \cap L|.$$

II.2.A.3. $|\mathcal{H} \cap L'| < d$ and $|\mathcal{H} \cap L''| \ge d$. Recall that $\mathcal{C} \cap L' \ne \emptyset$ and $\mathcal{C} \cap L'' \ne \emptyset$, by the defining condition of case II.2.A. Then by induction hypothesis all intervals in $\mathcal{U} \cap L'$ are uncoloured, but the intervals in $\mathcal{C} \cap L'$ are coloured. Since the pair $(\mathcal{C}, \mathcal{U})$ is d-previsible, we have $\mathcal{U} \cap L'' = \emptyset$. Therefore, $|\mathcal{H} \cap L''| = |\mathcal{C} \cap L''|$, and by condition (1.3) of the (η, d)-homogeneity for $\mathcal{C}$, we get $|\mathcal{H}_i \cap L''| = |\mathcal{C}_i \cap L''| \ge 1$ and $|\mathcal{C}_i \cap L''|$, $1 \le i \le d$, satisfy (2.1).

If $\mathcal{U} \cap L' = \emptyset$ as well, then all intervals in $\mathcal{H} \cap L$ come from $\mathcal{C} \cap L$, and there is nothing to do.

Let $|\mathcal{U} \cap L'| = x > 0$. We need to colour the x intervals in $\mathcal{U} \cap L'$. To simplify notation, let $m = |\mathcal{C} \cap L'|$. Note that $1 \le m + x < d$. Let

$$S = \{i \colon |\mathcal{C}_i \cap L'| = 1\} \quad \text{and} \quad T = \{i \colon |\mathcal{C}_i \cap L'| = 0\}.$$

Let $t_1, \dots, t_{d-m}$ be an ordering of T such that

$$|\mathcal{C}_{t_1} \cap L''| \le \dots \le |\mathcal{C}_{t_{d-m}} \cap L''|. \tag{2.2}$$

Since $x < d - m$, there are more colours in T than intervals in $\mathcal{U} \cap L'$. Now attach the colours $t_1, \dots, t_x$, bijectively, to intervals in $\mathcal{U} \cap L'$. Then $|\mathcal{H}_i \cap L'| \le 1$. Consequently, $|\mathcal{H}_i \cap K| \le 1$ for each $K \subset L'$.

The colouring of $\mathcal{H} \cap L'' = \mathcal{C} \cap L''$ is (η, d)-homogeneous by the assumption. It remains to check that the $|\mathcal{H}_i \cap L|$ satisfy (2.1). Since $\mathcal{U} \cap L'' = \emptyset$ and $|\mathcal{H} \cap L''| \ge d$ we have

$$|\mathcal{C} \cap L| \ge |\mathcal{C} \cap L''| = |\mathcal{H} \cap L''| \ge d.$$

Consequently, since the colouring of $\mathcal{C}$ is (η, d)-homogeneous, we see that $|\mathcal{C}_i \cap L| \geq |\mathcal{C}_i \cap L''| \geq 1$ and

$$\eta \max_{1\le i\le d} |\mathcal{C}_i \cap L''| \le \min_{1\le i\le d} |\mathcal{C}_i \cap L''|, \tag{2.3}$$

$$\eta \max_{1\le i\le d} |\mathcal{C}_i \cap L| \le \min_{1\le i\le d} |\mathcal{C}_i \cap L|. \tag{2.4}$$

Moreover,

$$|\mathcal{H}_i \cap L| = |\mathcal{C}_i \cap L| = |\mathcal{C}_i \cap L''| + 1 \qquad \text{for } i \in S, \tag{2.5}$$

$$|\mathcal{H}_i \cap L| = |\mathcal{C}_i \cap L| + 1 = |\mathcal{C}_i \cap L''| + 1 \qquad \text{for } i = t_1, \dots, t_x, \tag{2.6}$$

$$|\mathcal{H}_i \cap L| = |\mathcal{C}_i \cap L| = |\mathcal{C}_i \cap L''| \qquad \text{for } i = t_{x+1}, \dots, t_{d-m}. \tag{2.7}$$

Let k be such that $\max_i |\mathcal{H}_i \cap L| = |\mathcal{H}_k \cap L|$. Then $k \in S$ or $k \in T$.

If $k \in S$, then $|\mathcal{H}_k \cap L| = |\mathcal{C}_k \cap L|$ by (2.5), and (2.1) is satisfied for L and $\mathcal{H}$ in view of (2.4) and the inequality $|\mathcal{C}_i \cap L| \le |\mathcal{H}_i \cap L|$.

If $k \in T$, then we have either $k \in \{t_1, \dots, t_x\}$ or $k \in \{t_{x+1}, \dots, t_{d-m}\}$. The ordering defined by (2.2) implies that $k = t_x$ in the former case and $k = t_{d-m}$ in the latter. If $k = t_{d-m}$, then (2.1) is satisfied for L and $\mathcal{H}$ in view of (2.3), (2.7) and the inequality $|\mathcal{C}_i \cap L''| \le |\mathcal{H}_i \cap L|$. If $k = t_x$ and $|\mathcal{H}_{t_x} \cap L| > |\mathcal{H}_{t_{d-m}} \cap L|$ then we need to check that

$$\eta |\mathcal{H}_{t_x} \cap L| \le |\mathcal{H}_i \cap L| \qquad \text{for } 1 \le i \le d. \tag{2.8}$$

For $i \in S$ inequality (2.8) is satisfied by (2.3) combined with (2.5) and (2.6). For $i = t_1, \dots, t_x$ it is satisfied by (2.4) and (2.6). When $|\mathcal{H}_{t_x} \cap L| > |\mathcal{H}_{t_{d-m}} \cap L|$, then (2.6) and (2.7) combined with the ordering (2.2) imply

$$|\mathcal{C}_{t_x} \cap L''| = |\mathcal{C}_{t_{x+1}} \cap L''| = \dots = |\mathcal{C}_{t_{d-m}} \cap L''|.$$

Therefore $|\mathcal{H}_{t_x} \cap L| = |\mathcal{C}_{t_x} \cap L''| + 1 = |\mathcal{H}_i \cap L| + 1$ for all $i = t_{x+1}, \dots, t_{d-m}$, so inequality (2.8) is satisfied, even with $1/2$ on the left-hand side, for $i = t_{x+1}, \dots, t_{d-m}$.

II.2.A.4. $|\mathcal{H} \cap L'| \ge d$ and $|\mathcal{H} \cap L''| < d$. This case is analogous to II.2.A.3.

Next, we proceed with case II.2.B.

II.2.B.1. $|\mathcal{H} \cap L'| \ge d$ and $|\mathcal{H} \cap L''| \ge d$. Recall that—by the condition defining case II.2.B—$\mathcal{C} \cap L' = \emptyset$ and $\mathcal{C} \cap L'' \ne \emptyset$. Then by the induction hypothesis all intervals in $\mathcal{U} \cap L''$ are already coloured, but those in $\mathcal{U} \cap L'$ are uncoloured.

We need to colour the intervals in $\mathcal{U} \cap L'$. It is enough to colour them modulo d (see (1.5)). After this, we get a colouring of $\mathcal{H} \cap L$ such that both L' and L'' satisfy (1.3). To check that L satisfies (1.3) as well, we proceed as in case II.2.A.1.

II.2.B.2. $|\mathcal{H}\cap L'| < d$ and $|\mathcal{H}\cap L''| < d$. The induction hypothesis states that the intervals in $\mathcal{U}\cap L'$ and in $\mathcal{U}\cap L''$ are uncoloured. Now we proceed as in case II.2.A.2.

II.2.B.3. $|\mathcal{H}\cap L'| < d$ and $|\mathcal{H}\cap L''| \geq d$. Recall that $\mathcal{C}\cap L'' \neq \emptyset$. Therefore, by the previsibility assumption, $\mathcal{U}\cap L'' = \emptyset$. Since $\mathcal{C}\cap L' = 0$, by the induction hypothesis the intervals in $\mathcal{U}\cap L'$ are uncoloured, and we need to colour them in case $\mathcal{U}\cap L' \neq \emptyset$. This is done as in case II.2.A.3.

II.2.B.4. $|\mathcal{H}\cap L'| \geq d$ and $|\mathcal{H}\cap L''| < d$. Recall that $\mathcal{C}\cap L' = \emptyset$. In this case the induction hypothesis says that the intervals in both $\mathcal{U}\cap L'$ and $\mathcal{U}\cap L''$ are uncoloured. We need to colour them all. First, we colour the intervals in $\mathcal{U}\cap L''$ by giving each of them a different colour which was not used to colour $\mathcal{C}\cap L''$. This is possible since $|\mathcal{H}\cap L''| < d$. Then we colour the intervals in $\mathcal{U}\cap L'$ by the modulo d method as in (1.5), but starting with colours which have not been used to colour intervals in $\mathcal{H}\cap L''$. In this way we get a modulo d colouring of $\mathcal{H}\cap L$, which is $(1/2, d)$-homogeneous, hence also (η, d)-homogeneous.

This completes the proof of Theorem 2.2. ∎

3. A colouring problem without solution. Here we analyze the role of the previsibility assumption in Theorem 2.2.

Throughout this section we take $d = 2^a$, $a \in \mathbb{N}$, and $\eta = 1/n$ with $n \in \mathbb{N}$ and $j \geq n + a + 1$.

We will define a sequence of collections $\mathcal{C}(0) \subset \mathcal{C}(1) \subset \cdots \subset \mathcal{C}(n) \subset \mathcal{D}_j$, of size $|\mathcal{C}(k)| = k + d$. The initial collection $\mathcal{C}(0)$ is of size d, hence—up to permutation—it has a unique (η, d)-homogeneous colouring. Then we will check that for $1 \leq k \leq n-1$, there is a unique (η, d)-homogeneous colouring of $\mathcal{C}(k)$ keeping the previously determined (η, d)-homogeneous colouring of $\mathcal{C}(k-1)$. Finally, we will see that there is no (η, d)-homogeneous colouring of $\mathcal{C}(n)$ keeping the previous (η, d)-homogeneous colouring of $\mathcal{C}(n-1)$.

In our example below, the parameter d determines the size of the initial collection $\mathcal{C}(0)$, while the parameter η determines the number of steps needed to arrive at a problem of consistent colouring without solution.

To define the desired sequence of collections, take a chain of dyadic intervals

$$L_1 \subset \cdots \subset L_{n+2}, \qquad L_i \in \mathcal{D}_{j-a-i+1}.$$

Then $|L_i| = \frac{1}{2}|L_{i+1}|$, and let P_i be the dyadic brother of L_i in L_{i+1}, $i = 1, \ldots, n+1$. Thus $P_i = L_{i+1} \setminus L_i$.

Now, take two sets of intervals from $\mathcal{D}_j$:

$$I_1, \ldots, I_{d-1} \in \mathcal{D}_j \quad \text{such that} \quad I_i \subset L_1 \text{ for } i = 1, \ldots, d-1,$$
$$J_1, \ldots, J_{n+1} \in \mathcal{D}_j \quad \text{such that} \quad J_i \subset P_i \text{ for } i = 1, \ldots, n+1.$$

Consider the following sequence of collections:

$$\mathcal{C}(k) = \{I_1, \dots, I_{d-1}\} \cup \{J_{n-k+1}, \dots, J_{n+1}\} \quad \text{for } k = 0, \dots, n. \tag{3.1}$$

PROPOSITION 3.1. *The sequence of collections $\mathcal{C}(k)$, $0 \le k \le n$, defined by* (3.1) *is increasing and it has the following properties:*

(A) Stage 0. *There exists exactly one—up to permutation—(η, d)-homogeneous colouring of $\mathcal{C}(0)$,*

$$\mathcal{C}(0) = \mathcal{C}_1(0) \cup \dots \cup \mathcal{C}_d(0).$$

(B) Stage k, $1 \le k \le n-1$. *Let*

$$\mathcal{C}(k-1) = \mathcal{C}_1(k-1) \cup \dots \cup \mathcal{C}_d(k-1)$$

be the (η, d)-homogeneous colouring of $\mathcal{C}(k-1)$, obtained at stage $k-1$. Then there exists exactly one (η, d)-homogeneous colouring of $\mathcal{C}(k)$,

$$\mathcal{C}(k) = \mathcal{C}_1(k) \cup \dots \cup \mathcal{C}_d(k),$$

such that

$$\mathcal{C}_i(k-1) \subset \mathcal{C}_i(k) \quad \textit{for } 1 \le i \le d.$$

(C) Stage n. *Let*

$$\mathcal{C}(n-1) = \mathcal{C}_1(n-1) \cup \dots \cup \mathcal{C}_d(n-1)$$

be the (η, d)-homogeneous colouring of $\mathcal{C}(n-1)$, obtained at stage $n-1$. There does not exist an (η, d)-homogeneous colouring of $\mathcal{C}(n)$,

$$\mathcal{C}(n) = \mathcal{C}_1(n) \cup \dots \cup \mathcal{C}_d(n),$$

such that

$$\mathcal{C}_i(n-1) \subset \mathcal{C}_i(n) \quad \textit{for } 1 \le i \le d.$$

Proof. Verification of (A). Consider possible colourings of $\mathcal{C}(0)$. Take L_{n+2} as a testing interval. Observe that $|\mathcal{C}(0)| = |\mathcal{C}(0) \cap L_{n+2}| = d$, so if we want to have (η, d)-homogeneity, we must have (1.4) and therefore $|\mathcal{C}_i(0) \cap L_{n+2}| = 1$ for $i = 1, \dots, d$. Without loss of generality we can assume that J_{n+1} has colour 1, and each I_i with $i = 1, \dots, d-1$ has colour $i+1$. Therefore for $\mathcal{C}(0)$ and each testing interval $L \subset L_{n+2}$ we have $|\mathcal{C}_i(0) \cap L| \le 1$ for $1 \le i \le d$.

The basic observation. Our example is based on iterating the following basic observation. Let $k \le n$. Assume that $\mathcal{C}(k)$ has an (η, d)-homogeneous decomposition into $\mathcal{C}_1(k), \dots, \mathcal{C}_d(k)$, so that

$$\mathcal{C}_1(0) \subset \mathcal{C}_1(k), \quad \dots, \quad \mathcal{C}_d(0) \subset \mathcal{C}_d(k).$$

Then necessarily

$$J_{n-k+1} \text{ must have colour } 1. \tag{3.2}$$

Verification of (3.2). We already know that J_{n+1} has to have colour 1. To check the claim for J_{n-k+1}, $k = 1, \ldots, n$, we consider the pair of collections $\mathcal{C}(0) \subset \mathcal{C}(k)$:

$$\mathcal{C}(k) = \mathcal{C}(0) \cup \{J_{n-k+1}, \ldots, J_n\},$$

and the testing interval L_{n-k+2}. Elements of $\mathcal{C}(0)$ included in L_{n-k+2} are $I_1, \ldots, I_{d-1}$. In addition, $J_{n-k+1} \subset P_{n-k+1} \subset L_{n-k+2}$, while $J_{n-k+2}, \ldots, J_n \not\subset L_{n-k+2}$. Therefore

$$|\mathcal{C}(0) \cap L_{n-k+2}| = d-1, \qquad |\mathcal{C}(k) \cap L_{n-k+2}| = d,$$
$$\mathcal{C}_1(0) \cap L_{n-k+2} = \emptyset, \qquad |\mathcal{C}_i(0) \cap L_{n-k+2}| = 1 \quad \text{for } i = 2, \ldots, d.$$

Therefore, (1.4) of the (η, d)-homogeneity condition for $\mathcal{C}(k)$ implies that J_{n-k+1} is of colour 1.

Verification of (B). Recall that for $0 \le k \le n$,

$$\mathcal{C}(k) = \{I_1, \ldots, I_{d-1}\} \cup \{J_{n-k+1}, \ldots, J_{n+1}\}.$$

Moreover, by (3.2), the only possible $(1/n, d)$-homogeneous decomposition of $\mathcal{C}(k)$ is

$$\mathcal{C}_1(k) = \{J_{n-k+1}, \ldots, J_{n+1}\}, \qquad \mathcal{C}_i(k) = \{I_{i-1}\} \quad \text{for } 2 \le i \le d.$$

Let us check that for $0 \le k \le n-1$, the above decomposition of $\mathcal{C}(k)$ is indeed $(1/n, d)$-homogeneous. We present a detailed proof for $k = n-1$, since the cases $k \le n-1$ are fully analogous.

First, take L_s, $s = 3, \ldots, n+2$, as a testing interval. Then the elements of $\mathcal{C}(n-1)$ included in L_s are $I_1, \ldots, I_{d-1}$ and $J_2, \ldots, J_{s-1}$. Therefore

$$|\mathcal{C}(n-1) \cap L_s| = s + d - 3,$$

and

$$|\mathcal{C}_1(n-1) \cap L_s| = s - 2, \qquad |\mathcal{C}_i(n-1) \cap L_s| = 1 \quad \text{for } i = 2, \ldots, d.$$

Therefore

$$\frac{1}{n} \max_{1 \le i \le d} |\mathcal{C}_i(n-1) \cap L_s| \le \min_{1 \le i \le d} |\mathcal{C}_i(n-1) \cap L_s|, \qquad s = 3, \ldots, n+2.$$

Next take L_2 as a testing interval. Then the elements of $\mathcal{C}(n-1)$ included in L_2 are $I_1, \ldots, I_{d-1}$, so $|\mathcal{C}(n-1) \cap L_2| = d-1$,

$$\mathcal{C}_1(n-1) \cap L_2 = \emptyset, \qquad |\mathcal{C}_i(n-1) \cap L_2| = 1 \quad \text{for } i = 2, \ldots, d.$$

Therefore L_2 also satisfies (1.4) of the $(1/n, d)$-homogeneity condition for $\mathcal{C}(n-1)$. Consequently, $L_1, P_1 \subset L_2$ also satisfy these conditions.

Finally, take P_k, $k = 2, \ldots, n+1$, as a testing interval. The only element of $\mathcal{C}(n-1)$ included in P_k is J_k, so $|\mathcal{C}(n-1) \cap P_k| = 1$, and more precisely

$$|\mathcal{C}_1(n-1) \cap P_k| = 1, \qquad \mathcal{C}_i(n-1) \cap P_k = \emptyset \quad \text{for } i = 2, \ldots, d.$$

Thus, P_k (and hence each testing interval included in P_k) satisfies (1.4) of the $(1/n, d)$-homogeneity condition for $\mathcal{C}(n-1)$.

Verification of (C). Consider $\mathcal{C}(n-1)$ and $\mathcal{C}(n) = \mathcal{C}(n-1) \cup \{J_1\}$. Recall that

$$\mathcal{C}(n) = \{I_1, \ldots, I_{d-1}\} \cup \{J_1, \ldots, J_{n+1}\}.$$

Take L_{n+2} as a testing interval. All intervals from $\mathcal{C}(n)$ are included in L_{n+2}, and the colouring yields

$$|\mathcal{C}_1(n) \cap L_{n+2}| = n+1, \qquad |\mathcal{C}_i(n) \cap L_{n+2}| = 1 \quad \text{for } i = 2, \ldots, d.$$

For $\mathcal{C}(n)$ and L_{n+2} we have to consider (1.3) of the $(1/n, d)$-homogeneity condition. But the above formulae mean that for $\mathcal{C}(n)$ and the testing interval L_{n+2}, the condition (1.3) is satisfied with $\eta' = 1/(n+1)$, but not with $\eta = 1/n$. ■

REMARK. For $0 \le k \le n-1$, the pair of collections $(\mathcal{C}(k), \mathcal{U}(k))$, where $\mathcal{U}(k) = \mathcal{C}(k+1) \setminus \mathcal{C}(k)$, is not d-previsible. Nevertheless, the colouring problem has a solution for $0 \le k \le n-2$.

The examples of Proposition 3.1 grew out of counterexamples to classical martingale inequalities (see [12, p. 105], or [8, p. 156]).

4. A two-person game. The problem of consistent colourings gives rise to the following two-person game. The game is played by two players with collections of coloured dyadic intervals in $\mathcal{D}_j$ for a fixed $j \in \mathbb{N}$. It starts by fixing $\eta > 0$, $d \in \mathbb{N}$, and a subcollection

$$\mathcal{C}(0) \subset \mathcal{D}_j$$

with an (η, d)-homogeneous colouring

$$\mathcal{C}_1(0), \ldots, \mathcal{C}_d(0),$$

according to Definition 1.1. The rules of the game are as follows:

1. In the first stage, Player A chooses a collection $\mathcal{C}(1)$ with $\mathcal{C}(1) \supsetneq \mathcal{C}(0)$ and $\mathcal{C}(1) \subset \mathcal{D}_j$. Player B determines an (η, d)-homogeneous colouring of $\mathcal{C}(1)$ that preserves the colours of $\mathcal{C}(0)$.
2. At stage n, Player A chooses $\mathcal{C}(n)$ with $\mathcal{C}(n) \supsetneq \mathcal{C}(n-1)$ and $\mathcal{C}(n) \subset \mathcal{D}_j$. Player B determines an (η, d)-homogeneous colouring of $\mathcal{C}(n)$ preserving the colours of $\mathcal{C}(n-1)$.
3. The game stops at stage n if either $\mathcal{C}(n-1) = \mathcal{D}_j$, and then Player B is the winner, or else if there does not exist an (η, d)-homogeneous colouring of $\mathcal{C}(n)$ that preserves the colours of $\mathcal{C}(n-1)$, in which case Player A is the winner.

The results of this paper enable us to predict the outcome of the game as follows. If we do not pose any constraints on the choice of the collections

$\mathcal{C}(k)$, then the example in Section 3 and Proposition 3.1 describe a realization of our game where Player A has a winning strategy.

However, if we restrict the moves of Player A by imposing that $(\mathcal{C}(k-1), \mathcal{C}(k) \setminus \mathcal{C}(k-1))$ is d-previsible, then with the aid of Theorem 2.2 and its proof, Player B always has a winning strategy. In case the moves of Player A are restricted by d-previsibility, we modify the stopping rule accordingly: Player B is the winner at stage n if there does not exist $\mathcal{C}(n) \subset \mathcal{D}_j$ such that $\mathcal{C}(n) \supsetneq \mathcal{C}(n-1)$ and the pair $(\mathcal{C}(n-1), \mathcal{C}(n) \setminus \mathcal{C}(n-1))$ is d-previsible.

Acknowledgements. A. Kamont was partially supported by SPADE2 program at IMPAN and NCN grant N N201 607840. Paul F. X. Müller was partially supported by FWF P 20166-N18 and FWF P 23987.

References

[1] T. Figiel, *On equivalence of some bases to the Haar system in spaces of vector-valued functions*, Bull. Polish Acad. Sci. Math. 36 (1988), 119–131.

[2] T. Figiel, *Singular integral operators: a martingale approach*, in: Geometry of Banach Spaces (Strobl, 1989), London Math. Soc. Lecture Note Ser. 158, Cambridge Univ. Press, Cambridge, 1990, 95–110.

[3] J. B. Garnett, *Bounded Analytic Functions*, Pure Appl. Math. 96, Academic Press, New York, 1981.

[4] P. W. Jones, *Carleson measures and the Fefferman–Stein decomposition of* BMO($\mathbb{R}$), Ann. of Math. (2) 111 (1980), 197–208.

[5] P. W. Jones, *BMO and the Banach space approximation problem*, Amer. J. Math. 107 (1985), 853–893.

[6] A. Kamont and P. F. X. Müller, *Rearrangements with supporting trees, isomorphisms and shift operators*, Math. Z. 274 (2013), 57–83.

[7] J. Lee, P. F. X. Müller and S. Müller, *Compensated compactness, separately convex functions and interpolatory estimates between Riesz transforms and Haar projections*, Comm. Partial Differential Equations 36 (2011), 547–601.

[8] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces. II. Function Spaces*, Ergeb. Math. Grenzgeb. 97, Springer, Berlin, 1979.

[9] P. F. X. Müller, *Rearrangements of the Haar system that preserve BMO*, Proc. London Math. Soc. (3) 75 (1997), 600–618.

[10] P. F. X. Müller, *Isomorphisms between H^1 Spaces*, IMPAN Monogr. Mat. (N.S.) 66, Birkhäuser, Basel, 2005.

[11] K. Smela, *Continuous rearrangements of the Haar system in H_p for* $0 < p < \infty$, Studia Math. 189 (2008), 189–199.

[12] E. M. Stein, *Topics in Harmonic Analysis Related to the Littlewood–Paley Theory*, Ann. of Math. Stud. 63, Princeton Univ. Press, Princeton, NJ, and Univ. of Tokyo Press, Tokyo, 1970.

Anna Kamont
Institute of Mathematics
Polish Academy of Sciences
Wita Stwosza 57
80-952 Gdańsk, Poland
E-mail: A.Kamont@impan.gda.pl

Paul F. X. Müller
Department of Analysis
J. Kepler University
A-4040 Linz, Austria
E-mail: paul.mueller@jku.at

Received December 10, 2013;
received in final form July 22, 2014 (7973)