BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 62, No. 2, 2014

On Exponential Stability of Volterra Difference Equations with Infinite Delay

by

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Summary. General nonlinear Volterra difference equations with infinite delay are considered. A new explicit criterion for global exponential stability is given. Furthermore, we present a stability bound for equations subject to nonlinear perturbations. Two examples are given to illustrate the results obtained.

1. Introduction. Volterra difference equations are widely used in the modeling of processes in continuous mechanics and biomechanics, problems of control and estimations, and some schemes of numerical solutions of integral and integro-differential equations (see e.g. [BH], [E05], [L], [MW]).

In particular, problems of stability of Volterra difference equations have attracted much attention during the last twenty years (see e.g. [ACF]–[ES], [KCT], [M97], [NNSM]–[SB] and references therein).

In the literature, various methods have been used to investigate stability of Volterra difference equations, such as Lyapunov functions, comparison theorems, Bohl–Perron type theorems, topological methods, etc. (see e.g. [BK12], [CKRV98], [CKRV00], [E05], [E09], [KCT]). Recently, E. Braverman and I. M. Karabash [BK12] gave some Bohl–Perron-type stability theorems which show relations between exponential stability and (l^p, l^q)-stability of Volterra difference equations with infinite delay. Using Lyapunov functionals, Y. N. Raffoul and Y. M. Dib [RD] derived conditions for asymptotic stability and exponential stability of Volterra discrete-time equations with finite delay subject to nonlinear perturbations. For a nice survey, see [KCT].

2010 *Mathematics Subject Classification*: Primary 39A30; Secondary 93D20.
Key words and phrases: global exponential stability, infinite delay, phase spaces, Volterra difference equations.

DOI: 10.4064/ba62-2-3

To the best of our knowledge, there are not many explicit criteria for exponential stability of nonlinear Volterra difference equations with infinite delay (see [BK12], [E09], [KCT]). In this paper, we present a new approach to global exponential stability of Volterra difference equations with infinite delay. Our approach is based on the celebrated Perron–Frobenius theorem and the comparison principle. Consequently, we get a new explicit criterion for global exponential stability of the zero solution of the general time-varying Volterra difference equation with infinite delay. Furthermore, we derive an explicit stability bound for equations subject to nonlinear time-varying perturbations. Two examples are given to illustrate the results obtained.

We now present some notation and preliminary results. For a natural number k, denote $\underline{k} := \{1, \dots, k\}$. Write $\mathbb{Z}_+ := \{k \in \mathbb{Z} : k \geq 0\}$ and $\mathbb{Z}_- := \{k \in \mathbb{Z} : k \leq 0\}$. For $l, q \in \mathbb{Z}_+$, let $\mathbb{R}^{l\times q}$ be the set of all real $l \times q$-matrices and $\mathbb{R}^{l\times q}_+$ be the set of all $l \times q$-matrices with nonnegative entries. Inequalities between real matrices or vectors will be understood componentwise, i.e. $A \geq B$ if and only if $a_{ij} \geq b_{ij}$, $i \in \underline{l}$, $j \in \underline{q}$, where $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{l\times q}$. Furthermore, $A \gg B$ means that $a_{ij} > b_{ij}$ for all $i \in \underline{l}$, $j \in \underline{q}$. If $x = (x_1, \dots, x_m)^T \in \mathbb{R}^m$ and $P = (p_{ij}) \in \mathbb{R}^{l\times q}$ we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. For any matrix $A \in \mathbb{R}^{m\times m}$, the spectral radius of A is denoted by $\rho(A) = \max\{|z| : z \in \sigma(A)\}$, where $\sigma(A) := \{z \in \mathbb{C} : \det(zI_m - A) = 0\}$ is the set of all eigenvalues of A.

A norm $\|\cdot\|$ on $\mathbb{R}^m$ is said to be *monotonic* if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in \mathbb{R}^m$. Throughout the paper, the norm of vectors is assumed to be monotonic and the norm $\|M\|$ of a matrix $M \in \mathbb{R}^{l\times q}$ is always understood as the operator norm defined by $\|M\| = \max_{\|y\|=1} \|My\|$, where $\mathbb{R}^l$ and $\mathbb{R}^q$ are provided with some monotonic vector norms. The operator norm $\|\cdot\|$ has the following property (see e.g. [HS]):

$$\|P\| \leq \big\||P|\big\| \leq \|Q\| \quad \text{whenever} \quad |P| \leq Q, \qquad \text{for } P \in \mathbb{R}^{l\times q},\ Q \in \mathbb{R}^{l\times q}_+.$$

Finally, for given $\alpha > 1$, set

$$l^\alpha(\mathbb{R}^{m\times m}) := \Big\{(U(k))_k : U(k) \in \mathbb{R}^{m\times m}\ (k \in \mathbb{Z}),\ \sum_{k=0}^{\infty} \|U(k)\|\alpha^k < \infty\Big\}.$$

The next theorem summarizes some basic properties of nonnegative matrices which will be used in what follows.

Theorem 1.1 ([HS]). *Let $M \in \mathbb{R}^{m\times m}_+$ and $t \in \mathbb{R}$. Then*

- (i) (Perron–Frobenius) *$\rho(M)$ is an eigenvalue of M and there exists a nonnegative eigenvector $x \in \mathbb{R}^m$, $x \neq 0$, such that $Mx = \rho(M)x$.*
- (ii) *$(tI_m - M)^{-1}$ exists and is nonnegative if and only if $t > \rho(M)$.*

The following is immediate from Theorem 1.1.

THEOREM 1.2 ([NH]). *Let $M \in \mathbb{R}_+^{m\times m}$. Then the following statements are equivalent:*

(i) $\rho(M) < 1$.
(ii) $\exists p \in \mathbb{R}^m$, $p \gg 0 : Mp \ll p$.
(iii) $(I_m - M)^{-1} \geq 0$.

2. An explicit criterion for global exponential stability. Consider a nonlinear Volterra difference equation with infinite delay of the form

$$x(n+1) = F\Big(n, x(n), \sum_{k=-\infty}^{n} G(n,k,x(k))\Big), \qquad n \geq n_0, \tag{2.1}$$

where $F(\cdot,\cdot,\cdot) : \mathbb{Z}_+ \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ and $G(\cdot,\cdot,\cdot) : \mathbb{Z}_+ \times \mathbb{Z} \times \mathbb{R}^m \to \mathbb{R}^m$ are given functions such that $F(n,0,0) = 0$ for all $n \in \mathbb{Z}_+$ and $G(n,k,0) = 0$ for $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, $n \geq k$ (i.e. $\xi = 0$ is an equilibrium point of the equation (2.1)).

For given $\gamma > 1$, denote

$$\mathcal{B}_\gamma := \Big\{\varphi(\cdot) : \mathbb{Z}_- \to \mathbb{R}^m : \sup_{k\in\mathbb{Z}_-} \|\varphi(k)\|\gamma^k < \infty\Big\},$$

the phase space, a Banach space with the norm $\|\varphi\|_\gamma = \sup_{k\in\mathbb{Z}_-} \|\varphi(k)\|\gamma^k$ (see e.g. [ACF]). With fixed $n_0 \in \mathbb{Z}_+$ and given $\varphi \in \mathcal{B}_\gamma$, consider for (2.1) an initial condition of the form

$$x(n_0 + k) = \varphi(k), \qquad k \in \mathbb{Z}_-. \tag{2.2}$$

Equation (2.1) can be written as a functional difference equation of the form

$$x(n+1) = \mathcal{N}\big(n, x_n\big), \qquad n \geq n_0, \tag{2.3}$$

where $x_n \in \mathcal{B}_\gamma$ is defined by $x_n(k) := x(n+k)$, $k \in \mathbb{Z}_-$, and $\mathcal{N}(\cdot,\cdot) : \mathbb{Z}_+ \times \mathcal{B}_\gamma \to \mathbb{R}^m$ is given by

$$\mathcal{N}(n,\varphi) := F\Big(n, \varphi(0), \sum_{k=-\infty}^{n} G(n,k,\varphi(k-n))\Big), \qquad (n,\varphi) \in \mathbb{Z}_+ \times \mathcal{B}_\gamma.$$

Throughout we assume that

(H_1) There exist $A, B \in \mathbb{R}_+^{m\times m}$ such that

$$|F(n,x,y)| \leq A|x| + B|y|, \qquad \forall n \in \mathbb{Z}_+,\ \forall x,y \in \mathbb{R}^m.$$

(H_2) There exists $C(\cdot) : \mathbb{Z}_+ \to \mathbb{R}_+^{m\times m}$ such that

$$\begin{cases} |G(n,k,x)| \leq C(n-k)|x|, & \forall n \in \mathbb{Z}_+,\ \forall k \in \mathbb{Z},\ n \geq k,\ \forall x,y \in \mathbb{R}^m; \\ (C(n))_n \in l^\gamma(\mathbb{R}^{m\times m}) & \text{for some } \gamma > 1. \end{cases}$$

By virtue of (H_1) and (H_2), $\mathcal{N}(\cdot,\cdot)$ is well-defined on $\mathbb{Z}_+ \times \mathcal{B}_\gamma$. Furthermore, the initial value problem (2.1)–(2.2) always has a unique solution, denoted by $x(\cdot, n_0, \varphi)$.

Definition 2.1. The zero solution of (2.1) is said to be *globally exponentially stable* (briefly, GES) in $\mathcal{B}_\gamma$ if there exist $K > 0$ and $\lambda \in (0,1)$ such that

$$\|x(n, n_0, \varphi)\| \le K\lambda^{n-n_0}\|\varphi\|_\gamma, \qquad \forall n \ge n_0,\, \forall \varphi \in \mathcal{B}_\gamma. \tag{2.4}$$

We are now in a position to state the first result of this paper; its proof is given in the Appendix.

Theorem 2.2. *Suppose* (H_1) *and* (H_2) *hold. If* $\rho(A + B\sum_{k=0}^{\infty} C(k)) < 1$ *then the zero solution of* (2.1) *is GES in* $\mathcal{B}_{\gamma_0}$ *for some* $\gamma_0 \in (1, \gamma]$.

Remark 2.3. Roughly speaking, (H_1)–(H_2) means that the nonlinear time-varying equation (2.1) is "bounded above" by the linear time-invariant equation

$$y(n+1) = Ay(n) + \sum_{k=-\infty}^{n} BC(n-k)x(k), \qquad n \in \mathbb{Z}_+. \tag{2.5}$$

Then Theorem 2.2 says that if (2.5) is GES in $\mathcal{B}_{\gamma_0}$ for some $\gamma_0 \in (1,\gamma]$ ($(C(n))_n \in l^\gamma(\mathbb{R}^{m\times m})$, $\gamma > 1$ and $\rho(A + B\sum_{k=0}^{\infty} C(k)) < 1$), then the zero solution of (2.1) is GES in $\mathcal{B}_{\gamma_0}$, too.

In particular, for the time-varying Volterra difference equation with infinite delay

$$y(n+1) = A(n)y(n) + \sum_{k=-\infty}^{n} B\big(n, k, y(k)\big), \qquad n \in \mathbb{Z}_+, \tag{2.6}$$

the following is immediate from Theorem 2.2.

Corollary 2.4. *Suppose there exist* $A \in \mathbb{R}_+^{m\times m}$ *and* $C(\cdot) : \mathbb{Z}_+ \to \mathbb{R}_+^m$ *such that*

$$|A(n)| \le A \text{ and } |B(n,k,x)| \le C(n-k)|x|, \quad \forall n, k \in \mathbb{Z}_+,\, n \ge k,\, \forall x \in \mathbb{R}^m.$$

If $(C(n))_n \in l^\gamma(\mathbb{R}^{m\times m})$ *for some* $\gamma > 1$ *and* $\rho(A + \sum_{k=0}^{\infty} C(k)) < 1$ *then* (2.6) *is GES in* $\mathcal{B}_{\gamma_0}$ *for some* $\gamma_0 \in (1,\gamma]$.

Remark 2.5. In particular, when $A(\cdot) \equiv A \in \mathbb{R}^{m\times m}$ and $B(n,k,x) := C(n-k)x$, $n \ge k$, $x \in \mathbb{R}^m$, (2.6) reduces to a linear time-invariant equation of convolution type

$$y(n+1) = Ay(n) + \sum_{k=-\infty}^{n} C(n-k)y(k), \qquad n \in \mathbb{Z}_+. \tag{2.7}$$

Assume that $(C(n))_n \in l^1(\mathbb{R}^{m\times m})$. It is well-known that (2.7) is uniformly asymptotically stable if and only if $\det\big(I_m z - A - \sum_{i=0}^{\infty} z^{-i}C(i)\big) \neq 0$ for all $z \in \mathbb{C}$ with $|z| \geq 1$ (see e.g. [E09, Theorem 3.2]). Note that this condition holds if $\rho(|A|+\sum_{i=0}^{\infty}|C(i)|) < 1$ (see e.g. [NNSM, Theorem 3.10]). Moreover, if (2.7) is uniformly asymptotically stable then it is GES if and only if $((C(n))_n \in l^{\gamma}(\mathbb{R}^{m\times m})$ for some $\gamma > 1$ (see e.g. [ES]). Therefore, Corollary 2.4 can be seen as a "generalization" of [ES, Theorem 5] to semilinear time-varying equations of the form (2.6).

We illustrate Theorem 2.2 by an example.

Example 2.6. Consider equation (2.1) in $\mathbb{R}^2$ with $F(\cdot,\cdot,\cdot)$ and $G(\cdot,\cdot,\cdot)$ defined by

$$F(n,x,y) = \begin{pmatrix} \sqrt{\frac{x_1^2}{144}\sin^2(nx_2) + \frac{x_2^2}{64} + y_1^2 e^{-ny_2^2}} \\ \arctan\big(\frac{nx_1}{6(n+1)} + \frac{11y_1}{6} + 3y_2\big) \end{pmatrix},$$

and

$$G(n,k,x) = \begin{pmatrix} \ln\big(1 + \frac{3e^{-k^2}a^{n-k}|x_1|}{4[3(n-k)+1][3(n-k)+4]}\big) \\ \frac{x_1}{16}\big(\frac{a}{2}\big)^{n-k} + \frac{3a^{n-k}x_2}{2[2(n-k)+3][2(n-k)+5]} \end{pmatrix},$$

where $a \in (0,1)$, $x := (x_1,x_2)^T$, $y := (y_1,y_2)^T \in \mathbb{R}^2$, $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, $n \geq k$.

Since

$$\sqrt{\frac{x_1^2}{144}\sin^2(nx_2) + \frac{x_2^2}{64} + y_1^2 e^{-ny_2^2}} \leq \frac{1}{12}|x_1| + \frac{1}{8}|x_2| + |y_1|,$$

$$\left|\arctan\left(\frac{nx_1}{6(n+1)} + \frac{11y_1}{6} + 3y_2\right)\right| \leq \frac{1}{6}|x_1| + \frac{11}{6}|y_1| + 3|y_2|,$$

$$\ln\left(1 + \frac{3e^{-k^2}a^{n-k}|x_1|}{4[3(n-k)+1][3(n-k)+4]}\right) \leq \frac{3a^{n-k}}{4[3(n-k)+1][3(n-k)+4]}|x_1|,$$

for all $x_i, y_i \in \mathbb{R}$, $i \in \{1,2\}$, $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, $n \geq k$, it follows that

$$|F(n,x,y)| \leq A|x| + B|y|, \qquad |G(n,k,x)| \leq C(n-k)|x|,$$

where

$$A := \begin{pmatrix} 1/12 & 1/8 \\ 1/6 & 0 \end{pmatrix}, \qquad B := \begin{pmatrix} 1 & 0 \\ 11/6 & 3 \end{pmatrix},$$

$$C(k) := a^k \begin{pmatrix} \frac{3}{4(3k+1)(3k+4)} & 0 \\ \frac{1}{2^{k+4}} & \frac{3}{2(2k+3)(2k+5)} \end{pmatrix}, \qquad k \in \mathbb{Z}_+.$$

Furthermore, since

$$\sum_{k=0}^{\infty} C(k) \leq \begin{pmatrix} 1/4 & 0 \\ 1/8 & 1/4 \end{pmatrix}, \qquad A + B\sum_{k=0}^{\infty} C(k) \leq M := \begin{pmatrix} 1/3 & 1/8 \\ 1 & 3/4 \end{pmatrix},$$

we have

$$\rho\Big(A + B\sum_{k=0}^{\infty} C(k)\Big) \le \rho(M) = \frac{1}{24}(13+\sqrt{97}) < 1.$$

Let $\mathbb{R}^{2\times 2}$ be endowed with the 1-norm (see e.g. [E05]), that is, for $A = (a_{ij}) \in \mathbb{R}^{2\times 2}$, $\|A\|_1 = \max_{1\le j\le 2}\sum_{i=1}^{2}|a_{ij}|$. Note that $\|C(k)\|_1 \le \frac{1}{4}a^k$ for all $k \in \mathbb{Z}_+$. Choosing $\beta \in (1, a^{-1})$, we have $\sum_{k=0}^{\infty}\|C(k)\|_1\beta^k < \infty$. Thus, $(C(k))_k \in l^{\beta}(\mathbb{R}^{2\times 2})$ and so the zero solution of (2.1) is GES in $\mathcal{B}_{\beta_0}$ for some $\beta_0 \in (1,\beta]$, by Theorem 2.2.

3. Stability of perturbed systems. Suppose (H_1)–(H_2) hold and $\rho(A + B\sum_{k=0}^{\infty}C(k)) < 1$. Thus, the zero solution of (2.1) is GES in $\mathcal{B}_{\gamma_0}$ for some $\gamma_0 \in (1,\gamma]$, by Theorem 2.2. Consider a perturbed equation of the form

$$\begin{aligned}(3.1)\qquad & x(n+1)\\ &= F\Big(n, x(n), \sum_{k=-\infty}^{n} G(n,k,x(k))\Big) + \widetilde{F}\Big(n, x(n), \sum_{k=-\infty}^{n} \widetilde{G}(n,k,x(k))\Big).\end{aligned}$$

Here $F(\cdot,\cdot,\cdot)$ and $G(\cdot,\cdot,\cdot)$ are as in (2.1), whereas

$$\widetilde{F}(\cdot,\cdot,\cdot) : \mathbb{Z}_+ \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \quad \text{and} \quad \widetilde{G}(\cdot,\cdot,\cdot) : \mathbb{Z}_+ \times \mathbb{Z} \times \mathbb{R}^m \to \mathbb{R}^m$$

are unknown. Furthermore, we assume that

$$(3.2)\quad \begin{cases} |\widetilde{F}(n,x,y)| \le \varDelta_1|x| + \varDelta_2|y|, & \forall n \in \mathbb{Z}_+,\ \forall x,y \in \mathbb{R}^m,\\ |\widetilde{G}(n,k,x)| \le \varDelta_G(n-k)|x|, & \forall n \in \mathbb{Z}_+,\ k \in \mathbb{Z},\ n \ge k,\ \forall x \in \mathbb{R}^m,\\ (\varDelta_G(n))_n \in l^{\beta}(\mathbb{R}^{m\times m}) & \text{for some } \beta > 1.\end{cases}$$

where $\varDelta_1 \in \mathbb{R}_+^{m\times m}$, $\varDelta_2 \in \mathbb{R}_+^{m\times m}$, $\varDelta_G(\cdot) : \mathbb{Z}_+ \to \mathbb{R}_+^{m\times m}$.

We are now in a position to state the main result of this section; its proof is given in the Appendix.

THEOREM 3.1. *Assume that* (3.2) *holds. If*

$$(3.3)\qquad \|\varDelta_1\| + \sum_{k=0}^{\infty}\|\varDelta_2\varDelta_G(k)\| < \frac{1}{\|(I_m - A - B\sum_{k=0}^{\infty}C(k))^{-1}\|},$$

then the zero solution of (3.1) *remains GES in* $\mathcal{B}_\omega$ *for some* $\omega \in (1,\beta_0]$ *with* $\beta_0 := \min\{\gamma,\beta\}$.

REMARK 3.2. When $\gamma = \beta = 1$, a similar result to Theorem 3.1, but for uniform asymptotical stability of linear time-invariant Volterra difference equations of convolution type, can be found in [NNSM, Theorem 5.2].

We illustrate Theorem 3.1 by an example.

Example 3.3. Consider a scalar Volterra difference equation with infinite delay

$$x(n+1) = F\Big(n, x(n), \sum_{k=-\infty}^{n} G(n,k,x(k))\Big), \qquad n \in \mathbb{Z}_+, \tag{3.4}$$

where

$$\begin{aligned} F(n,x,y) &= \arctan\Big(\frac{nx}{4n+1} + y\Big), \\ G(n,k,x) &= \frac{2a^{n-k}e^{-x^2}x}{[3(n-k)+1][3(n-k)+4]}, \end{aligned} \tag{3.5}$$

where $a \in (0,1)$, $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, $n \geq k$, $x, y \in \mathbb{R}$.

Since

$$\Big|\arctan\Big(\frac{nx}{4n+1} + y\Big)\Big| \leq \frac{1}{4}|x| + |y| \quad \text{and} \quad |2a^{n-k}e^{-x^2}| \leq 2a^{n-k}$$

for all $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, $x, y \in \mathbb{R}$, it follows that

$$|F(n,x,y)| \leq A|x| + B|y| \quad \text{and} \quad |G(n,k,x)| \leq C(n-k)|x|$$

for all $x, y \in \mathbb{R}$, $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, $n \geq k$, where

$$A = \frac{1}{4}, \qquad B = 1, \qquad C(n) = \frac{2a^n}{(3n+1)(3n+4)}, \qquad n \in \mathbb{Z}_+.$$

Clearly,

$$A + B\sum_{n=0}^{\infty} C(n) \leq \frac{1}{4} + \sum_{n=0}^{\infty} \frac{2}{(3n+1)(3n+4)} = \frac{1}{4} + \frac{2}{3} < 1$$

and $C(n) \leq \frac{1}{2}a^n$ for all $n \in \mathbb{Z}_+$. Choosing $\gamma \in (1, a^{-1})$, we have $a\gamma \in (0,1)$ and $\sum_{n=0}^{\infty} C(n)\gamma^n < \infty$. Thus, the zero solution of (3.4) is GES in $\mathcal{B}_{\gamma_0}$ for some $\gamma \in (1, \gamma]$, by Theorem 2.2.

Consider a perturbed Volterra difference equation with infinite delay of the form

$$\begin{aligned} x(n+1) \\ = F\Big(n, x(n), \sum_{k=-\infty}^{n} G(n,k,x(k))\Big) + \widetilde{F}\Big(n, x(n), \sum_{k=-\infty}^{n} \widetilde{G}(n,k,x(k))\Big), \end{aligned} \tag{3.6}$$

where $F(\cdot,\cdot,\cdot)$ and $G(\cdot,\cdot,\cdot)$ are as in (3.4)–(3.5), and $\widetilde{F}(\cdot,\cdot)$ and $\widetilde{G}(\cdot,\cdot,\cdot)$ are given by

$$\begin{aligned} \widetilde{F}(n,x,y) &= \frac{2^{-\sin^2(nx)}\Delta_1(n)x}{5} + \sin(\Delta_2 y), \\ \widetilde{G}(n,k,x) &= \cos k \ln(1 + \Delta_G(n-k)|x|), \end{aligned}$$

where $x, y \in \mathbb{R}$, $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, $n \geq k$; $\mathit{\Delta}_1(\cdot) : \mathbb{Z}_+ \to \mathbb{R}$ and $\mathit{\Delta}_G(\cdot) : \mathbb{Z}_+ \to \mathbb{R}_+$ are unknown functions satisfying $\sup_{n\in\mathbb{Z}_+} |\mathit{\Delta}_1(n)| < \infty$ and $(\mathit{\Delta}_G(n))_n$ is in $l^{\beta_1}(\mathbb{R})$ for some $\beta_1 > 1$; $\mathit{\Delta}_2$ is a positive real parameter.

It is clear that

$$|\widetilde{F}(n, x, y)| \leq \mathit{\Delta}_1|x| + \mathit{\Delta}_2|y| \quad \text{and} \quad |\widetilde{G}(n, k, x)| \leq \mathit{\Delta}_G(n-k)|x|,$$

for all $x \in \mathbb{R}$, $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, $n \geq k$, where $\mathit{\Delta}_1 := \frac{1}{5}\sup_{n\in\mathbb{Z}_+} |\mathit{\Delta}_1(n)|$.

By Theorem 3.1, the zero solution of (3.6) is GES in $\mathcal{B}_{\beta_2}$ for some $\beta_2 > 1$ provided

$$\mathit{\Delta}_1 + \mathit{\Delta}_2 \sum_{k=0}^{\infty} \mathit{\Delta}_G(k) < \frac{1}{\left(1 - \dfrac{1}{4} - \displaystyle\sum_{k=0}^{\infty} \frac{2a^k}{(3k+1)(3k+4)}\right)^{-1}},$$

or equivalently,

$$\mathit{\Delta}_1 + \mathit{\Delta}_2 \sum_{k=0}^{\infty} \mathit{\Delta}_G(k) < 1 - p,$$

where

$$p := \frac{1}{4} + \sum_{k=0}^{\infty} \frac{2a^k}{(3k+1)(3k+4)} \in (0,1) \quad \text{and} \quad a \in (0,1).$$

4. Appendix: Proofs of Theorems 2.2 and 3.1

4.1. Proof of Theorem 2.2. We show that the zero solution of (2.1) is GES in $\mathcal{B}_{\gamma_0}$ for some $\gamma_0 \in (1, \gamma]$ provided $\rho(A + B\sum_{k=0}^{\infty} C(k)) < 1$.

Since $\rho(A + B\sum_{k=0}^{\infty} C(k)) < 1$ and $(C(n))_n \in l^{\gamma}(\mathbb{R}^{m\times m})$ with $\gamma > 1$, it follows that

$$\rho\Big(A + B\sum_{k=0}^{\infty} C(k)\gamma_1^k\Big) < 1 \qquad \text{for some } \gamma_1 \in (1, \gamma],$$

by continuity of the spectral radius. Furthermore, since $A + B\sum_{k=0}^{\infty} C(k)\gamma_1^k \geq 0$ and $\rho(A+B\sum_{k=0}^{\infty} C(k)\gamma_1^k) < 1$, by Theorem 1.2(ii), there exists $p \in \mathbb{R}^m_+$, $p \gg 0$, such that $(A + B\sum_{k=0}^{\infty} C(k)\gamma_1^k)p \ll p$. By continuity, there exists $\gamma_2 > 1$ such that

$$\Big(A + B\sum_{k=0}^{\infty} C(k)\gamma_1^k\Big)p \leq \gamma_2^{-1}p.$$

Let $\gamma_0 := \min\{\gamma_1, \gamma_2\}$. Clearly, $\gamma_0 > 1$ and

$$\Big(A + B\sum_{k=0}^{\infty} C(k)\gamma_0^k\Big)p \leq \gamma_0^{-1}p. \tag{4.1}$$

We first show that if $\gamma_0 \geq \gamma$ then the zero solution of (2.1) is GES in $\mathcal{B}_{\gamma}$. The proof consists of two steps.

Step I. We show that

$$\|x(n,n_0,\varphi)\| \le K\lambda^{n-n_0}, \quad \forall n,n_0 \in \mathbb{Z}_+, n\ge n_0, \forall \varphi \in \mathcal{B}_\gamma, \|\varphi\|_\gamma \le 1,$$

for some $K>0$ and $\lambda \in (0,1)$.

Since $p \gg 0$, there exists $M>1$ (independent of n_0) such that $|\varphi(n)|\gamma^n \ll Mp$ for all $n\in\mathbb{Z}_-$ and $\varphi\in\mathcal{B}_\gamma$ with $\|\varphi\|_\gamma\le 1$, or equivalently,

$$|\varphi(n-n_0)| \ll M\gamma^{-(n-n_0)}p, \quad \forall n\in\mathbb{Z},\ n\le n_0,\ \forall\varphi\in\mathcal{B}_\gamma,\ \|\varphi\|_\gamma\le 1. \tag{4.2}$$

Set $\lambda := \gamma_0^{-1}$ and $u(n) := M\lambda^{n-n_0}p$, $n\in\mathbb{Z}$. From (2.2) and (4.2), it follows that

$$|x(n)| \ll u(n), \quad \forall n\in\mathbb{Z},\ n\le n_0. \tag{4.3}$$

It is worth noticing that

$$\begin{aligned}
|x(n_0+1)| &\overset{(2.1),(\mathrm{H}_1),(\mathrm{H}_2)}{\le} A|x(n_0)| + B\sum_{k=-\infty}^{n_0} C(n_0-k)|x(k)| \\
&\overset{(4.3)}{\le} Au(n_0) + B\sum_{k=-\infty}^{n_0} C(n_0-k)u(k) \\
&= AMp + B\sum_{k=-\infty}^{n_0} C(n_0-k)M\lambda^{k-n_0}p \\
&= M\Big(A + B\sum_{k=0}^{\infty} C(k)\gamma_0^k\Big)p \\
&\overset{(4.1)}{\le} M\gamma_0^{-1}p \overset{\gamma\le\gamma_0}{\le} M\gamma^{-1}p \\
&= u(n_0+1).
\end{aligned}$$

By induction, we can show that $|x(n)| = |x(n,n_0,\varphi)| \le M\lambda^{n-n_0}p$ for all $n\ge n_0$. By the monotonicity of vector norms, it follows that

$$\|x(n,n_0,\varphi)\| \le M\|p\|\lambda^{n-n_0} = K\lambda^{n-n_0}, \quad \forall n\ge n_0,$$

where $K := M\|p\|$.

Step II. We show that (2.4) holds.

Consider the linear Volterra difference equation

$$y(n+1) = Ay(n) + \sum_{k=-\infty}^{n} BC(n-k)y(k), \quad n\in\mathbb{Z}_+. \tag{4.4}$$

where $A, B, C(\cdot)$ are as in (H_1) and (H_2).

By Step I, we have $\|y(n,n_0,\varphi)\| \le K\lambda^{n-n_0}$ for all $n\ge n_0$ and any $\varphi\in\mathcal{B}_\gamma$ with $\|\varphi\|_\gamma\le 1$. By the linearity of (4.4),

$$\Big\|y\Big(n,n_0,\frac{\varphi}{\|\varphi\|_\gamma}\Big)\Big\| = \frac{1}{\|\varphi\|_\gamma}\|y(n,n_0,\varphi)\| \le K\lambda^{n-n_0}, \quad \forall n\ge n_0,$$

for all $\varphi \in \mathcal{B}_\gamma, \varphi \neq 0$. Therefore,

$$\|y(n, n_0, \varphi)\| \leq K\lambda^{n-n_0}\|\varphi\|_\gamma, \quad \forall n \geq n_0, \forall \varphi \in \mathcal{B}_\gamma. \tag{4.5}$$

For a given $\varphi \in \mathcal{B}_\gamma$, let $x(\cdot) := x(\cdot, n_0, \varphi)$ be the solution of (2.1)–(2.2) and let $y(\cdot) := y(\cdot, n_0, |\varphi|)$ be the solution of (4.4) with the initial condition $y(n_0 + k) = |\varphi|(k), \forall k \in \mathbb{R}_-$, where $|\varphi|(n) := |\varphi(n)|$, $n \in \mathbb{Z}$. Since $A, B, C(n) \geq 0$ for $n \in \mathbb{Z}_+$ and $|\varphi| \geq 0$, it follows that $y(n) \geq 0$ for all $n \geq n_0$.

Note that

$$\begin{aligned} |x(n_0+1)| &\overset{(2.1),(\mathrm{H}_1),(\mathrm{H}_2)}{\leq} A|x(n_0)| + B\sum_{k=-\infty}^{n_0} C(n_0-k)|x(k)| \\ &\overset{(2.2)}{=} A|\varphi(0)| + \sum_{k=-\infty}^{n_0} BC(n_0-k)|\varphi(k-n_0)| \\ &\overset{(4.4)}{=} Ay(n_0) + \sum_{k=-\infty}^{n_0} BC(n_0-k)y(k) \\ &= y(n_0+1). \end{aligned}$$

By induction, $|x(n)| \leq y(n)$ for all $n \geq n_0$. By the monotonicity of vector norms,

$$\|x(n)\| \leq \big\||x(n)|\big\| \leq \|y(n)\|, \quad \forall n \geq n_0. \tag{4.6}$$

Thus, (2.4) follows from (4.5) and (4.6). Hence, the zero solution of (2.1) is GES in $\mathcal{B}_\gamma$.

If $1 < \gamma_0 < \gamma$ then (H_2) holds for $\gamma_0 \in (1, \gamma)$. Note that $\mathcal{B}_{\gamma_0} \subset \mathcal{B}_\gamma$ for $\gamma_0 \in (1, \gamma]$. Therefore, $\mathcal{N}(\cdot, \cdot)$ is well-defined on $\mathbb{Z}_+ \times \mathcal{B}_{\gamma_0}$. Then (2.1) is GES in $\mathcal{B}_{\gamma_0}$, by the above result. This completes the proof. ∎

4.2. Proof of Theorem 3.1. Let

$$M := A + \sum_{k=0}^{\infty} BC(k).$$

By the assumption, $\rho(M) < 1$. Since $M, A, B, \varDelta_1, \varDelta_2, \varDelta_G(k)$, $k \in \mathbb{Z}_+$, are nonnegative, so is $M + \varDelta_1 + \sum_{k=0}^{\infty} \varDelta_2\varDelta_G(k)$. We first show that (3.2) and (3.3) imply

$$\rho_0 := \rho\Big(M + \varDelta_1 + \sum_{k=0}^{\infty} \varDelta_2\varDelta_G(k)\Big) < 1.$$

Assume on the contrary that $\rho_0 \geq 1$. By the Perron–Frobenius Theorem (Theorem 1.1(i)), there exists $x \in \mathbb{R}^m_+$, $x \neq 0$, such that

$$\Big(M + \varDelta_1 + \sum_{k=0}^{\infty} \varDelta_2\varDelta_G(k)\Big)x = \rho_0 x,$$

or equivalently,

$$(\rho_0 I_m - M)x = \Big(\varDelta_1 + \sum_{k=0}^{\infty} \varDelta_2 \varDelta_G(k)\Big)x. \tag{4.7}$$

Let $H(t) := (tI_m - M)$, $t \in \mathbb{R}$. Since $\rho(M) < 1 \le \rho_0$, $H(\rho_0)^{-1}$ and $H(1)^{-1}$ exist and are nonnegative, by Theorem 1.1(ii). From (4.7), it follows that

$$H(\rho_0)^{-1}\Big(\varDelta_1 + \sum_{k=0}^{\infty} \varDelta_2 \varDelta_G(k)\Big)x = x.$$

Taking norms of both sides of the last equation, we have

$$\|H(\rho_0)^{-1}\|\Big(\|\varDelta_1\| + \sum_{k=0}^{\infty} \|\varDelta_2 \varDelta_G(k)\|\Big)\|x\| \ge \|x\|. \tag{4.8}$$

Since $x \neq 0$, (4.8) implies

$$\|\varDelta_1\| + \sum_{k=0}^{\infty} \|\varDelta_2 \varDelta_G(k)\| \ge \frac{1}{\|H(\rho_0)^{-1}\|}. \tag{4.9}$$

On the other hand, the resolvent identity gives

$$H(1)^{-1} - H(\rho_0)^{-1} = (\rho_0 - 1)(H(\rho_0)^{-1} H(1)^{-1} \ge 0.$$

This yields $H(1)^{-1} \ge H(\rho_0)^{-1} \ge 0$. Therefore, $\|H(1)^{-1}\| \ge \|H(\rho_0)^{-1}\| \ge 0$. It follows from (4.9) that

$$\|\varDelta_1\| + \sum_{k=0}^{\infty} \|\varDelta_2 \varDelta_G(k)\| \ge \frac{1}{\|H(1)^{-1}\|} = \frac{1}{\|(I_m - A - B\sum_{k=0}^{\infty} C(k))^{-1}\|}.$$

However, this conflicts with (3.3). Therefore, $\rho_0 < 1$.

We now show that the zero solution of (3.1) is GES in $\mathcal{B}_\omega$ for some $\omega \in (1, \beta_0]$, with $\beta_0 := \min\{\gamma, \beta\}$. Since $(C(n))_n \in l^\gamma(\mathbb{R}^{m\times m})$ and $(\varDelta_G(n))_n \in l^\beta(\mathbb{R}^{m\times m})$, we have $(C(n) + \varDelta_G(n))_n \in l^{\beta_0}(\mathbb{R}^{m\times m})$. The remainder of the proof is similar to that of Theorem 2.2 and is omitted. ∎

References

[ACF] R. P. Agarwal, C. Cuevas and M. V. S. Frasson, *Semilinear functional difference equations with infinite delay*, Math. Computer Modelling 55 (2012), 1083–1105.

[BK12] E. Braverman and I. Karabash, *Bohl–Perron-type stability theorems for linear difference equations with infinite delay*, J. Difference Equations Appl. 18 (2012), 909–939.

[BK13] E. Braverman and I. Karabash, *Structured stability radii and exponential stability tests for Volterra difference systems*, Comput. Math. Appl. 66 (2013), 2259–2280.

[BH] H. Brunner and P. J. Houwen, *The Numerical Solution of Volterra Equations*, CWI Monographs, North-Holland, Amsterdam, 1986.

[CKRV98] M. R. Crisci, V. B. Kolmanovskii, E. Russo and A. Vecchio, *Stability of difference Volterra equations: direct Liapunov method and numerical procedure*, Comput. Math. Appl. 36 (1998), 77–97.

[CKRV00] M. R. Crisci, V. B. Kolmanovskii, E. Russo and A. Vecchio, *On the exponential stability of discrete Volterra equations*, J. Difference Equations Appl. 6 (2000), 667–680.

[CDCS] C. Cuevas, F. Dantas, M. Choquehuanca and H. Soto, *l^p-boundedness properties for Volterra difference equations*, Appl. Math. Comput. 219 (2013), 6986–6999.

[E05] S. Elaydi, *An Introduction to Difference Equations*, Springer, 2005.

[E09] S. Elaydi, *Stability and asymptoticity of Volterra difference equations: A progress report*, J. Comput. Appl. Math. 228 (2009), 504–513.

[ES] S. Elaydi and S. Murakami, *Asymptotic stability versus exponential stability in linear Volterra difference equations of convolution type*, J. Difference Equations Appl. 2 (1996), 401–410.

[HS] D. Hinrichsen and N. K. Son, *Stability radii of positive discrete-time equations under affine parameter perturbations*, Int. J. Robust Nonlinear Control 8 (1998), 1169–1188.

[KCT] V. B. Kolmanovskii, E. Castellanos-Velasco and J. A. Torres-Munoz, *A survey: stability and boundedness of Volterra difference equations*, Nonlinear Anal. 53 (2003), 861–928.

[L] N. Levinson, *A nonlinear Volterra equation arising in the theory of superfluidity*, J. Math. Anal. Appl. 1 (1960), 1–11.

[MW] W. R. Mann and F. Wolf, *Heat transfer between solids and gases under nonlinear boundary conditions*, Quart. Appl. Math. 9 (1951), 163–184.

[MM] H. Matsunaga and S. Murakami, *Asymptotic behavior of solutions of functional difference equations*, J. Math. Anal. Appl. 305 (2005), 391–410.

[M97] S. Murakami, *Representation of solutions of linear functional difference equations in phase space*, Nonlinear Anal. 30 (1997), 1153–1164.

[NH] P. H. A. Ngoc and L. T. Hieu, *New criteria for exponential stability of nonlinear difference systems with time-varying delay*, Int. J. Control 86 (2013), 1646–1651.

[NNSM] P. H. A. Ngoc, T. Naito, J. S. Shin and S. Murakami, *Stability and robust stability of positive linear Volterra difference equations*, Int. J. Robust Nonlinear Control 19 (2008), 552–568.

[PP] C. G. Philos and I. K. Purnaras, *The behavior of solutions of linear Volterra difference equations with infinite delay*, Comput. Math. Appl. 47 (2004), 1555–1563.

[RD] Y. N. Raffoul and Y. M. Dib, *Boundedness and stability in nonlinear discrete systems with nonlinear perturbation*, J. Difference Equations Appl. 9 (2003), 853–862.

[SB] Y. Song and C. T. H. Baker, *Perturbation of Volterra difference equations*, J. Difference Equations Appl. 10 (2004), 379–397.

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Received July 12, 2014;
received in final form October 10, 2014 (7975)