BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 62, No. 2, 2014

OPERATOR THEORY

Generalizations of Kaplansky's Theorem Involving Unbounded Linear Operators

by

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Summary. We are mainly concerned with the result of Kaplansky, [10], on the composition of two normal operators in the case in which at least one of the operators is unbounded.

1. Introduction. Normal operators are a major class of bounded and unbounded operators. Among their virtues, they are the largest class of single operators for which the spectral theorem is proved (cf. [17]). There are other classes of interesting non-normal operators, such as hyponormal and subnormal operators. They have been of interest to many mathematicians and have been extensively investigated, so that even monographs have been devoted to them—see for instance [4] and [12].

In this paper we are mainly interested in generalizing the following result to unbounded normal and bounded hyponormal operators:

THEOREM 1.1 (Kaplansky, [10]). *Let A and B be two bounded operators on a Hilbert space such that AB and A are normal. Then B commutes with AA^* iff BA is normal.*

Before recalling some essential background, we make the following convention:

All operators are linear and are defined on a separable complex Hilbert space, which we will denote henceforth by H.

A bounded operator A on H is said to be *normal* if $AA^* = A^*A$, and *hyponormal* if $AA^* \leq A^*A$, that is, $\|A^*x\| \leq \|Ax\|$ for all $x \in H$. Hence a

2010 *Mathematics Subject Classification*: Primary 47B20; Secondary 47A05.
Key words and phrases: products of operators, bounded and unbounded normal, hyponormal, subnormal operators, Kaplansky theorem, Fuglede–Putnam theorem.

DOI: 10.4064/ba62-2-6

normal operator is always hyponormal. Obviously, a hyponormal operator need not be normal. However, in a finite-dimensional setting, a hyponormal operator is normal. This is proved via a nice and simple trace argument (see e.g. [8]). An operator is said to be *subnormal* if it has a normal extension, and *co-subnormal* if its adjoint is subnormal. Another important class of non-normal operators is that of paranormal ones. We recall that an operator A is said to be *paranormal* if $\|A^2x\| \geq \|Ax\|^2$ for any unit vector x in H (it can easily be shown that hyponormality implies paranormality). Similarly, an operator A is called *co-paranormal* if its adjoint is paranormal.

Since the paper is also concerned with unbounded operators, and for the reader's convenience, we recall some known notions and results about unbounded operators.

If A and B are two unbounded operators with domains $D(A)$ and $D(B)$ respectively, then B is said to be an *extension* of A (or A is a *suboperator* of B), and we write $A \subset B$, if $D(A) \subset D(B)$ and A and B coincide on each element of $D(A)$. An operator A is said to be *densely defined* if $D(A)$ is dense in H. The (Hilbert) adjoint of A is denoted by A^* and it is known to be unique if A is densely defined. An operator A is said to be *closed* if its graph is closed in $H \times H$. We say that an unbounded operator A is *self-adjoint* if $A = A^*$, and *normal* if A is closed and $AA^* = A^*A$.

A densely defined operator A is said to be *formally normal* if $D(A) \subset D(A^*)$ and $\|Ax\| = \|A^*x\|$ for all $x \in D(A)$. It is easy to see that a densely defined suboperator of a normal operator is formally normal.

Recall also that the product BA is closed if for instance B is closed and A is bounded, and that if A, B and AB are densely defined, then only $B^*A^* \subset (AB)^*$ holds; and if further A is assumed to be bounded, then $B^*A^* = (AB)^*$.

The notion of hyponormality extends naturally to unbounded operators. An unbounded A is called *hyponormal* if:

(1) $D(A) \subset D(A^*)$,
(2) $\|A^*x\| \leq \|Ax\|$ for all $x \in D(A)$.

Any other result or notion (such as the classical Fuglede–Putnam theorem, polar decomposition, subnormality etc.) will be assumed to be known by readers. For more details, the interested reader is referred to [2], [3], [7], [16] and [17]. For other works related to products of normal (bounded and unbounded) operators, the reader may consult [6], [11], [13], [14] and [15], and the references therein.

2. Main results: the bounded case. The following was proved in [10]:

PROPOSITION 2.1. *If A and B are bounded operators on a Hilbert space H, A is normal and B commutes with A^*A, then the operators AB and BA are unitarily equivalent.*

So, under the assumptions of Corollary 2.1 below, BA being unitarily equivalent to the hyponormal operator AB is automatically hyponormal. Clearly, hyponormality can be replaced by any other property of AB written in terms of an inner product. In particular, one can consider (co-) subnormality, (co-) paranormality and so on.

COROLLARY 2.1. *Let A and B be two bounded operators on a Hilbert space such that A is normal and AB is hyponormal. Then*

$$AA^*B = BAA^* \Rightarrow BA \text{ is hyponormal.}$$

The reverse implication in Corollary 2.1 does not hold (even if A is self-adjoint), as shown by the following example:

EXAMPLE 1. Let A and B act on the standard basis (e_n) of $\ell^2(\mathbb{N})$ by:

$$Ae_n = \alpha_n e_n \quad \text{and} \quad Be_n = e_{n+1}, \qquad \forall n \geq 1,$$

respectively. Assume further that α_n is bounded, *real-valued* and *positive*, for all n. Hence A is self-adjoint (hence normal!) and positive. Then

$$ABe_n = \alpha_{n+1}e_{n+1} \quad \text{and} \quad BAe_n = \alpha_n e_{n+1}, \ \forall n \geq 1,$$

meaning that both AB and BA are weighted shifts with weights $\{\alpha_n\}_{n=2}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ respectively.

Now recall (see e.g. [8]) that a weighted shift with weight $\{\alpha_n\}_{n=1}^{\infty}$ is hyponormal if and only if $\{|\alpha_n|\}_{n=1}^{\infty}$ is increasing.

Hence if $\{\alpha_n\}_{n=1}^{\infty}$ is increasing, then AB and BA are both hyponormal. Moreover, $AB = BA$ (equivalently, $A^2B = BA^2$) if and only if the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is constant. Taking any nonconstant increasing sequence $\{\alpha_n\}_{n=1}^{\infty}$ gives the desired example.

Another interesting consequence of Theorem 1.1, Proposition 2.1 and Ando's theorem (see [1]) is the following result:

COROLLARY 2.2. *Let A, B be two bounded operators such that A is also normal. Assume that AB is paranormal and that BA is co-paranormal. Assume further that the kernels of AB and $(AB)^*$ (or those of BA and $(BA)^*$) coincide. Then*

$$AA^*B = BA^*A \Leftrightarrow BA \text{ and } AB \text{ are normal.}$$

3. Main results: the unbounded case. We start by giving an example to show that the same assumptions as in Theorem 1.1 do not yield the same result if B is an unbounded operator, let alone when both A and B are unbounded.

What we want is a normal bounded operator A and an unbounded (and closed) operator B such that BA is normal, $A^*AB \subset BA^*A$ but AB is not normal.

Example 2. Let

$$Bf(x) = e^{x^2} f(x) \quad \text{and} \quad Af(x) = e^{-x^2} f(x)$$

with domains

$$D(B) = \{f \in L^2(\mathbb{R}) : e^{x^2} f \in L^2(\mathbb{R})\} \quad \text{and} \quad D(A) = L^2(\mathbb{R}).$$

Then it is well known that A is bounded and self-adjoint (hence normal), and that B is self-adjoint (hence closed).

Now AB is not normal for it is not closed, as $AB \subset I$. On the other hand, BA is normal as $BA = I$ (on $L^2(\mathbb{R})$). Hence $AB \subset BA$, which implies that

$$AAB \subset ABA \;\Rightarrow\; AAB \subset ABA \subset BAA.$$

The anonymous referee kindly suggested the following result. It is a generalization of the second result in Corollary 3.1 which was proved earlier by the authors using a different proof.

Proposition 3.1. *If A is a bounded normal operator and B is a closed operator such that AB is normal and $A^*AB \subset BA^*A$, then*

(1) *BA is closed and densely defined, and $(BA)^*$ is formally normal.*
(2) *BA is normal provided $D(BA) \subset D[(BA)^*]$.*

Remark. The assumption $D(BA) \subset D[(BA)^*]$ is essentially weaker than the hyponormality of BA (which in turn is essentially weaker than the subnormality of BA).

Proof of Proposition 3.1. (1) We wish to adapt the proof of Kaplansky's theorem [10] (with U unitary and $A = U|A| = |A|U$) to the context of unbounded operators. However, there is a delicate moment: we have to know that $|A|^2B \subset B|A|^2$ implies $|A|B \subset B|A|$. But this follows from Lemma 2.1 in [9]. Then we get $C \subset BA$, where $C = U^*ABU$ is normal. Since AB is densely defined, so is BA. Clearly BA is closed. Taking adjoints, we see that $(BA)^*$ is densely defined and $(BA)^* \subset C^*$. Since C^* is normal, $(BA)^*$ is formally normal.

(2) Apply (1). ∎

Corollary 3.1. *Let B be a closed operator and let A be a bounded operator such that both AB and A are normal. If BA is normal, then $A^*AB \subset BA^*A$.*

If BA is hyponormal and $A^*AB \subset BA^*A$, then BA is normal.

Proof. We only prove the first assertion of the corollary. The proof in this case is a direct adaptation of the Kaplansky theorem. Since AB and BA are normal, the equality

$$A(BA) = (AB)A$$

implies that

$$A(BA)^* = (AB)^*A$$

by the Fuglede–Putnam theorem (see e.g. [3]). Hence

$$AA^*B^* \subset B^*A^*A \text{ or } A^*AB \subset BA^*A. \blacksquare$$

Imposing another commutativity condition allows us to generalize Theorem 1.1 to unbounded normal operators by bypassing hyponormality and subnormality:

THEOREM 3.1. *Let B be an unbounded operator and let A be a bounded one such that both A and B are normal. If $A^*AB \subset BA^*A$, $AB^*B \subset B^*BA$ and BA is densely defined, then BA is normal.*

The proof is partly based on the following interesting result on maximality of self-adjoint operators:

PROPOSITION 3.2 (Devinatz–Nussbaum–von Neumann [5]). *Let A, B and C be self-adjoint operators. Then*

$$A \subseteq BC \;\Rightarrow\; A = BC.$$

Proof of Theorem 3.1. First, BA is closed as A is bounded and B is closed. So $BA(BA)^*$ (and $(BA)^*BA$) is self-adjoint. Then we have

$$A^*ABB^* \subset BA^*AB^* = BAA^*B^* \subset BA(BA)^*$$

and hence

$$BA(BA)^* \subset (A^*ABB^*)^* = BB^*A^*A,$$

so that Proposition 3.2 gives us

$$BA(BA)^* = BB^*A^*A,$$

for both BB^* and A^*A are self-adjoint since B is closed and A is bounded respectively.

Similarly

$$A^*AB^*B \subset A^*B^*BA \subset (BA)^*BA.$$

Adjointing the above "inclusion" and applying again Proposition 3.2 yields

$$(BA)^*BA = B^*BA^*A = BB^*A^*A,$$

establishing the normality of BA. ∎

Acknowledgements. The authors wish to warmly thank the referee for all his suggestions and for his meticulousness.

References

[1] T. Ando, *Operators with a norm condition*, Acta Sci. Math. (Szeged) 33 (1972), 169–178.

[2] S. K. Berberian, *Introduction to Hilbert Space*, reprint of the 1961 original, Chelsea, New York, 1976.
[3] J. B. Conway, *A Course in Functional Analysis*, 2nd ed., Springer, 1990.
[4] J. B. Conway, *The Theory of Subnormal Operators*, Math. Surveys Monogr. 36, Amer. Math. Soc., Providence, RI, 1991.
[5] A. Devinatz, A. E. Nussbaum and J. von Neumann, *On the permutability of self-adjoint operators*, Ann. of Math. (2) 62 (1955), 199–203.
[6] A. Gheondea, *When are the products of normal operators normal?*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 52 (100) (2009), 129–150.
[7] I. Gohberg, S. Goldberg and M. A. Kaashoek, *Basic Classes of Linear Operators*, Birkhäuser, Basel, 2003.
[8] P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Springer, 1982.
[9] Z. J. Jabłoński, I. B. Jung and J. Stochel, *Unbounded quasinormal operators revisited*, Integral Equations Operator Theory 79 (2014), 135–149.
[10] I. Kaplansky, *Products of normal operators*, Duke Math. J. 20 (1953), 257–260.
[11] F. Kittaneh, *On the normality of operator products*, Linear Multilinear Algebra 30 (1991) 1–4.
[12] M. Martin and M. Putinar, *Lectures on Hyponormal Operators*, Operator Theory Adv. Appl. 39, Birkhäuser, Basel, 1989.
[13] M. H. Mortad, *On the closedness, the self-adjointness and the normality of the product of two unbounded operators*, Demonstratio Math. 45 (2012), 161–167.
[14] M. H. Mortad, *Commutativity of unbounded normal and self-adjoint operators and applications*, Operators and Matrices 8 (2014), 563–571.
[15] A. B. Patel and P. B. Ramanujan, *On sum and product of normal operators*, Indian J. Pure Appl. Math. 12 (1981), 1213–1218.
[16] W. Rudin, *Functional Analysis*, 2nd ed., McGraw-Hill, 1991.
[17] K. Schmüdgen, *Unbounded Self-Adjoint Operators on Hilbert Space*, Grad. Texts Math. 265, Springer, 2012.

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Received March 3, 2014;
received in final form July 25, 2014 (7966)