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FUNCTIONAL ANALYSIS

Functionals on Banach Algebras with Scattered Spectra

by

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Summary. Let A be a complex, commutative Banach algebra and let M_A be the structure space of A. Assume that there exists a continuous homomorphism $h : L^1(G) \to A$ with dense range, where $L^1(G)$ is a group algebra of the locally compact abelian group G. The main results of this note can be summarized as follows:

(a) If every weakly almost periodic functional on A with compact spectra is almost periodic, then the space M_A is scattered (i.e., M_A has no nonempty perfect subset).
(b) Weakly almost periodic functionals on A with compact scattered spectra are almost periodic.
(c) If M_A is scattered, then the algebra A is Arens regular if and only if $A^* = \overline{\operatorname{span}}\, M_A$.

1. Introduction. Throughout the paper A will denote a complex, commutative Banach algebra. We shall denote by M_A the structure space of A. As is well known, M_A is a locally compact, Hausdorff space and the Gelfand transform $\Gamma : a \mapsto \widehat{a}$ identifies A with a subalgebra of $C_0(M_A)$, the Banach algebra of all complex-valued continuous functions on M_A which vanish at infinity. For $\varphi \in A^*$ and $a \in A$, the functional $\varphi \cdot a$ on A is defined by $\langle \varphi \cdot a, b\rangle = \langle \varphi, ab\rangle$, $b \in A$. This operation turns A^* into a Banach A-module. Let $O_*(\varphi)$ denote the weak*-closure of the set $\{\varphi \cdot a : a \in A\}$. Recall that the w^*-spectrum of a functional $\varphi \in A^*$, written $\sigma_*(\varphi)$, is defined by $O_*(\varphi) \cap M_A$. We can readily see that $\sigma_*(\varphi) = \operatorname{hull}(I_\varphi)$, where $I_\varphi = \{a \in A : \varphi \cdot a = 0\}$ is a closed ideal in A.

Let G be a locally compact abelian group and $L^1(G)$ be the group algebra of G. The well known Loomis theorem [9] states that if the w^*-spectrum of $\varphi \in L^\infty(G)$ is compact and scattered, then φ is an almost periodic function, namely $\varphi \in \overline{\operatorname{span}}\, \sigma_*(\varphi)$. Recall that a closed subset S of a topological

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Hausdorff space is said to be *scattered* if S does not contain a nonempty perfect subset.

For $1 < p < \infty$, let $A_p(G)$ denote the space of functions on G which can be represented as

$$f = \sum_{n=1}^{\infty} u_n * v_n^{\vee},$$

where the u_n's are in $L^p(G)$, the v_n's are in $L^q(G)$ $(1/p + 1/q = 1)$, $v_n^{\vee}(g) = v_n(-g)$, and

$$\sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q < \infty.$$

The norm of f is the infimum of the above sums over all such representations of f. The space $A_p(G)$ is a commutative Banach algebra [4], often called the *Herz algebra*. We recall that $L^1(\widehat{G})$ is isometrically isomorphic to $A_2(G)$ via the Fourier transform F. Here $\widehat{G}$ is the dual group of G. A generalization of the Loomis theorem to Herz algebras has been proved by Lust-Piquard in [10].

As functions which are continuous on G with compact support are dense in $L^p(G)$, $A_2(G)$ is dense in $A_p(G)$. It follows that $F : L^1(\widehat{G}) \to A_p(G)$ is a continuous homomorphism with dense range. This suggests the question: Assume that there exists a continuous homomorphism $h : L^1(G) \to A$ with dense range. Is there a generalization of the Loomis theorem to the algebra A? In this paper we give a partial answer to this question.

We first note that the class of Banach algebras A satisfying the above conditions is fairly large. In general, these algebras arise in the following way: Let $g \mapsto T_g$ be a bounded continuous representation of G on a Banach space X. For $f \in L^1(G)$, define $T_f = \int_G f(g) T_g \, dg$. We see that T_f is a bounded linear operator on X. Let $L_T(G)$ denote the closure of $\{T_f : f \in L^1(G)\}$ with respect to the operator-norm topology. Then the algebras $L_T(G)$ satisfy the conditions imposed on A.

If there exists a continuous homomorphism $h : L^1(G) \to A$ with dense range, the spectrum of h, written $\mathrm{sp}(h)$, is defined as the hull of the ideal $\ker(h)$. The standard technique of Banach algebras shows that h^* homeomorphically identifieds M_A with $\mathrm{sp}(h)$. Moreover, the Gelfand transform of $h(f)$ is just $\widehat{f}(\chi)$ $(\chi \in \mathrm{sp}(h))$, where $\widehat{f}$ is the Fourier transform of $f \in L^1(G)$. If $\varphi \in A^*$, for notational simplicity we will write $h^*\varphi$ for $\varphi^{\vee}$. We shall also need the following notations: X is a Banach space, X^* is its dual, X^{**} is its second dual, and X_1 is the closed unit ball in X. We shall regard X as naturally embedded into X^{**}. For $\varphi \in X^*$ and $x \in X$, by $\langle \varphi, x \rangle$, and also by $\varphi(x)$, we denote the natural duality between X^* and X. We will denote by w and w^* the weak topology in X and the weak* topology in X^*, respectively. By $\bar{E}^w$ and $\bar{E}$ we will denote the weak closure and the norm closure, respectively, of a set $E \subset X$. $\bar{E}^*$ will denote the weak* closure of a set $E \subset X^*$.

2. Preliminaries. Let A be a complex, commutative Banach algebra. The functional $\varphi \in A^*$ is said to be (*weakly*) *almost periodic* on A if the set $\{\varphi \cdot a : a \in A_1\}$ is relatively (weakly) compact. This is equivalent to saying that the linear operator $T_\varphi : A \to A^*$ defined by $T_\varphi(a) = \varphi \cdot a$ is (weakly) compact. For example, if $A = L^1(G)$ then this reduces to the classical notion of (weak) almost periodicity for $\varphi \in L^\infty(G)$. We will denote by $\mathrm{ap}(A)$ (resp. $\mathrm{wap}(A)$) the set of all almost periodic (resp. weakly almost periodic) functionals on A. Both $\mathrm{ap}(A)$ and $\mathrm{wap}(A)$ are norm-closed A-submodules of A^*. As is known [2], $\mathrm{ap}(L^1(G)) = \mathrm{AP}(G)$ and $\mathrm{wap}(L^1(G)) = \mathrm{WAP}(G)$, where $\mathrm{AP}(G)$ and $\mathrm{WAP}(G)$ are the spaces of almost periodic and weakly almost periodic functions on G respectively.

We can endow A^{**} with a product (making A^{**} a Banach algebra) which is a natural extension of the original product in A (cf. [1]). This product is defined as follows: If $\varphi \in A^*$ and $F, H \in A^{**}$, then we set $\langle F \circ H, \varphi\rangle = \langle F, H\cdot\varphi\rangle$, where $H\cdot\varphi$ is the functional on A defined by $\langle H\cdot\varphi, a\rangle = \langle H, \varphi\cdot a\rangle$, $a \in A$. The algebra A is said to be *Arens regular* if A^{**} is commutative. This is equivalent to the condition that $\mathrm{wap}(A) = A^*$ (see [1]).

Let μ be an arbitrary bounded regular Borel measure on M_A. Then μ can be considered as an element of A^* with respect to the duality

$$\langle \mu, a\rangle = \int_{M_A} \widehat{a}(\phi)\, d\mu(\phi), \qquad a \in A.$$

It is easy to see that $\sigma_*(\mu)$ and $\operatorname{supp}\mu$ in the usual terms are the same.

LEMMA 2.1. *If μ is a bounded regular Borel measure on M_A, then $\mu \in \mathrm{wap}(A)$.*

Proof. We follow basically the proof by Dunkl–Ramirez [2], given there for the Fourier algebra. It is enough to show that if μ is a positive measure on M_A with compact support, then the operator T_μ is weakly compact. Define the map $S : L^2(M_A, d\mu) \to A^*$ by $Sf = f(\phi)\, d\mu(\phi)$ ($\phi \in M_A$). We see that S is a weakly compact operator and $T_\mu = S \circ \Gamma$. It follows that the operator T_μ is also weakly compact. ∎

The following lemma was proved in [8, Lemma 6.1] for Arens regular Banach algebras.

LEMMA 2.2. *If A has a bounded approximate identity, then every $\varphi \in \mathrm{wap}(A)$ (resp. $\varphi \in \mathrm{ap}(A)$) can be represented as $\varphi = \psi \cdot a$ for some $\psi \in \mathrm{wap}(A)$ (resp. $\psi \in \mathrm{ap}(A)$) and $a \in A$.*

Proof. Let $\varphi \in \mathrm{wap}(A)$. Note that $\mathrm{wap}(A)$ is a Banach A-module. It follows from the Cohen–Hewitt Factorization Theorem [5, 32.22] that the set $\{\psi \cdot a : \psi \in \mathrm{wap}(A),\ a \in A\}$ is a norm-closed linear subspace of A^*. Let $(e_i)_{i\in I}$ be a bounded approximate identity for A. Then $\varphi \cdot e_i \to \varphi$ in the w^*-topology. On the other hand, since the set $\{\varphi \cdot e_i : i \in I\}$ is relatively weakly compact, $\varphi \cdot e_i \to \varphi$ weakly. Hence we have

$$\varphi \in \overline{\{\psi \cdot a : \psi \in \mathrm{wap}(A),\, a \in A\}}^{w} = \{\psi \cdot a : \psi \in \mathrm{wap}(A),\, a \in A\}.$$

For $\mathrm{ap}(A)$ a similar argument works. ∎

A *Banach G-module* X is a Banach space X which is a G-module such that:

(i) $e \cdot x = x$ for all $x \in X$, where e is the identity of G.
(ii) $\|g \cdot x\| \le C\|x\|$ for some constant $C > 0$, for all $x \in X$ and $g \in G$.
(iii) For all $x \in X$, the map $g \mapsto g \cdot x$ is continuous from G into X.

In this case, we can define for each $\varphi \in X^*$, $g \in G$, the element $g \cdot \varphi \in X^*$ by $\langle g \cdot \varphi, x\rangle = \langle \varphi, g \cdot x\rangle$, $x \in X$. A Banach G-module X is said to be *almost periodic* if the set $\{g \cdot x : g \in G\}$ is relatively compact for every $x \in X$. It follows from the Peter–Weyl theory [11, Chap. 4, Sect. 3] that, if X is an almost periodic Banach G-module, then X is generated by the eigenvectors of G, i.e., by those $x \in X$ that satisfy $g \cdot x = \chi(g)x$ for some $\chi \in \widehat{G}$ and for all $g \in G$.

LEMMA 2.3. *If there exists a continuous homomorphism $h : L^1(G) \to A$ with dense range, then A is a Banach G-module and furthermore $\langle g\cdot\varphi, a\rangle = (\varphi \cdot a)^\vee(g)$ for every $\varphi \in A^*$, $a \in A$ and $g \in G$.*

Proof. Let $g \in G$ and $f \in L^1(G)$. Define $g \cdot h(f) = h(f_g)$, where $f_g(s) = f(s - g)$. Let $(f_i)_{i\in I}$ be an approximate identity in $L^1(G)$ bounded by one. Since $h((f_i)_g)h(f) \to g \cdot h(f)$, we have $\|g \cdot h(f)\| \le \|h\|\|h(f)\|$. Thus since $h(L^1(G))$ is dense in A the module operation can be extended to all A, after which the algebra A becomes a Banach G-module. Now let $\varphi \in A^*$ and $a \in A$ be given. It is easy to verify that $\int_G f(g)(g \cdot \varphi)\,dg = \varphi \cdot h(f)$. Using this we have

$$\int_G f(g)\langle g \cdot \varphi, a\rangle\,dg = \langle \varphi \cdot h(f), a\rangle = \langle (\varphi \cdot a)^\vee, f\rangle.$$

Since this is true for all $f \in L^1(G)$, we obtain

$$\langle g \cdot \varphi, a\rangle = (\varphi \cdot a)^\vee(g). \ \blacksquare$$

Let A be an arbitrary commutative Banach algebra. If $\phi \in M_A$ then $\phi \cdot a = \widehat{a}(\phi)\phi$, and consequently, $\phi \in \mathrm{ap}(A)$. Hence, $\overline{\mathrm{span}}\, M_A \subseteq \mathrm{ap}(A)$.

LEMMA 2.4. *If there exists a continuous homomorphism $h : L^1(G) \to A$ with dense range, then $\mathrm{ap}(A) = \overline{\mathrm{span}}\, M_A$.*

Proof. Let $\varphi \in \mathrm{ap}(A)$. By Lemma 2.2, φ is of the form $\varphi = \psi \cdot a$ for some $\psi \in \mathrm{ap}(A)$ and $a \in A$. Since the set $\{g \cdot a : g \in G\}$ is bounded, from the identity $g\cdot\varphi = \psi\cdot(g\cdot a)$ we deduce that the set $\{g \cdot \varphi : g \in G\}$ is relatively compact. On the other hand, since the map $g \mapsto g \cdot \varphi$ is w^*-continuous, it follows that $g \mapsto g \cdot \varphi$ is norm-continuous. Thus $\mathrm{ap}(A)$ is an almost periodic Banach G-module. Hence, $\mathrm{ap}(A)$ is generated by the eigenvectors of G. Let us now find the eigenvectors of G. Assume that $g \cdot \psi = \overline{\chi}(g)\psi$ for some $\chi \in \widehat{G}$, $\psi \in A^*\backslash\{0\}$ and for all $g \in G$. Then for any $f \in L^1(G)$, we can write

$$\psi \cdot h(f) = \int_G f(g)(g \cdot \psi)\, dg = \widehat{f}(\chi)\psi.$$

It follows that $\chi \in \mathrm{sp}(h)$. Since $h(L^1(G))$ is dense in A, we have $\psi\cdot a = \phi(a)\psi$ for some $\phi \in M_A$ and all $a \in A$. Thus since $\ker\phi \subseteq \ker\psi$, we obtain $\psi = c\phi$ for some $c \neq 0$. The proof is complete. ∎

3. The results. The first main result of this note is the following theorem.

THEOREM 3.1. *Assume that there exists a continuous homomorphism $h : L^1(G) \to A$ with dense range. If $\varphi \in \overline{\mathrm{span}}\,\sigma_*(\varphi)$ for every $\varphi \in \mathrm{wap}(A)$ with compact spectra, then M_A is scattered.*

Proof. It suffices to show that every compact subset of M_A is scattered. Let μ be an arbitrary bounded regular Borel measure on M_A. Then μ can be considered as a measure on $\widehat{G}$ with $\mathrm{supp}\,\mu \subseteq \mathrm{sp}(h)$. First we claim that $\mu^\vee(g) = \widehat{\mu}(-g)$, where $\widehat{\mu}$ is the Fourier–Stieltjes transform of μ. To see this, let $f \in L^1(G)$. Then we can write

$$\begin{aligned}\langle h^*\mu, f\rangle = \langle \mu, h(f)\rangle &= \int_{\widehat{G}} \widehat{f}(\chi)\, d\mu(\chi) = \int_{\widehat{G}} \Big(\int_G f(g)\overline{\chi}(g)\, dg\Big)\, d\mu(\chi)\\ &= \int_G \Big(\int_{\widehat{G}} \overline{\chi}(g)\, d\mu(\chi)\Big) f(g)\, dg = \int_G \widehat{\mu}(g) f(g)\, dg.\end{aligned}$$

Since this is true for all $f \in L^1(G)$, we obtain $\mu^\vee(g) = \widehat{\mu}(-g)$.

Now let K be an arbitrary compact subset of M_A and μ be an arbitrary continuous regular Borel measure supported on K. To prove that K is scattered, in view of [7, p. 52, Theorem 10] it is enough to show that μ is identically zero. By Lemma 2.1, $\mu \in \mathrm{wap}(A)$. Hence, by the assumption we have $\mu \in \overline{\mathrm{span}}\,K$. It follows that $\mu^\vee(g)$ can be approximated in the $\|\cdot\|_\infty$ norm by linear combinations of the characters in K. Consequently, $\widehat{\mu}(-g)$ is an almost periodic function on G. Let Φ be the invariant mean on $\mathrm{AP}(G)$. Since $\langle \Phi, \chi(g)\rangle = 1$ if $\chi = 1$ and $\langle \Phi, \chi(g)\rangle = 0$ if $\chi = 0$, we have

$$\langle \Phi, \overline{\chi}(g)\widehat{\mu}(-g)\rangle = \mu\{\chi\} = 0.$$

This shows that all Fourier–Bohr coefficients of the function $\widehat{\mu}(-g)$ are zero. By the uniqueness theorem we obtain $\widehat{\mu}(-g) \equiv 0$, and so $\mu = 0$. This proves the theorem. ■

The next theorem is the second main result of this note.

Theorem 3.2. *Assume that there exists a continuous homomorphism $h : L^1(G) \to A$ with dense range. If the w^*-spectrum of $\varphi \in \mathrm{wap}(A)$ is compact and scattered, then $\varphi \in \overline{\mathrm{span}}\, \sigma_*(\varphi)$.*

Before giving the proof of this theorem, we need the following facts. For $\chi \in \widehat{G}$, $C_\chi[f]$ denotes the Fourier–Bohr coefficient of a function $f \in \mathrm{AP}(G)$. As is known, $C_\chi[f] = \int_\Sigma \overline{f}(\sigma)\, d\sigma$, where Σ is the Bohr compactification of G and $\overline{f}(\sigma)$ the Bohr extension of f. The *Bohr spectrum* $\sigma_{\mathrm{B}}(f)$ of $f \in \mathrm{AP}(G)$ is defined as the set of all $\chi \in \widehat{G}$ such that $C_\chi[f] \neq 0$. We also note that if $f \in \mathrm{AP}(G)$ and $k \in L^1(G)$, then $f * k \in \mathrm{AP}(G)$ and $C_\chi[f * k] = \widehat{k}(\chi) C_\chi[f]$. It is well known that if $f \in \mathrm{AP}(G)$, then $\sigma_{\mathrm{B}}(f) \subseteq \sigma_*(f)$ and moreover $\overline{\sigma_{\mathrm{B}}(f)}^* = \sigma_*(f)$.

We also need the following lemma.

Lemma 3.3. *If there exists a continuous homomorphism $h : L^1(G) \to A$ with dense range, then $\sigma_*(\varphi^\vee) \subseteq \sigma_*(\varphi)$ for every $\varphi \in A^*$.*

Proof. Let $\varphi \in A^*$. Suppose that there is $\chi_0 \in \sigma_*(\varphi^\vee)$ but $\chi_0 \notin \sigma_*(\varphi)$. Then there exists a $k \in L^1(G)$ such that $\widehat{k}(\chi_0) \neq 0$ and $\widehat{k} = 0$ on some neighborhood of $\sigma_*(\varphi)$. Let $\pi : A \to A/I_\varphi$ be the canonical map and $\overline{h} = \pi \circ h$. Then $\mathrm{sp}(\overline{h}) = \sigma_*(\varphi)$. It follows that k belongs to the smallest ideal in $L^1(G)$ whose hull is $\mathrm{sp}(\overline{h})$. Hence $\overline{h}(k) = 0$, so that $h(k) \in I_\varphi$. Consequently, $\varphi \cdot h(k) = 0$. It follows from the relation $\varphi^\vee * k = h^*(\varphi \cdot h(k))$ (which can readily be verified) that $k \in I_{\varphi^\vee}$. Since $\chi_0 \in \sigma_*(\varphi^\vee)$, we obtain $\widehat{k}(\chi_0) = 0$. This is a contradiction. ■

Proof of Theorem 3.2. Assume that the w^*-spectrum of $\varphi \in \mathrm{wap}(A)$ is compact and scattered. By Lemma 3.3, $\sigma_*(\varphi^\vee)$ is also compact and scattered. The Loomis theorem implies that $\varphi^\vee \in \mathrm{AP}(G)$. Since $(\varphi \cdot h(f))^\vee = \varphi^\vee * f$, we have $\varphi^\vee * f \in \mathrm{AP}(G)$ for all $f \in L^1(G)$. Also since $h(L^1(G))$ is dense in A, this clearly implies that $(\varphi \cdot a)^\vee \in \mathrm{AP}(G)$ for all $a \in A$.

Now let $F \in A^{**}$ be such that $F(\chi) = 0$ for all $\chi \in \sigma_*(\varphi)$. To prove the theorem, it suffices to show that $F(\varphi) = 0$. Let Σ be the Bohr compactification of G, and $\overline{f}(\sigma)$ $(\sigma \in \Sigma)$ the Bohr extension of a function $f \in \mathrm{AP}(G)$. For any given $\sigma \in \Sigma$, define $\sigma \cdot \varphi \in A^*$ as follows: Since G is dense in Σ, there exists a net $(g_\lambda)_{\lambda \in \Lambda}$ in G such that $g_\lambda \to \sigma$ in Σ. Taking into account Lemma 2.3, we can write

$$\lim_\lambda \langle g_\lambda \cdot \varphi, a\rangle = \lim_\lambda (\varphi \cdot a)^\vee(g_\lambda) = \overline{(\varphi \cdot a)^\vee}(\sigma), \qquad a \in A.$$

Since the set $\{g_\lambda \cdot \varphi\}_{\lambda\in\Lambda}$ is bounded, we can define $\sigma \cdot \varphi \in A^*$ by

$$\sigma \cdot \varphi = w^*\text{-}\lim_\lambda g_\lambda \cdot \varphi. \tag{3.1}$$

Note that

$$\langle \sigma \cdot \varphi, a\rangle = \overline{(\varphi \cdot a)^\vee}(\sigma), \qquad a \in A. \tag{3.2}$$

By Lemma 2.2, φ can be represented as $\varphi = \psi \cdot a$ for some $\psi \in \mathrm{wap}(A)$ and $a \in A$. Since the set $\{g \cdot a : g \in G\}$ is bounded, from the identity $g\cdot\varphi = \psi\cdot(g\cdot a)$ we deduce that the set $\{g \cdot \varphi : g \in G\}$ is relatively weakly compact. Using (3.1) we get

$$\sigma \cdot \varphi = w\text{-}\lim_\lambda g_\lambda \cdot \varphi.$$

Hence, the set $\{\sigma \cdot \varphi : \sigma \in \Sigma\}$ is relatively weakly compact (since Σ is compact, this set is relatively norm-compact). Using relation (3.2) in the same way, we can see that the map $\sigma \mapsto \sigma\cdot\varphi$ is weakly continuous on Σ. Therefore, the function $g \mapsto \langle F, g\cdot\varphi\rangle$ is in $\mathrm{AP}(G)$. We claim that $\langle F, g\cdot\varphi\rangle = (F\cdot\varphi)^\vee(g)$. To see this, let $f \in L^1(G)$. Since the map $g \mapsto g \cdot \varphi$ is weakly continuous, we have

$$\begin{aligned}\int_G \langle F, g\cdot\varphi\rangle f(g)\,dg &= \Big\langle F, \int_G (g\cdot\varphi) f(g)\,dg\Big\rangle = \langle F, \varphi\cdot h(f)\rangle \\ &= \langle F\cdot\varphi, h(f)\rangle = \langle (F\cdot\varphi)^\vee, f\rangle.\end{aligned}$$

Since this is true for all $f \in L^1(G)$, our claim follows. We also note that

$$\langle F, \sigma\cdot\varphi\rangle = \overline{(F\cdot\varphi)^\vee}(\sigma), \qquad \sigma \in \Sigma. \tag{3.3}$$

Let us now find the Fourier–Bohr coefficients of the function $(F\cdot\varphi)^\vee$. For this purpose consider the following vector-valued integral:

$$\varphi_\chi = \int_\Sigma \overline{\chi}(\sigma)(\sigma\cdot\varphi)\,d\sigma.$$

Then we have

$$\begin{aligned}\langle \varphi_\chi, a\rangle &= \int_\Sigma \overline{\chi}(\sigma)\langle\sigma\cdot\varphi, a\rangle\,d\sigma = \int_\Sigma \overline{\chi}(\sigma)\overline{(\varphi\cdot a)^\vee}(\sigma) \\ &= C_\chi[(\varphi\cdot a)^\vee] = \widehat{a}(\chi)C_\chi[\varphi^\vee], \qquad a \in A.\end{aligned}$$

It follows that $\varphi_\chi = \chi C_\chi[\varphi^\vee]$. Further, since the mapping $\sigma \mapsto \sigma\cdot\varphi$ is weakly continuous, in view of (3.3) we obtain

$$\begin{aligned}C_\chi[(F\cdot\varphi)^\vee] &= \int_\Sigma \overline{\chi}(\sigma)\overline{(F\cdot\varphi)^\vee}(\sigma)\,d\sigma = \int_\Sigma \overline{\chi}(\sigma)\langle F, \sigma\cdot\varphi\rangle\,d\sigma \\ &= \langle F, \varphi_\chi\rangle = F(\chi)C_\chi[\varphi^\vee].\end{aligned}$$

Since $\sigma_*(\varphi^\vee) \subseteq \sigma_*(\varphi)$ and $F(\chi) = 0$ for all $\chi \in \sigma_*(\varphi)$, it follows that all Fourier–Bohr coefficients of the function $(F\cdot\varphi)^\vee$ are zero. By the uniqueness theorem, $(F \cdot \varphi)^\vee \equiv 0$. In particular, we have $F(\varphi) = (F \cdot \varphi)^\vee(e) = 0$. This proves the theorem. ∎

Let us record a consequence of this theorem.

Corollary 3.4. *Assume that there exists a continuous homomorphism $h : L^1(G) \to A$ with dense range and M_A is scattered. Then the algebra A is Arens regular if and only if $A^* = \overline{\operatorname{span}}\, M_A$.*

Proof. It is clear that if $A^* = \overline{\operatorname{span}}\, M_A$, then A is Arens regular. Now assume that A is Arens regular. Then $\operatorname{wap}(A) = A^*$ by Theorem 1 of [1]. Let $\varphi \in A^*$. By Lemma 6.1 of [8] (see also Lemma 2.2 of this note), φ is of the form $\varphi = \psi \cdot a$ for some $\psi \in A^*$ and $a \in A$. It suffices to show that $\psi \cdot h(f) \in \overline{\operatorname{span}}\, M_A$ for every $f \in L^1(G)$. Let $(f_i)_{i\in I}$ be a bounded approximate identity for $L^1(G)$ such that $\operatorname{supp} \widehat{f_i}$ $(i \in I)$ is compact. Then $(h(f_i))_{i\in I}$ is a bounded approximate identity for A. Since $\sigma_*(\psi)$ is scattered, from the relation

$$\sigma_*(\psi \cdot h(f)h(f_i)) \subseteq \sigma_*(\psi) \cap \operatorname{supp} \widehat{f} \cap \operatorname{supp} \widehat{f_i}$$

(which can be readily verified), we deduce that $\sigma_*(\psi \cdot h(f)h(f_i))$ is compact and scattered. By Theorem 3.2, we have $\psi \cdot h(f)h(f_i) \in \overline{\operatorname{span}}\, M_A$. Since $\psi \cdot h(f)h(f_i) \to \psi \cdot h(f)$ in norm, we obtain $\psi \cdot h(f) \in \overline{\operatorname{span}}\, M_A$. The proof is complete. ∎

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