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FUNCTIONAL ANALYSIS

Tame Köthe Sequence Spaces are Quasi-Normable

by

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Summary. We show that every tame Fréchet space admits a continuous norm and that every tame Köthe sequence space is quasi-normable.

1. Introduction. First we recall definitions and basic properties of the above mentioned classes of spaces. Let X be a Fréchet space with the topology defined by an increasing sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of seminorms. We call X *tame* if the following condition holds: there is an increasing function $S:\mathbb{N}\to\mathbb{N}$ such that for every continuous linear operator $T:X\to X$ there is a natural k_0 such that for every $k\geq k_0$ there is a constant C_k such that

$$\|Tx\|_k \leq C_k\|x\|_{S(k)} \quad \text{for every } x\in X.$$

This class of spaces was defined by D. Vogt and E. Dubinsky in [3]. They proved that in a tame infinite type power series space every complemented subspace has a basis. For other papers related to the notion of tameness see [7]–[9]. It is known that every finite type power series space is tame (see [10, Lemma 5.1]). The aim of this paper is to analyze which Köthe sequence spaces are tame.

We call X *quasi-normable* if for every 0-neighbourhood U there exists another 0-neighbourhood V such that for every $\varepsilon>0$ we can find a bounded set B in X such that

$$V\subset \varepsilon U + B.$$

The class of quasi-normable spaces was introduced by A. Grothendieck in [4]. See also [2], [6]. By $L(X)$ we denote the linear space of all continuous linear

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operators acting on X. For any operator $A \in L(X)$ we define

$$\sigma_A(k) = \inf\{n \in \mathbb{N} : \sup_{\|x\|_n \le 1} \|Ax\|_k < \infty\}.$$

Let I be an arbitrary index set and $A = (a^n)_{n\in\mathbb{N}}$ a sequence of nonnegative functions defined on I with the property that $a_i^n \le a_i^{n+1}$ for all $n \in \mathbb{N}, i \in I$. Let us recall that for $1 \le p < \infty$ a *Köthe sequence space* is defined as follows:

$$\lambda_p(I, A) = \Big\{x = (x_1, x_2, \ldots) : \|x\|_k := \Big(\sum_{i\in I}(a_i^k|x_i|)^p\Big)^{1/p} < \infty \ \forall k \in \mathbb{N}\Big\}$$

and

$$\lambda_\infty(I, A) = \{x = (x_1, x_2, \ldots) : \|x\|_k := \sup_{i\in I} a_i^k |x_i| < \infty \ \forall k \in \mathbb{N}\}$$

(see [5, 27]). For other notions from functional analysis used in this paper see [5].

2. Preliminary results

LEMMA 1. *The space ω of all sequences is not tame.*

Proof. Recall that

$$\omega = \{x = (x_1, x_2, \ldots) : \|x\|_k := \max_{j\le k} |x_j| < \infty\}.$$

Let $S : \mathbb{N} \to \mathbb{N}$ be an arbitrary increasing function and let $A : \omega \to \omega$ be an operator defined as

$$A((x_j)_{j\in\mathbb{N}}) = (x_{S(j+1)})_{j\in\mathbb{N}}.$$

Let

$$x^{(n)} = (0, \ldots, 0, \underset{\substack{\downarrow \\ \text{place } S(k+1)}}{n}, 0, \ldots).$$

Then $\|Ax^{(n)}\|_k = n$ and $\|x^{(n)}\|_{S(k)} = 0$. Therefore there is no constant C such that $\|Ax\|_k \le C\|x\|_{S(k)}$ for all $x \in \omega$, which proves that ω is not tame. ∎

LEMMA 2. *Tameness is inherited by complemented subspaces.*

Proof. Let $P : E \to X$ be a projection. If A is a continuous linear operator on X then the operator $A \circ P : E \to X$ is an element of $L(E)$. Thus

$$\|Ax\|_k = \|A \circ Px\|_k \le C_k \|x\|_{\sigma_{AP}(k)}$$

and $\sigma_A(k) \le \sigma_{AP}(k)$. If $\sigma_{AP}(k) \le S(k)$ then $\sigma_A(k) \le S(k)$ and thus if E is tame then X is tame as well. ∎

Let $\phi : \mathbb{N} \to \mathbb{N}$ be an arbitrary increasing function and define spaces of linear continuous operators

$$L_\phi(X) = \{A \in L(X) : \forall k \in \mathbb{N}\ \exists C_k\ \forall x \in X\ \ \|Ax\|_k \le C_k \|x\|_{\phi(k)}\},$$
$$L_{\phi,n}(X) = \{A \in L(X) : \forall k \ge n\ \exists C_k\ \forall x \in X\ \ \|Ax\|_k \le C_k \|x\|_{\phi(k)}\}.$$

If we put

$$\|A\|_{\phi(i),i} = \sup_{\|x\|_{\phi(i)} \le 1} \|Ax\|_i,$$

then $L_\phi(X)$ and $L_{\phi,n}(X)$ are Fréchet spaces with the sequences of seminorms defined as $\|\cdot\|_m = \max_{1\le i\le m} \|\cdot\|_{\phi(i),i}$ and $\|\cdot\|_m = \max_{n\le i\le m} \|\cdot\|_{\phi(i),i}$, respectively. Only completeness needs a comment. If $(A_p)_p$ is a Cauchy sequence in $L_\phi(X)$ then for every $x \in X$ the sequence $(A_p x)_p$ is a Cauchy sequence in the complete space X. This means that for the operator $Ax = \lim_{p\to\infty} A_p x$ we have

$$\forall k \in \mathbb{N}\ \exists P \in \mathbb{N} : \qquad \|(A - A_P)x\|_k \le \|x\|_{\phi(k)}.$$

This implies that $\|Ax\|_k \le (C_k^P + 1)\|x\|_{\phi(k)} = D_k \|x\|_{\phi(k)}$ for all k, which shows that $A \in L_\phi(X)$. The proof in the case of $L_{\phi,n}(X)$ is the same.

LEMMA 3. *In every tame Fréchet space X the following condition holds: there exists $\psi : \mathbb{N} \to \mathbb{N}$ such that for any $\phi : \mathbb{N} \to \mathbb{N}$ there exists $k \in \mathbb{N}$ such that for all $m \ge k$ there are $n \in \mathbb{N}$ and a constant $C_m > 0$ such that*

$$\forall x^* \in X^*, y \in X : \qquad \max_{k\le l\le m} \|x^*\|^*_{\psi(l)} \|y\|_l \le C_m \max_{1\le p\le n} \|x^*\|^*_{\phi(p)} \|y\|_p, \tag{1}$$

where $\|x^*\|^*_m = \sup_{\|x\|_m \le 1} |x^*(x)|$.

Proof. If the space X is tame with the function ψ then every continuous linear operator is an element of a certain $L_{\psi,k}$ so we may write $L(X) = \bigcup_{k\in\mathbb{N}} L_{\psi,k}(X)$. If we now endow the space $L(X)$ with the topology of pointwise convergence then for every increasing function $\phi : \mathbb{N} \to \mathbb{N}$ we obtain the following diagram where the arrows represent continuous linear mappings:

$$\begin{array}{ccc} \bigcup_k L_{\psi,k} & \xrightarrow{\text{id}} & L \\ & & \uparrow{\scriptstyle \text{id}} \\ & & L_\phi \end{array}$$

The continuity of the horizontal arrow comes from the following argument: for every 0-neighbourhood $U(0, x_1, \ldots, x_n, k, \varepsilon) = \{A \in L(X) : \forall 1 \le i \le n\ \|Ax_i\|_k < \varepsilon\}$ in L we define a 0-neighbourhood $V = \{A \in L_\phi : \|A\|_k < \varepsilon/M\}$ in L_ϕ, where $M = \max_{1\le i\le n} \|x_i\|_k$. As is easily seen, $\text{id}(V) \subset U$. The continuity of the vertical arrow is proved similarly. Using Grothendieck's Factorization Theorem [5, 24.33] we find a natural number k such that L_ϕ

embeds continuouosly in $L_{\psi,k}$. In other words in the tame Fréchet space the following holds:

$$\text{(2)}\qquad \exists\psi \nearrow \infty\ \forall\phi \nearrow \infty\ \exists k\ \forall m \ge k\ \exists n, C_m\ \forall T \in L_\phi(X):$$
$$\max_{k\le l\le m} \|T\|_{\psi(l),l} \le C_m \max_{1\le p\le n} \|T\|_{\phi(p),p}.$$

In particular, for one-dimensional operators T, $Tx = x^*(x)y$, $x^* \in X$, $y \in X$, we get (1). ∎

Lemma 4. *Let $\lambda_p(I,A)$ be an arbitrary Köthe sequence space. If it is not quasi-normable then, without loss of generality, we may assume that A satisfies the following conditions: $a_i^1 = 1$ for all i, and for every natural number m there exists an index subset $J_m = \{i(m,j) : j \in \mathbb{N}\}$ such that*

$$\text{(3)}\qquad \sup_j a^m_{i(m,j)} = c_m < \infty \quad \textit{and} \quad \lim_j a^{m+1}_{i(m,j)} = \infty.$$

Proof. From [2, Th. 17] it follows that if $\lambda_p(I,A)$ is not quasi-normable then

$$\exists n\ \forall m \ge n\ \exists J \subset I: \quad \inf_{i\in J} \frac{a_i^n}{a_i^m} > 0 \quad \text{and} \quad \inf_{i\in J} \frac{a_i^n}{a_i^k} = 0 \quad \text{for some } k(m) \ge m.$$

Firstly, we may assume that $n = 1$ and $a_i^1 = 1$ for all i (by dividing by a_i^1). Secondly, every set J_m is infinite so we may write $J_m = \{i(m,j) : j \in \mathbb{N}\}$. Finally, omitting rows of the matrix A suitably, numbers $k(m)$ can be chosen as $k(m) = m+1$ for $m \in \mathbb{N}$. ∎

2. Main results

Proposition 5. *Every tame Fréchet space has a continuous norm.*

Proof. If the space does not admit a continuous norm then from [1, Lemmas 1 and 2] it contains ω as a complemented subspace; but then from our assumption and Lemma 2, ω is tame, which contradicts Lemma 1. ∎

Theorem 6. *Tame Köthe sequence spaces are quasi-normable.*

Proof. By Proposition 5 we may assume that $a_i^k > 0$ for all $i \in I$, $k \in \mathbb{N}$. Suppose that $\lambda_p(I,A)$ is a tame Köthe space which is not quasi-normable. Using Lemma 3 we may write

$$\|x^*\|^*_{\psi(k)} \|y\|_k \le C_k \max_{1\le p\le n} \|x^*\|^*_{\phi(p)} \|y\|_p.$$

Without losing of generality we may assume that $n \ge k$. For all $j, v \in \mathbb{N}$ define

$$x_v^* x = x_{i(\phi(k-1),v)} \quad \text{and} \quad y_j = e_{i(k-1,j)},$$

where x_i denotes the ith coordinate of the vector x, e_i is the ith vector of the standard basis, and $i(k,j)$ denotes the index of number j from the index set J_k. Since $\|y_j\|_p = a^p_{i(k-1,j)}$ and $\|x^*_v\|^*_l = (a^l_{i(\phi(k-1),v)})^{-1}$, we obtain for all $j, v \in \mathbb{N}$ the inequality

(4) $$\frac{a^k_{i(k-1,j)}}{a^{\psi(k)}_{i(\phi(k-1),v)}} \le C_k \max_{1\le p\le n} \frac{a^p_{i(k-1,j)}}{a^{\phi(p)}_{i(\phi(k-1),v)}}.$$

The function ϕ has been arbitrary so far but from now on we choose $\phi(k-1) = \psi(k)$. Without loss of generality we may assume that ψ is strictly increasing, which, combined with Lemma 4, gives us

$$a^{\psi(k)}_{i(\phi(k-1),v)} = a^{\phi(k-1)}_{i(\phi(k-1),v)} \le c_{\phi(k-1)}$$

for all v and

(5) $$a^k_{i(k-1,j)} \xrightarrow[j\to\infty]{} \infty.$$

Equivalently we may write

(6) $$\frac{1}{c_{\phi(k-1)}}\, a^k_{i(k-1,j)} \le \frac{a^k_{i(k-1,j)}}{a^{\psi(k)}_{i(\phi(k-1),v)}}.$$

The estimates of the right hand side of (4) will be divided into two cases. If $p \le k-1$ then

$$a^p_{i(k-1,j)} \le a^{k-1}_{i(k-1,j)} \le c_{k-1} \quad \text{and} \quad a^{\phi(p)}_{i(\phi(k-1),v)} \ge a^1_{i(\phi(k-1),v)} = 1,$$

for all j, v. If $p \ge k$ then also $\phi(p) \ge \phi(k) \ge \phi(k-1)+1$, which leads to

$$a^{\phi(p)}_{i(\phi(k-1),v)} \ge a^{\phi(k-1)+1}_{i(\phi(k-1),v)} \xrightarrow[v\to\infty]{} \infty$$

and

$$a^p_{i(k-1,j)} \ge a^k_{i(k-1,j)} \xrightarrow[j\to\infty]{} \infty.$$

This implies that for every natural number j there is an index $v_j \in \mathbb{N}$ depending on k but not on p such that $a^{\phi(p)}_{i(\phi(k-1),v_j)} \ge a^p_{i(k-1,j)}$. If we now extract from $\{x^*_v\}_{v=1}^\infty$ the subsequence $(x^*_{v_j})_{j\in\mathbb{N}}$ then we obtain the inequality

(7) $$\max_{1\le p\le n} \frac{a^p_{i(k-1,j)}}{a^{\phi(p)}_{i(\phi(k-1),v_j)}} \le \max\{c_{k-1}, 1\} = d_k.$$

Combining the inequalities (4), (6) and (7) we finally get

$$a^k_{i(k-1,j)} \le C_k c_{\phi(k-1)} d_k < \infty \quad \text{for all } j;$$

but, by (5), $\lim_j a^k_{i(k-1,j)} = \infty$, a contradiction. This completes the proof. ■

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