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NUMBER THEORY

Exponential Sums with Farey Fractions

by

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Summary. For positive integers m and N, we estimate the rational exponential sums with denominator m over the reductions modulo m of elements of the set

$$\mathcal{F}(N) = \{s/r : r, s \in \mathbb{Z},\ \gcd(r,s) = 1,\ N \ge r > s \ge 1\}$$

of Farey fractions of order N (only fractions s/r with $\gcd(r,m) = 1$ are considered).

1. Introduction. For any integer $N \ge 1$, we consider the set

$$\mathcal{F}(N) = \{s/r : r, s \in \mathbb{Z},\ \gcd(r,s) = 1,\ N \ge r > s \ge 1\}$$

of Farey fractions of order N. There is an extensive literature where various distributional properties of Farey fractions are investigated. In particular, some of these properties are directly related to the Riemann Hypothesis. (See, for example, the survey [4] as well as the more recent works [1, 2, 3, 7, 8] and references therein.)

Here we consider an apparently new problem of the distribution of $\mathcal{F}(N)$ in residue classes modulo an integer $m \ge 2$. In particular, we show that for any interval $\mathcal{I} = [k, k+h-1] \subseteq [0, m-1]$ and any integer $N \ge m^{\varepsilon}$, where $\varepsilon > 0$ is fixed, the number $R_m(N, \mathcal{I})$ of Farey fractions $s/r \in \mathcal{F}(N)$ with $\gcd(r,m) = 1$ and such that $s/r \equiv z \pmod m$ for some $z \in \mathcal{I}$ is close to its expected value, provided that m is large enough.

Naturally, our main tool is bounds for exponential sums. For any integer $m \ge 2$ and any real z we put

$$\mathbf{e}_m(z) = \exp(2\pi i z/m)$$

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and consider the exponential sums

$$S_m(a;N) = \sum_{\substack{s/r\in\mathcal{F}(N)\\ \gcd(r,m)=1}} \mathbf{e}_m(as/r), \qquad a \in \mathbb{Z}.$$

(Note that each fraction $s/r \in \mathcal{F}(N)$ is reduced modulo m before it is used in the sum $S_m(a;N)$.)

We remark that for larger values of N, namely for $N \ge m^{1/2+\varepsilon}$, one can use the results in [12, 13] to show that almost all residue classes from the reduced residue system modulo m are represented by the elements of $\mathcal{F}(N)$ asymptotically the same number of times. A variant of such a result is also given in [5].

We write $V = U^{o(1)}$ to indicate the quantity which satisfies

$$\lim_{U\to\infty} \frac{\log V}{\log U} = 0.$$

Theorem 1. *For any integer* $a \not\equiv 0 \pmod m$,

$$|S_m(a;N)| \le N(Nm)^{o(1)} \qquad \textit{as } N, m \to \infty.$$

Combining Theorem 1 with some standard technique, we also derive the following asymptotic formula for $R_m(N,\mathcal{I})$.

Theorem 2. *Uniformly over the intervals* $\mathcal{I} = [k, k+h-1] \subseteq [0, m-1]$,

$$\left| R_m(N,\mathcal{I}) - \frac{6}{\pi^2}\prod_{p\mid m}\left(1+\frac{1}{p}\right)^{-1}\frac{h}{m}\,N^2\right| \le N(Nm)^{o(1)}$$

as $N, m \to \infty$, *where the product is taken over all primes* $p \mid m$.

2. Double sums. Here we obtain a rather general statement which may be of independent interest.

We use the notations $U = O(V)$ and $U \ll V$ to indicate that $|U| \le cV$ for some absolute constant $c > 0$.

Lemma 3. *Let* $Y \ge 1$ *be an arbitrary integer. Assume that for each integer* x *we are given two integers* L_x *and* U_x *with* $0 \le L_x < m$ *and* $U_x \le Y$. *Then for any integer* $a \not\equiv 0 \pmod m$,

$$\Bigg| \sum_{\substack{x=1\\ \gcd(x,m)=1}}^{X} \sum_{y=L_x+1}^{U_x} \mathbf{e}_m(ay/x)\Bigg| \le (X+Y)(Xm)^{o(1)}$$

as $X, m \to \infty$.

Proof. Let

$$W = \sum_{\substack{x=1 \\ \gcd(x,m)=1}}^{X} \sum_{y=L_x+1}^{U_x} \mathbf{e}_m(ay/x).$$

We may assume that

$$0 < a \le m-1.$$

For a rational number $\alpha = u/v$ with $\gcd(v,m)=1$, we denote by $\rho(\alpha)$ the unique integer w with

$$w \equiv u/v \pmod m \quad \text{and} \quad -m/2 < w < (m+1)/2.$$

Using the bound

$$\sum_{y=L+1}^{L+H} \mathbf{e}_m(\alpha y) \ll \min\left\{H, \frac{m}{|\rho(\alpha)|}\right\},$$

which holds for any rational α and integers L and H (see [10, Bound (8.6)]), we obtain

$$W \ll \sum_{\substack{x=1 \\ \gcd(x,m)=1}}^{X} \min\left\{Y, \frac{m}{\rho(a/x)}\right\}.$$

We now put $J = \lceil \log(m/2) \rceil$ and define the sets

$$\mathcal{X}_j = \{x : 1 \le x \le X,\ \gcd(x,m)=1,\ e^j \le |\rho(a/x)| < e^{j+1}\},$$

where $j = 0, \ldots, J$. Therefore

$$W \ll \sum_{j=0}^{J} \#\mathcal{X}_j \min\left\{Y, \frac{m}{e^j}\right\} \ll Y \sum_{j=0}^{J_0} \#\mathcal{X}_j + m \sum_{j=J_0+1}^{J} \#\mathcal{X}_j e^{-j}, \tag{1}$$

where J_0 is the largest $j \le J$ with $e^j \le m/Y$.

To estimate $\#\mathcal{X}_j$ we note that if $e^j \le |\rho(a/x)| < e^{j+1}$, then $xz \equiv a \pmod m$ for some integer z with $0 < |z| < e^{j+1}$. Thus $xz = a + mk$ for some integer k with $|k| < e^{j+1}X/m + 1$. Hence there are at most $O(e^jX/m+1)$ possible values of k and for each fixed $k \ll e^jX/m$ there are

$$\tau(|a+mk|) = (Xm)^{o(1)}$$

nonzero integers x and z with $xz = a + mk$, where $\tau(u)$ is the number of positive integer divisors of an integer $u \ne 0$ (see [9, Theorem 317]). Therefore we obtain the estimate

$$\#\mathcal{X}_j \le (e^jX/m+1)m^{o(1)},$$

which after inserting into (1) gives

$$(2)\qquad |W| \le Y \sum_{j=0}^{J_0} (e^j X/m + 1) m^{o(1)} + m^{1+o(1)} \sum_{j=J_0+1}^{J} (e^j X/m + 1) e^{-j}.$$

We now have

$$\sum_{j=0}^{J_0} (e^j X/m + 1) \ll e^{J_0} X/m + J_0 \ll X/Y + m^{o(1)}$$

and also

$$\sum_{j=J_0+1}^{J} (e^j X/m + 1) e^{-j} \le JX/m + e^{-J_0} \le X m^{-1+o(1)} + Y/m.$$

Substituting the above bounds into (2), we obtain the desired result. ∎

3. Proof of Theorem 1. For an integer $d \ge 1$ we use $\mu(d)$ to denote the Möbius function. We recall that $\mu(1) = 1$, $\mu(d) = 0$ if $d \ge 2$ is not square-free, and $\mu(d) = (-1)^{\omega(d)}$ otherwise, where $\omega(d)$ is the number of prime divisors of d. Then by the inclusion-exclusion principle,

$$\begin{aligned} S_m(a; N) &= \sum_{d=1}^{N} \mu(d) \sum_{\substack{r=1 \\ \gcd(r,m)=1 \\ d \mid r}}^{N} \sum_{\substack{s=r+1 \\ d \mid s}} \mathbf{e}_m(as/r) \\ &= \sum_{d=1}^{N} \mu(d) \sum_{\substack{x=1 \\ \gcd(x,m)=1}}^{\lfloor N/d \rfloor} \sum_{y=x+1}^{\lfloor N/d \rfloor} \mathbf{e}_m(ay/x). \end{aligned}$$

Now, for each $d = 1, \ldots, N$ we apply Lemma 3 with $L_x = x$ and $U_x = \lfloor N/d \rfloor$, and after a short calculation we obtain the desired result.

4. Proof of Theorem 2. For any integer $N \ge 1$, let

$$\mathcal{F}_m(N) = \{s/r \in \mathcal{F}(N) : \gcd(r, m) = 1\}.$$

Now using the *Erdős–Turán inequality* (see [6, 11]), which links the discrepancy with exponential sums, we immediately deduce from Theorem 1 that

$$\left| R_m(N, \mathcal{I}) - \frac{h}{m} \#\mathcal{F}_m(N) \right| \le N^{1+o(1)} m^{o(1)}.$$

It remains to approximate $\#\mathcal{F}_m(N)$. We follow the proof of the well-known asymptotic formula for $\#\mathcal{F}(N)$ given in [9, Theorem 330].

We have

$$\#\mathcal{F}_m(N) = \sum_{\substack{r=1\\ \gcd(r,m)=1}}^{N} \varphi(r) = \sum_{\substack{r=1\\ \gcd(r,m)=1}}^{N} r \sum_{d|r} \frac{\mu(d)}{d},$$

where $\varphi(r)$ is the Euler function (see [9, Equation (16.3.1)]). Interchanging the order of summation, we obtain

$$\#\mathcal{F}_m(N) = \sum_{\substack{r=1\\ \gcd(r,m)=1}}^{N} \varphi(r) = \sum_{\substack{d=1\\ \gcd(d,m)=1}}^{N} \frac{\mu(d)}{d} \sum_{\substack{r=1\\ \gcd(r,m)=1\\ r\equiv 0 \pmod d}}^{N} r.$$

Then replacing r with dt, we deduce

$$\#\mathcal{F}_m(N) = \sum_{\substack{d=1\\ \gcd(d,m)=1}}^{N} \mu(d) \sum_{\substack{1\le t\le N/d\\ \gcd(t,m)=1}}^{N} t. \tag{3}$$

We note that by the inclusion-exlusion principle, for any real T,

$$\begin{aligned}\sum_{\substack{1\le t\le T\\ \gcd(t,m)=1}} t &= \sum_{e|m} \mu(e) \sum_{\substack{1\le t\le T\\ e|t}}^{N} t = \sum_{e|m} \mu(e) e \sum_{1\le u\le T/e}^{N} u\\ &= \sum_{e|m} \mu(e) e \left(\frac{T^2}{2e^2} + O(T/e)\right) = \frac{1}{2} T^2 \sum_{e|m} \frac{\mu(e)}{e} + O(T\tau(m)).\end{aligned}$$

Using [9, Theorem 317 and Equation (16.3.1)], we obtain

$$\sum_{\substack{1\le t\le T\\ \gcd(t,m)=1}} t = \frac{\varphi(m)}{2m} T^2 + O(Tm^{o(1)}),$$

which after substitution into (3) yields

$$\begin{aligned}\#\mathcal{F}_m(N) &= \sum_{\substack{d=1\\ \gcd(d,m)=1}}^{N} \mu(d)\left(\frac{\varphi(m)}{m} \cdot \frac{N}{2d^2} + O\left(\frac{Nm^{o(1)}}{d}\right)\right)\\ &= \frac{\varphi(m)}{2m} N^2 \sum_{\substack{d=1\\ \gcd(d,m)=1}}^{N} \frac{\mu(d)}{d^2} + O(Nm^{o(1)})\\ &= \frac{\varphi(m)}{2m} N^2 \sum_{\substack{d=1\\ \gcd(d,m)=1}}^{\infty} \frac{\mu(d)}{d^2} + O(Nm^{o(1)}).\end{aligned}$$

Obviously

$$\sum_{\substack{d=1\\ \gcd(d,m)=1}}^{\infty} \frac{\mu(d)}{d^2} = \prod_{p\nmid m}\left(1-\frac{1}{p^2}\right) = \frac{m}{\varphi(m)}\prod_{p\mid m}\left(1+\frac{1}{p}\right)^{-1}\prod_{p}\left(1-\frac{1}{p^2}\right),$$

where the product is taken over all primes p (see [9, Theorem 285]). Then recalling that

$$\prod_{p}\left(1-\frac{1}{p^2}\right) = \sum_{\substack{d=1\\ \gcd(d,m)=1}}^{\infty} \frac{\mu(d)}{d^2} = \zeta(2)^{-1} = \frac{6}{\pi^2}$$

(see [9, Theorem 287 and Equation (17.2.2)]), we obtain

$$\#\mathcal{F}_m(N) = \frac{6}{\pi^2}\prod_{p\mid m}\left(1+\frac{1}{p}\right)^{-1} N^2 + O(Nm^{o(1)}).$$

This completes the proof of the theorem.

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