BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 57, No. 2, 2009

Almost Properness of Extremal Mappings

by

Armen EDIGARIAN and Przemysław KLIŚ

Presented by Józef SICIAK

Summary. We give a simple proof of almost properness of any extremal mapping in the sense of Lempert function or in the sense of Kobayashi–Royden pseudometric.

Let $D \subset \mathbb{C}^n$ be a domain. For any $z, w \in D$ (resp. $z \in D$, $X \in \mathbb{C}^n$) we denote by $\sigma_1(z, w)$ (resp. $\sigma_2(z, X)$) the set of all points $r \in [0, 1)$ (resp. $r \geq 0$) such that we can find a holomorphic mapping from the unit disc $\mathbb{D}$ to D with $f(0) = z$ and $f(r) = w$ (resp. $rf'(0) = X$). We put

$$\widetilde{k}_D(z, w) = \inf_{r \in \sigma_1(z,w)} r \quad \text{and} \quad \kappa_D(z, X) = \inf_{r \in \sigma_2(z,X)} r. \tag{1}$$

We call $\widetilde{k}_D$ the *Lempert function* and κ_D the *Kobayashi–Royden pseudometric* (see e.g. [3]). A holomorphic mapping $f : \mathbb{D} \to D$ is a $\widetilde{k}_D$*-extremal* (resp. κ_D*-extremal*) for $z, w \in D, z \neq w$ (resp. $z \in D$, $X \in \mathbb{C}^n \setminus \{0\}$) if $f(0) = z$ and $f(\widetilde{k}_D(z, w)) = w$ (resp. $f(0) = z$ and $\kappa_D(z, X)f'(0) = X$).

The Lempert function and the Kobayashi–Royden pseudometric play an essential role in complex analysis, especially in problems related to boundary properties of biholomorphic (more generally, proper holomorphic) mappings (see e.g. [2]). In many cases, the primary problem is to show that appropriate bounded extremal functions $f : \mathbb{D} \to D$ are *almost proper*, i.e., $f^*(\zeta) \subset \partial D$ for a.a. $\zeta \in \mathbb{T}$, where $\mathbb{T}$ denotes the unit circle and f^* denotes the non-tangential boundary value of f (see e.g. [6]). This problem was studied for example in [4], [5], [1], [3]. The main idea of the paper is to give a truly elementary proof of a result from [5] in a more general setting.

2000 *Mathematics Subject Classification*: Primary 32F45.
Key words and phrases: Kobayashi–Royden metric, Lempert function, extremal mapping, almost proper mapping.

DOI: 10.4064/bp57-2-6

THEOREM 1. *Let $D \Subset \mathbb{C}^n$ be a weakly Runge domain (see below) and let $f : \mathbb{D} \to D$ be a holomorphic mapping such that for some $\gamma > 0$ we have*

$$\operatorname{dist}(f(\lambda), \partial D) \geq \gamma(1 - |\lambda|), \quad \lambda \in \mathbb{D}. \tag{2}$$

Assume that f is $\widetilde{k}_D$- or κ_D-extremal. Then for any $\alpha > 0$ and any $\beta < 1$ the set

$$\{\lambda \in \mathbb{T} : \operatorname{dist}(f(t\lambda), \partial D) \geq \alpha(1-t)^\beta,\ t \in (0,1)\} \tag{3}$$

has Lebesgue measure zero in $\mathbb{T}$. In particular, $f^*(\zeta) \in \partial D$ for a.a. $\zeta \in \mathbb{T}$.

We say that $D \Subset \mathbb{C}^n$ is a *weakly Runge domain* if there exists a domain $G \supset \overline{D}$ such that for any bounded holomorphic mapping $f : \mathbb{D} \to G$ with $f^*(\mathbb{T}) \Subset D$ we have $f(\mathbb{D}) \Subset D$.

Proof of Theorem 1. For $\alpha > 0$ and $\beta < 1$ we put

$$Q(\alpha, \beta) = \{\lambda \in \mathbb{T} : \operatorname{dist}(f(t\lambda), \partial D) \geq \alpha(1-t)^\beta,\ t \in (0,1)\}.$$

Note that for any $\beta_1 < \beta_2$ we have $Q(\alpha, \beta_1) \subset Q(\alpha, \beta_2)$.

So, without loss of generality we may assume that for some $\alpha > 0$ and $\beta \in (0,1)$ the set $Q(\alpha,\beta)$ has positive measure. In the following we denote the set $Q(\alpha, \beta)$ by P. We may assume that $0 < (2\pi)^{-1}\int_P d\theta < 1$ (otherwise we take as P any smaller subset of $Q(\alpha,\beta)$ of positive measure). We put

$$\varphi(z) = \frac{1}{2\pi}\int_P \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\theta.$$

Note that $0 < \Re\varphi(\lambda) < 1$ for any $\lambda \in \mathbb{D}$. Without loss of generality, we may assume that f is $\widetilde{k}_D$-extremal for points $f(0), f(\sigma)$ (resp. κ_D-extremal for $f(0), f'(0)$). For a fixed $t \in (0,1)$ consider a mapping

$$g_t(\lambda) = f(t\lambda) + e^{\gamma_t(\varphi(\lambda) - \varphi(\sigma))}\,\frac{\lambda}{\sigma}\,(f(\sigma) - f(t\sigma))$$

(resp. $g_t(\lambda) = f(t\lambda) + e^{\gamma_t(\varphi(\lambda)-\varphi(0))}\lambda(1-t)f'(0)$), where $\gamma_t \in \mathbb{R}$ will be chosen later. Note that $g_t(0) = f(0)$ and $g_t(\sigma) = f(\sigma)$ (resp. $g_t(0) = f(0)$ and $g_t'(0) = f'(0)$). Our aim is to show that for all $t \in (0,1)$ sufficiently close to 1 we can choose γ_t in such a way that $g_t(\mathbb{D}) \Subset D$, which contradicts the extremality of f. To get this we only have to show that $g_t^*(\mathbb{T}) \Subset D$. We will prove this for $\widetilde{k}_D$-extremal mappings (for κ_D-extremals one can use similar arguments).

It is sufficient to show that for any t close to 1 we have

$$\left\| e^{\gamma_t(\varphi^*(\lambda) - \varphi(\sigma))}\,\frac{\lambda}{\sigma}\,(f(\sigma) - f(t\sigma)) \right\| \leq \frac{\alpha}{2}(1-t)^\beta \quad \text{for } \lambda \in P$$

and

$$\left\| e^{\gamma_t(\varphi^*(\lambda) - \varphi(\sigma))}\,\frac{\lambda}{\sigma}\,(f(\sigma) - f(t\sigma)) \right\| \leq \frac{\gamma}{2}(1-t) \quad \text{for } \lambda \in \mathbb{T} \setminus P.$$

Since $\|f(\sigma) - f(t\sigma)\| \le \rho|\sigma|(1-t)$, it suffices to have

$$e^{\gamma_t(1-\Re\varphi(\sigma))}\rho \le \frac{\alpha}{2}\,(1-t)^{\beta-1}$$

and

$$e^{-\gamma_t\Re\varphi(\sigma)}\rho \le \frac{\gamma}{2}. \tag{4}$$

Take γ_t such that

$$e^{\gamma_t(1-\Re\varphi(\sigma))}\rho = \frac{\alpha}{2}\,(1-t)^{\beta-1}.$$

Then for t sufficiently close to 1 we also have inequality (4). Moreover,

$$\|g_t - f(t\,\cdot)\|_{\mathbb{D}} \to 0 \quad \text{ as } t \to 1.$$

Since D is a weakly Runge domain, $g_t(\mathbb{D}) \Subset D$ for t close enough to 1.

To end the proof suppose that there exists a set $P \subset \mathbb{T}$ of positive measure such that for all $\zeta \in P$ we have

$$\operatorname{dist}(f^*(\zeta), \partial D) > \epsilon > 0.$$

Put

$$P_n = \{\lambda \in \mathbb{T} : \operatorname{dist}(f(t\lambda), \partial D) > \epsilon \text{ for any } t \in (1-1/n, 1)\}, \quad n \in \mathbb{N}.$$

Note that $P \subset \bigcup_{n\in\mathbb{N}} P_n$. Hence, for some n_0 the set P_{n_0} is of positive measure. ∎

Remark 2. (i) Note that any Runge domain is weakly Runge.

(ii) Take any domain $G \subset \mathbb{C}^n$ and let u be a plurisubharmonic function in G. Assume that $D = \{z \in G : u(z) < 0\} \Subset G$. Then D is weakly Runge.

Let us show that (2) holds for any analytic disc in a large class of domains.

Definition 3 (see [5]). A domain $D \subset \mathbb{C}^n$ is called *ρ-pseudoconvex* if there is a $\rho \in \mathrm{PSH} \cap \mathcal{C}(\overline{D})$ such that $\rho|_{\partial D} = 0$, $\rho < 0$ on D and $\operatorname{dist}(z, \partial D) \ge |\rho(z)|$.

Proposition 4. *Let $D \subset \mathbb{C}^n$ be a ρ-pseudoconvex domain and let $f : \mathbb{D} \to D$ be an analytic disc. Then* (2) *is satisfied.*

Proof. Let ρ be a plurisubharmonic function given by the definition of the ρ-psedoconvex domain. Consider the subharmonic function $v = \rho \circ f$. Note that for some $C > 0$ we have $|\rho(f(\zeta))| \ge C(1 - |\zeta|)$ (see e.g. [5]), and therefore $\operatorname{dist}(f(\zeta), \partial D) \ge C(1 - |\zeta|)$. ∎

Note that if D_1, D_2 are bounded domains and D is any connected component of $D_1 \cap D_2$, and if $f : \mathbb{D} \to D$ is such that $\operatorname{dist}(f(\lambda), \partial D_j) \ge \gamma_j(1 - |\lambda|)$ for $j = 1, 2$ and $\lambda \in \mathbb{D}$ then $\operatorname{dist}(f(\lambda), \partial D) \ge \min\{\gamma_1, \gamma_2\}(1 - |\lambda|)$. The class of ρ-pseudoconvex domains contains in particular the strongly pseudoconvex domains and the analytic polyhedra, i.e., bounded connected components of sets $\{z \in \mathbb{C}^n : |f_j(z)| < 1, j = 1, \ldots, m\}$, where f_j, $j = 1, \ldots, m$, are holomorphic functions in $\mathbb{C}^n$.

REMARK 5. In the proof of Theorem 2 in [5], E. Poletsky used the fact that if $D \Subset \mathbb{C}^n$ is a ρ-pseudoconvex domain and if $f : \mathbb{D} \to \mathbb{C}^n$ is a bounded holomorphic mapping such that $f^*(\zeta) \in D$ for a.a. $\zeta \in \mathbb{T}$, then $f(\mathbb{D}) \subset D$. Note that this is not true for annuli on the complex plane (which are ρ-pseudoconvex and weakly Runge). That is why in Theorem 1 we assume more, namely a Runge type property.

REMARK 6. W. Zwonek [7] constructed a pseudoconvex Reinhardt domain D and an extremal mapping $f : \mathbb{D} \to D$ for which (2) is not satisfied. Consider the domain $D = \{(z, w) \in \mathbb{D}^2 : |w| < e^{|z|/(|z|-1)}\}$ and the holomorphic mapping $f(\lambda) = (\lambda, 0)$. Then D is a pseudoconvex Reinhardt domain and $f : \mathbb{D} \to D$ is an extremal mapping. However, (2) is not satisfied.

COROLLARY 7. *Let $G \subset \mathbb{C}^n$ be a domain and let $f_1, \ldots, f_m$ be holomorphic functions such that*

$$\widetilde{G} = \{z \in G : |f_j(z)| < 1, j = 1, \ldots, m\} \Subset G.$$

If D is any connected component of $\widetilde{G}$ then any $\widetilde{k}_D$- and κ_D-extremal is almost proper.

REMARK 8. Note that if $D \subset \mathbb{C}$ is a taut domain, i.e., different from $\mathbb{C}$ and $\mathbb{C} \setminus \{a\}$, $a \in \mathbb{C}$, then any k_D- and κ_D-extremal $f : \mathbb{D} \to D$ is a covering (see e.g. [3]). Therefore, f is almost proper. We do not know whether for any taut domain in $\mathbb{C}^n$ its extremal mappings are almost proper.

Using the above technique we can show the following property of $\widetilde{k}_D$-extremals.

PROPOSITION 9. *Let $D \Subset \mathbb{C}^n$ be a domain and let $f : \mathbb{D} \to D$ be a holomorphic mapping such that for some $\gamma > 0$ we have*

$$\operatorname{dist}(f(\lambda), \partial D) \geq \gamma(1 - |\lambda|), \quad \lambda \in \mathbb{D}. \tag{5}$$

Assume that f is $\widetilde{k}_D$-extremal for $(f(0), f(\sigma))$. Then $f'(\sigma) \neq 0$.

Proof. For a fixed $t \in (0, 1)$ consider a mapping

$$g_t(\lambda) = f(t\lambda) + \frac{\lambda}{\sigma}(f(\sigma) - f(t\sigma)).$$

Note that $g_t(0) = f(0)$ and $g_t(\sigma) = f(\sigma)$. Assume that $f'(\sigma) = 0$. We want to show that for t sufficiently close to 1 we have $g_t(\mathbb{D}) \Subset D$. Indeed, put

$$\psi_t(\lambda) = \lambda \frac{f(\sigma) - f(t\sigma)}{\sigma(1 - t)}.$$

Then $\|\psi_t\|_{\mathbb{D}} \to 0$ as $t \to 1$. Hence, for t sufficiently close to 1 we have $\|\psi_t\|_{\mathbb{D}} \leq \gamma/2$. ∎

Acknowledgements. The authors thank Professor Włodzimierz Zwonek for helpful remarks.

The first author was supported in part by the Polish Ministry of Science and Higher Education Grant No. N N201 361436.

References

[1] A. Edigarian, *On extremal mappings in complex ellipsoids*, Ann. Polon. Math. 62 (1995), 83–96.
[2] F. Forstnerič and J.-P. Rosay, *Localizations of the Kobayashi metric and the boundary continuity of proper holomorphic mappings*, Math. Ann. 279 (1987), 239–252.
[3] M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, de Gruyter, 1993.
[4] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France 109 (1981), 427–474.
[5] E. Poletsky, *The Euler–Lagrange equations for extremal holomorphic mappings of the unit disk*, Michigan Math. J. 30 (1983), 317–333.
[6] W. Rudin, *Real and Complex Analysis*, 2nd ed., McGraw-Hill, New York, 1974.
[7] W. Zwonek, personal communication.

Armen Edigarian and Przemysław Kliś
Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
E-mail: armen.edigarian@im.uj.edu.pl
przemyslaw.klis@im.uj.edu.pl

Received May 25, 2009;
received in final form June 10, 2009 (7717)