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On the Relation between the S-matrix and the Spectrum of the Interior Laplacian

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Summary. The main results of this paper are:

1) a proof that a necessary condition for 1 to be an eigenvalue of the S-matrix is real analyticity of the boundary of the obstacle,

2) a short proof that if 1 is an eigenvalue of the S-matrix, then k^2 is an eigenvalue of the Laplacian of the interior problem, and that in this case there exists a solution to the interior Dirichlet problem for the Laplacian, which admits an analytic continuation to the whole space $\mathbb{R}^3$ as an entire function.

1. Introduction and statement of the result. We consider the obstacle scattering problem in $\mathbb{R}^3$, but the argument and the results remain valid in $\mathbb{R}^n$, $n \geq 2$.

Let the obstacle $D \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary S. Denote by $D' = \mathbb{R}^3 \setminus D$ the exterior domain and by N, the unit normal to S, pointing into D'. Let $k > 0$ be the wave number, and S^2 be the unit sphere in $\mathbb{R}^3$. The scattering matrix

$$\mathcal{S} = \mathcal{S}(k) = I - \frac{k}{2\pi i} A$$

for the obstacle scattering problem is a unitary operator in $L^2(S^2)$, I is the identity operator and A is an integral operator in $L^2(S^2)$, whose kernel $A(\beta, \alpha, k)$ is the scattering amplitude, which is defined in formula (5) below. The operator $\mathcal{S}$ has an eigenvalue 1 if and only if equation $Aw = 0$ has a non-trivial solution. The eigenvalues of $\mathcal{S}$ have 1 as an accumulation point, they all have absolute values equal to 1 since $\mathcal{S}$ is unitary.

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The following conjecture (the Doron–Smilansky (DS) conjecture) is known:

DS CONJECTURE. *A number $k^2 > 0$ is a Dirichlet eigenvalue of the Laplacian in a bounded domain D if and only if the corresponding $\mathcal{S}$-matrix for the scattering problem by the obstacle D has an eigenvalue* 1.

This conjecture is discussed in [1]–[3], and in [3] a counterexample to this conjecture is mentioned.

From the definition of the $\mathcal{S}$-matrix it follows that 1 is its eigenvalue if and only if 0 is an eigenvalue of A, that is, equation (12) (see below) has a non-trivial solution.

We prove (see Theorem 2) that if equation (12) has a non-trivial solution, then the boundary S of D is an analytic set. Since generically S is not an analytic set, it follows that the DS conjecture is incorrect. Our result gives a necessary condition for 1 to be an eigenvalue of the $\mathcal{S}$-matrix. This condition is not sufficient (and therefore not sufficient for the DS conjecture to hold for the domain D).

In [3] it is proved that if $D \subset \mathbb{R}^2$ is a bounded domain with a sufficiently smooth boundary S, and if 1 is a Dirichlet eigenvalue of $\mathcal{S}$, then k^2 is a Dirichlet eigenvalue of the Laplacian in D. An open problem, stated in [3], is to prove such a statement for $D \subset \mathbb{R}^n$ with $n > 2$. This is done in our paper by a method different from the one in [3]. Our proof is short and simple.

Let S^2_j, $j = 1, 2$, be arbitrary small fixed open subsets of S^2, and let the boundary conditions on S be either the Dirichlet, Neumann, or Robin conditions.

The following theorem is proved in [5, p. 85]:

THEOREM (Ramm). *The knowledge of $A(\beta, \alpha, k)$ for all $\alpha \in S^2_1$ and $\beta \in S^2_2$, and for a fixed $k > 0$, determines S and the boundary conditions on S uniquely.*

It follows that the knowledge of the S-matrix $\mathcal{S}(k)$ at a fixed $k > 0$ determines the boundary S of the obstacle and the boundary condition on S uniquely.

Therefore, the discrete spectrum of the Laplacian in D, corresponding to this boundary condition, is uniquely determined by the knowledge of $\mathcal{S}(k)$ at a fixed $k > 0$.

This conclusion establishes a relation between the S-matrix and the spectrum of the Laplacian in D.

Let us now formulate the obstacle scattering problem, introduce basic notions, and formulate our results.

The *scattering solution* $u(x,\alpha,k)$ is the solution to the following scattering problem:

$$Lu := (\nabla^2 + k^2)u = 0 \quad \text{in } D', \tag{1}$$

$$u|_S = 0, \tag{2}$$

$$u = u_0 + v, \quad u_0 := e^{ik\alpha\cdot x}, \tag{3}$$

$$\frac{\partial v}{\partial r} - ikr = o\left(\frac{1}{r}\right), \quad r := |x| \to \infty. \tag{4}$$

Here $\alpha \in S^2$ is the incident direction, i.e., the direction of the incident plane wave u_0, and v is the scattered field which satisfies the radiation condition (4). This condition implies that

$$v := v(x,\alpha,k) = A(\beta,\alpha,k)\,\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \beta := \frac{x}{r}. \tag{5}$$

The function $A := A(\beta,\alpha,k)$ is called the *scattering amplitude*. Let us denote by $A : L^2(S^2) \to L^2(S^2)$ the operator

$$Aw := \int_{S^2} A(\beta,\alpha,k)w(\alpha)\,d\alpha. \tag{6}$$

It is well known (see [5]) that problem (1)–(4) has a unique solution $u(x,\alpha,k)$,

$$A(\beta,\alpha,k) = -\frac{1}{4\pi}\int_S e^{-ik\beta\cdot s}u_N(s,\alpha,k)\,ds, \tag{7}$$

where $u_N(s,\alpha,k)$ is the normal derivative of the scattering solution $u(x,\alpha,k)$ on S, and the following relation holds:

$$u(x,\alpha,k) = e^{ik\alpha\cdot x} - \int_S g(x,s,k)u_N(s,\alpha,k)\,ds. \tag{8}$$

Here G, the resolvent kernel of the Dirichlet Laplacian in the exterior domain D', satisfies the following equation:

$$G(x,y,k) = g(x,y,k) - \int_S g(x,s,k)G_N(s,y,k)\,ds, \tag{9}$$

where

$$g(x,y,k) := \frac{e^{ik|x-y|}}{4\pi|x-y|}. \tag{10}$$

The function G solves the boundary value problem

$$LG = -\delta(x-y) \quad \text{in } D', \quad G|_S = 0, \tag{11}$$

and satisfies the radiation condition (4).

Let σ denote the set of eigenvalues of the Dirichlet Laplacian in D. This set is discrete.

It is proved in [5, pp. 52–57] that:

(a) The function $A(\beta, \alpha, k)$ admits a meromorphic continuation as a function of k from the ray $(0, \infty)$ to the whole complex k-plane.
(b) The scattering amplitude $A(\beta, \alpha, k)$ is analytic in the region $\operatorname{Im} k \geq 0$ (if $D \subset \mathbb{R}^{2n}$ then $k = 0$ is a logarithmic branch point).
(c) $A(\beta, \alpha, k)$ has infinitely many poles on the imaginary axis in the region $\operatorname{Im} k < 0$.
(d) As a function of α and β, the scattering amplitude $A(\beta, \alpha, k)$ admits analytic continuation from $S^2 \times S^2$ to the set $M \times M$, where $M := \{\Theta \in \mathbb{C}^3 : \Theta \cdot \Theta = 1\}$, where $\Theta \cdot \omega := \sum_{j=1}^{3} \Theta_j \, \omega_j$. The set M is a non-compact algebraic variety in $\mathbb{C}^3$.

Let us now state our basic results:

THEOREM 1. *If $\mathcal{S}(k)$ has an eigenvalue* 1, *that is*, *the equation*

$$Aw = \int_{S^2} A(\beta, \alpha, k) w(\alpha) d\alpha = 0 \tag{12}$$

has a non-trivial solution w, then $k^2 \in \sigma$, and there is a solution to the problem $(\nabla^2 + k^2)W = 0$ in D, $W|_S = 0$, which can be extended from D to $\mathbb{R}^3$ as a bounded entire function of x.

THEOREM 2. *If equation* (12) *has a non-trivial solution*, *then the boundary S is an analytic set.*

An *analytic set* is a set of zeros of (a finite collection of) analytic functions. One can find the definition and properties of analytic sets in [4, Section 1.4]). If S is an analytic set, then S is a piecewise real analytic surface. Since generically S is not piecewise real analytic surface, it follows from Theorem 2 that the DS conjecture is incorrect.

In Section 2 Theorems 1 and 2 are proved. In the proofs, the following result of the author is used:

LEMMA 1 ([5, p. 46]). *One has*

$$G(x, y, k) = \frac{e^{ik|y|}}{4\pi|y|} u(x, \alpha, k)[1 + o(1)], \qquad |y| \to \infty, \qquad \frac{y}{|y|} = -\alpha, \tag{13}$$

where $u(x, \alpha, k)$ is the scattering solution, *i.e.*, *the solution to* (1)–(4).

Lemma 1 yields formula (8) as a consequence of (9), while formula (9) is obtained by Green's formula. Formula (7) follows from (8).

2. Proofs

Proof of Theorem 1. Let us prove that if $w \not\equiv 0$ solves (12) *then $k^2 \in \sigma$.* Assume that equation (12) has a non-trivial solution w. Multiply (7) by

$w = w(\alpha)$ and integrate over S^2 with respect to α. The result is

$$(14)\qquad \int_S e^{-ik\beta\cdot s} p(s)\, ds = 0, \qquad p(s) := \int_{S^2} u_N(s,\alpha,k) w(\alpha)\, d\alpha.$$

Let us prove that $p(s) \not\equiv 0$. Indeed, if

$$(15)\qquad p(s) = \int_{S^2} u_N(s,\alpha,k) w(\alpha)\, d\alpha = 0 \quad \forall s \in S,$$

then $w(\alpha) \equiv 0$ because the set $\{u_N(s,\alpha,k)\}_{\alpha\in S^2}$ is total (dense) in $L^2(S)$ for any fixed $k > 0$ ([5, p. 162]).

Now equation (14) and Lemma 1 imply that

$$(16)\qquad \nu(x) := \int_S \frac{e^{ik|x-s|}}{4\pi|x-s|}\, p(s)\, ds$$

is identically zero in D'. Indeed, this ν solves equation (1), satisfies the radiation condition (4), and (14) implies

$$(17)\qquad \nu(x) = o(1/|x|), \qquad |x| \to \infty.$$

Relation (17) and Lemma 1 in [5, p. 25] imply that

$$(18)\qquad \nu(x) = 0 \quad \text{in } D'.$$

Therefore, by the jump formula for the normal derivative of the single layer potential (16) ([5, p. 14]), one gets

$$(19)\qquad \frac{\partial \nu}{\partial N_+} = p(s) \not\equiv 0,$$

where $\partial/\partial N_+$ denotes the limiting value on S of the normal derivative from inside of D.

This implies that $k^2 \in \sigma$. Indeed, $\nu(x)$ solves the equation

$$(20)\qquad (\nabla^2 + k^2)\nu = 0 \quad \text{in } D',$$

and satisfies the boundary condition

$$(21)\qquad \nu|_S = 0,$$

due to (18) and the continuity of ν across S. Finally, $\nu \not\equiv 0$ in D because of (19).

Now we prove the existence of a solution to problem (20)–(21) *which can be analytically continued to the whole space* $\mathbb{R}^3$ *as an entire function of* x.

The reciprocity relation $A(\beta,\alpha,k) = A(-\alpha,-\beta,k)$ (see [5, p. 53] and equation (12) imply

$$0 = \int_{S^2} A(\beta,\alpha,k)w(\alpha)\,d\alpha = -\frac{1}{4\pi}\int_S \Big(\int_{S^2} e^{ik\alpha\cdot s}w(\alpha)\,d\alpha\Big)u_N(s,-\beta)\,ds \quad \forall\beta\in S^2. \tag{22}$$

Since the set $\{u_N(s,\alpha,k)\}_{\alpha\in S^2}$ is total (dense) in $L^2(S)$ for any fixed $k>0$ ([5, p. 162]), relation (22) implies

$$\int_{S^2} e^{ik\alpha\cdot s}w(\alpha)\,d\alpha = 0 \quad \forall s\in S. \tag{23}$$

Therefore, the function

$$W(x) := \int_{S^2} e^{ik\alpha\cdot x}w(\alpha)\,d\alpha, \qquad x\in\mathbb{R}^3, \tag{24}$$

satisfies all the requirements mentioned in the last statement of Theorem 1.

Thus, Theorem 1 is proved. ■

Remark 1. A similar argument yields the following result:

Let $\sigma_{\mathcal{N}}$ be the set of eigenvalues of the Neumann Laplacian, and $A_{\mathcal{N}}(\beta,\alpha,k)$ the scattering amplitude, corresponding to the plane wave scattering by the obstacle D on the boundary of which the Neumann boundary condition holds. If equation (12), *with $A_{\mathcal{N}}$ in place of A, has a non-trivial solution, then $k^2\in\sigma_{\mathcal{N}}$.*

Remark 2. If $k^2\in\sigma$, then any non-trivial solution to (20)–(21) can be written in the form (16) with $p(s)$ defined in (19), and the boundary condition (18) holds. Taking $|x|\to\infty$, $x/|x|=\beta$, in (16) and using (18), one obtains

$$\int_S e^{-ik\beta\cdot s}p(s)\,ds = 0 \quad \forall\beta\in S^2, \qquad p(s)\not\equiv 0. \tag{25}$$

Thus, if $k^2\in\sigma$, then equation (25) has a non-trivial solution $p(s)$.

Proof of Theorem 2. Suppose equation (12) has a solution $\eta\in L^2(S^2)$, $\eta\not\equiv 0$. Then

$$\int_S ds\, u_N(s,\alpha)\int_{S^2} e^{-ik\beta\cdot s}\eta(\beta)\,d\beta = 0 \quad \forall\alpha\in S^2. \tag{26}$$

Since the set $\{u_N(s,\alpha)\}_{\alpha\in S^2}$ is total in $L^2(S)$, one concludes from (26) that

$$\psi(s) = 0 \quad \forall s\in S, \tag{27}$$

where

$$\psi(x) := \int_{S^2} e^{-ik\beta\cdot x}\eta(\beta)\,d\beta.$$

The function $\psi(x)$ is an entire function of x, that is, an analytic function of $x \in \mathbb{C}^3$. It vanishes on S, so S is an analytic set. Generically, the boundary S is not an analytic set.

Thus, Theorem 2 is proved. ■

REMARK 3. If one uses the reciprocity relation $A(\beta,\alpha,k)=A(-\alpha,-\beta,k)$, then one concludes that zero is an eigenvalue of A if either

$$\int_S e^{-ik\beta\cdot s}\int_{S^2} u_N(s,\alpha,k)w(\alpha)\,d\alpha = 0 \quad \forall\beta\in S^2, \quad w\not\equiv 0, \tag{28}$$

or

$$\int_S \Big(\int_{S^2} e^{ik\alpha\cdot s}w(\alpha)\,d\alpha\Big)u_N(s,-\beta)\,ds = 0 \quad \forall\beta\in S^2, \quad w\not\equiv 0. \tag{29}$$

The last relation implies equation (28) (with $\beta=-\alpha$ and $\eta(\beta)=w(\alpha)$).

Set $T_kp := \int_S g(s,t,k)p(t)\,dt$ and $U := U(x,k) := \int_S g(x,t,k)p(t)\,dt$, so $U|_S = T_kp$.

REMARK 4. The operator T_k^{-1} has simple poles at the points $k^2 = k_j^2$, where $k_j^2 \in \sigma$.

Remark 4 shows that the knowledge of the set of poles of the operator T_k^{-1} allows one to find the spectrum of the interior Dirichlet Laplacian in D.

Proof of Remark 4. Consider the equation $T_kp = f$. Then

$$U(x) = \int_S g(x,t,k)p(t)\,dt$$

solves the problem

$$(\nabla^2+k^2)U = 0 \quad \text{in } D, \qquad U|_S = f. \tag{30}$$

Let

$$(\nabla^2+k^2)\mathit{\Gamma} = -\delta(x-y) \quad \text{in } D, \qquad \mathit{\Gamma}|_S = 0.$$

Then Green's formula yields the following representation of the solution to problem (30):

$$U(x) = -\int_S f(t)\mathit{\Gamma}_{N_t}(t,x,k)\,dt, \qquad x\in D, \qquad k^2\neq k_j^2. \tag{31}$$

Since $\mathit{\Gamma}(x,y,k) = \sum_{j=1}^{\infty}\frac{\phi_j(x)\overline{\phi_j(y)}}{k^2-k_j^2}$ has a simple pole at $k^2=k_j^2$, the claim is proved. Here ϕ_j are the normalized eigenfunctions of the Dirichlet Laplacian in D.

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