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On Applications of Bing–Krasinkiewicz–Lelek Maps

by

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Summary. We characterize Peano continua using Bing–Krasinkiewicz–Lelek maps. Also we deal with some topics on Whitney preserving maps.

1. Introduction. In this note, all spaces are separable metrizable and maps are continuous. We denote the interval $[0,1]$ by I. A compact metric space is called a *compactum*, and *continuum* means a connected compactum. If X is a continuum, $C(X)$ denotes the space of all subcontinua of X with the topology generated by the Hausdorff metric. A continuum is said to be *indecomposable* if it is not the union of two proper subcontinua. A compactum is said to be a *Bing compactum* (or to be *hereditarily indecomposable*) if each of its subcontinua is indecomposable. A map between compacta is called a *light map* if each of its fibers is 0-dimensional. A map $f : X \to Y$ is called *n-dimensional* if $\dim f^{-1}(y) \leq n$ for each $y \in Y$.

A map between compacta is called a *Bing map* if each of its fibers is a Bing compactum (cf. [5], [7], [10], and [16]).

A map $f : X \to Y$ between compacta is called a *Krasinkiewicz map* if any continuum in X either contains a component of a fiber of f or is contained in a fiber of f (cf. [8], [13] and [14]).

Let $f : X \to Y$ be a map between compacta. For each $a > 0$, let $F(f,a)$ be the union of the components A of fibers with $\operatorname{diam} A \geq a$, and put

$$F(f) = \bigcup_{i=1}^{\infty} F(f, 1/i).$$

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Define the *Lelek dimension* of f to be the number $\dim_{\mathrm{L}} f = \dim F(f)$. For each $n \geq 1$, $f : X \to Y$ is called an *n-dimensional Lelek map* if $\dim_{\mathrm{L}} f \leq n$ (cf. [6] and [12]). In case $n \leq 0$, for convenience, a map $f : X \to Y$ is an n-dimensional Lelek map if and only if it is a 0-dimensional map. Note that an n-dimensional Lelek map is an n-dimensional map.

A map $f : X \to Y$ is called a *Bing–Krasinkiewicz map* if f is a Bing map and a Krasinkiewicz map. A map $g : X \to Y$ is called an *n-dimensional Bing–Krasinkiewicz–Lelek map* if g is a Bing map, a Krasinkiewicz map and an n-dimensional Lelek map.

In this paper we prove some theorems using these maps.

In Section 2 we characterize Peano continua using n-dimensional Bing–Krasinkiewicz–Lelek maps (Theorem 2.8 and Corollary 2.9), and give an affirmative answer to Problem 12 of [5].

In Section 3 we study Whitney preserving maps. If $f : X \to Y$ is a map between continua, then define a map $\widehat{f} : C(X) \to C(Y)$ by $\widehat{f}(A) = f(A)$ for each $A \in C(X)$. A map $f : X \to Y$ is said to be *Whitney preserving* if there exist Whitney maps (see [4]) $\mu : C(X) \to I$ and $\nu : C(Y) \to I$ such that for each $s \in [0, \mu(X)]$, $\widehat{f}(\mu^{-1}(s)) = \nu^{-1}(t)$ for some $t \in [0, \nu(Y)]$. In this case, we say that f is *μ, ν-Whitney preserving.*

The notion of a Whitney preserving map was introduced by Espinoza (cf. [2] and [3]). In Section 3 we generalize a result of Espinoza using Krasinkiewicz maps (Theorem 3.5). Also we give an application using 1-dimensional Bing–Krasinkiewicz–Lelek maps.

2. Bing–Krasinkiewicz–Lelek maps onto Peano continua. First we give some notations. Let X and Y be compacta. Then $C(X, Y)$ denotes the set of all continuous maps from X to Y endowed with the sup metric. Let $C_{\mathrm{s}}(X, Y)$ be the subset of $C(X, Y)$ which consists of all surjective maps from X onto Y. Note that $C_{\mathrm{s}}(X, Y)$ is a closed subset of $C(X, Y)$.

Let

$$\begin{aligned} B(X,Y) &= \{f \in C(X,Y) \mid f \text{ is a Bing map}\}, \\ B_s(X,Y) &= \{f \in C(X,Y) \mid f \text{ is a surjective Bing map}\}, \\ K(X,Y) &= \{f \in C(X,Y) \mid f \text{ is a Krasinkiewicz map}\}, \\ K_{\mathrm{s}}(X,Y) &= \{f \in C(X,Y) \mid f \text{ is a surjective Krasinkiewicz map}\}, \\ L_n(X,Y) &= \{f \in C(X,Y) \mid f \text{ is an } n\text{-dimensional Lelek map}\}, \\ BK(X,Y) &= B(X,Y) \cap K(X,Y), \\ BK_{\mathrm{s}}(X,Y) &= B_{\mathrm{s}}(X,Y) \cap K_{\mathrm{s}}(X,Y). \end{aligned}$$

LEMMA 2.1. *Let X be a compactum. Then $BK(X, I)$ is a dense G_δ-subset in $C(X, I)$.*

Proof. Since $B(X,I)$ and $K(X,I)$ are dense G_δ-subsets in $C(X,I)$ (cf. [7], [10], [14] and [16]), $BK(X,I)$ is a dense G_δ-subset in $C(X,I)$. ∎

LEMMA 2.2 (cf. [6], [12]). *Let X be a compactum and $Z \subset X$ a 0-dimensional F_σ-subset of X. Then $\{f \in C(X,I) \mid F(f) \cap Z = \emptyset\}$ is a dense G_δ-subset in $C(X,I)$.*

LEMMA 2.3 (cf. [6], [12]). *Let $n \geq 0$. Then for each $(n+1)$-dimensional compactum X, $L_n(X,I)$ is a dense G_δ-subset in $C(X,I)$.*

Let $f : X \to Y$ be a map between compacta. If $g : Y \to Z$ is a light map between compacta, it is easy to see that T is a component of a fiber of f if and only if T is a component of a fiber of $g \circ f$. This yields

LEMMA 2.4. *Let X, Y and Z be compacta. If $f : X \to Y$ is a map and $g : Y \to Z$ is a light map, then*

(1) $F(f) = F(g \circ f)$,
(2) *if f is a Bing–Krasinkiewicz map, then so is $g \circ f$.*

LEMMA 2.5. *There exists a map $F : I^2 \to I$ such that for each $t \in I$, $F|_{I\times\{t\}} : I \times \{t\} \to I$ and $F|_{\{t\}\times I} : \{t\} \times I \to I$ are surjective.*

Proof. Let $A = \{(x,x) \in I^2 \mid x \in I\}$ and $B = \{(x, x+1/2) \in I^2 \mid 0 \leq x \leq 1/2\} \cup \{(x, x-1/2) \in I^2 \mid 1/2 \leq x \leq 1\}$. Take a continuous function $F : I^2 \to I$ such that $F(A) = 0$ and $F(B) = 1$. Then F is as required. ∎

Before stating the next lemma we give some notations. For any natural numbers n and $i \leq n$, let $N_n^i = \{\{a_1, \ldots, a_i\} \subset \mathbb{N} \mid 1 \leq a_1 < \cdots < a_i \leq n\}$ and $N_n = \bigcup_{i=1}^n N_n^i$. Let $X_1, \ldots, X_n$ be spaces. Let $a \in N_n$. If $a \neq \{1, \ldots, n\}$ let $\widehat{p}_a : \prod_{j=1}^n X_j \to \prod_{j\in\{1,\ldots,n\}\setminus a} X_j$ be the projection. If $a = \{1, \ldots, n\}$ let $\widehat{p}_a : \prod_{j=1}^n X_j \to \{0\}$ be the constant map. Note that for each $a \in N_n$ and $x \in \widehat{p}_a(\prod_{j=1}^n X_j)$, $\widehat{p}_a^{-1}(x)$ is a topological copy of $\prod_{j\in a} X_j$.

LEMMA 2.6. *Let $n \geq 1$ and let $X_1, \ldots, X_n$ be nondegenerate continua. Then there exists a map $G : \prod_{j=1}^n X_j \to I$ such that for each $a \in N_n$ and $x \in \widehat{p}_a(\prod_{j=1}^n X_j)$, $G|_{\widehat{p}_a^{-1}(x)} : \widehat{p}_a^{-1}(x) \to I$ is surjective.*

Proof. We argue by induction on n. For $n = 1$, any surjective map $G : X_1 \to I$ has the required property.

Assume that the assertion holds when $1 \leq n \leq k$. Let $n = k+1$. Let $G : \prod_{j=1}^k X_j \to I$ be as in the statement of the lemma and let $s' : X_{k+1} \to I$ be surjective. Define $s'' : \prod_{j=1}^k X_j \times X_{k+1} \to I^2$ by $s''(x_1, x_2) = (G(x_1), s'(x_2))$ for each $(x_1, x_2) \in \prod_{j=1}^k X_j \times X_{k+1}$. Let $F : I^2 \to I$ be as in Lemma 2.5 and set $G' = F \circ s'' : \prod_{j=1}^{k+1} X_j \to I$. Then it is easy to see that G' is as required. ∎

LEMMA 2.7. *Let $n \geq 1$ and let $X_1, \dots, X_n$ be nondegenerate continua such that $\dim X_j \leq m_j$ for each $j = 1, \dots, n$. Then there exists a 0-dimensional F_σ-subset $Z \subset \prod_{j=1}^n X_j$ such that for each $a \in N_n$ and $x \in \widehat{p}_a(\prod_{j=1}^n X_j)$, $\dim(\widehat{p}_a^{-1}(x) \setminus Z) \leq \sum_{j \in a} m_j - 1$.*

Proof. Note that for each $a \in N_n$ and $x \in \widehat{p}_a(\prod_{j=1}^n X_j)$, $\dim \widehat{p}_a(\prod_{j=1}^n X_j) < \infty$ and $\dim \widehat{p}_a^{-1}(x) = \dim \prod_{j \in a} X_j \leq \sum_{j \in a} m_j$. Hence by Proposition 2 of [17] for each $a \in N_n$ there exists a 0-dimensional F_σ-subset Z_a of $\prod_{j=1}^n X_j$ such that $\dim(\widehat{p}_a^{-1}(x) \setminus Z_a) \leq \sum_{j \in a} m_j - 1$ for each $x \in \widehat{p}_a(\prod_{j=1}^n X_j)$.

Letting $Z = \bigcup_{a \in N_n} Z_a$ completes the proof. ∎

It is known that every nondegenerate Peano continuum is a light image of I (cf. [15, Corollary 13.4]).

THEOREM 2.8. *Let A be a nondegenerate continuum. Then the following conditions are equivalent.*

(1) *A is a Peano continuum.*
(2) *Let $n \geq 1$ and let $X_1, \dots, X_n$ be nondegenerate continua. Then for each 0-dimensional F_σ-subset $Z \subset \prod_{j=1}^n X_j$ there exists a Bing–Krasinkiewicz map $H : \prod_{j=1}^n X_j \to A$ such that*
   (a) *for each $a \in N_n$ and $x \in \widehat{p}_a(\prod_{j=1}^n X_j)$, $H|_{\widehat{p}_a^{-1}(x)} : \widehat{p}_a^{-1}(x) \to A$ is surjective,*
   (b) *$F(H) \cap Z = \emptyset$.*
(3) *Let $n \geq 1$ and let $X_1, \dots, X_n$ be nondegenerate continua such that $\dim X_j \leq m_j$ for each $j = 1, \dots, n$. Then there exists a Bing–Krasinkiewicz map $H : \prod_{j=1}^n X_j \to A$ such that*
   (a) *for each $a \in N_n$ and $x \in \widehat{p}_a(\prod_{j=1}^n X_j)$, $H|_{\widehat{p}_a^{-1}(x)} : \widehat{p}_a^{-1}(x) \to A$ is surjective,*
   (b) *$\dim_{\mathrm{L}} H|_{\widehat{p}_a^{-1}(x)} \leq \sum_{j \in a} m_j - 1$ (hence $H|_{\widehat{p}_a^{-1}(x)} : \widehat{p}_a^{-1}(x) \to A$ is a $(\sum_{j \in a} m_j - 1)$-dimensional Lelek map).*

Proof. (1)⇒(2). By Lemmas 2.1, 2.2 and 2.6 there exists a Bing–Krasinkiewicz map $G_1 : \prod_{j=1}^n X_j \to I$ such that

(a′) for each $a \in N_n$ and $x \in \widehat{p}_a(\prod_{j=1}^n X_j)$, $[1/4, 3/4] \subset G_1(\widehat{p}_a^{-1}(x))$,
(b′) $F(G_1) \cap Z = \emptyset$.

Since A is a Peano continuum, there exists a light map $\ell : I \to A$ such that $\ell([1/4, 3/4]) = A$. Let $H = \ell \circ G_1 : \prod_{j=1}^n X_j \to A$. Since G_1 is a Bing–Krasinkiewicz map and ℓ is a light map, by Lemma 2.4, H is a Bing–Krasinkiewicz map. Since $\ell([1/4, 3/4]) = A$, by (a′), H satisfies (a). Since ℓ is a light map, by (b′) and Lemma 2.4, H satisfies (b).

(2)⇒(3). Let $Z\subset\prod_{j=1}^{n}X_j$ be a 0-dimensional F_σ-subset as in Lemma 2.7. By assumption there exists a Bing–Krasinkiewicz map $H:\prod_{j=1}^{n}X_j\to A$ such that

(a) for each $a\in N_n$ and $x\in\widehat{p}_a(\prod_{j=1}^{n}X_j)$, $H|_{\widehat{p}_a^{-1}(x)}:\widehat{p}_a^{-1}(x)\to A$ is surjective,
(b) $F(H)\cap Z=\emptyset$.

Now we prove that for each $a\in N_n$ and $x\in\widehat{p}_a(\prod_{j=1}^{n}X_j)$, $\dim_{\mathrm{L}}H|_{\widehat{p}_a^{-1}(x)}\le\sum_{j\in a}m_j-1$. Note that $F(H|_{\widehat{p}_a^{-1}(x)})\subset F(H)\cap\widehat{p}_a^{-1}(x)$. By (b) we have $\dim(F(H)\cap\widehat{p}_a^{-1}(x))\le\dim(\widehat{p}_a^{-1}(x)\setminus Z)$. As $\dim(\widehat{p}_a^{-1}(x)\setminus Z)\le\sum_{j\in a}m_j-1$, we have $\dim(F(H)\cap\widehat{p}_a^{-1}(x))\le\sum_{j\in a}m_j-1$. This means $\dim F(H|_{\widehat{p}_a^{-1}(x)})\le\sum_{j\in a}m_j-1$. Hence $\dim_{\mathrm{L}}H|_{\widehat{p}_a^{-1}(x)}\le\sum_{j\in a}m_j-1$.

(3)⇒(1). To prove this, consider the case when $X_1=I$. Then A is a continuous image of I, which means that A is a Peano continuum (cf. Theorem 8.18 of [15]). This completes the proof. ■

Note that if $f:X\to Y$ is a Bing–Krasinkiewicz map between compacta and A is a closed subset of X, then $f|_A:A\to Y$ is a Bing–Krasinkiewicz map. So as a corollary of Theorem 2.8, we have the following result.

Corollary 2.9. *Let Y be a nondegenerate continuum. Then the following conditions are equivalent.*

(1) *Y is a Peano continuum.*
(2) *For each nondegenerate continuum X there exists a surjective Bing–Krasinkiewicz map from X onto Y.*
(3) *For each $n\ge 0$ and $(n+1)$-dimensional continuum X there exists a surjective n-dimensional Bing–Krasinkiewicz–Lelek map from X onto Y.*

In [5], Kato and the author posed the following problem.

Problem 2.10 (Problem 12 of [5]). *For each nondegenerate continuum X and each nondegenerate Peano continuum Y, does there exist an upper semicontinuous decomposition $\mathcal{D}$ of X such that each element $D\in\mathcal{D}$ is a Bing compactum and the quotient space $X/\mathcal{D}$ is homeomorphic to Y?*

Corollary 2.9 gives an affirmative answer to Problem 2.10. In fact if X is a nondegenerate continuum, Y is a nondegenerate Peano continuum and $f:X\to Y$ is a surjective Bing map, let $\mathcal{D}=\{f^{-1}(y)\mid y\in Y\}$. Then it is easy to see that $\mathcal{D}$ is the required upper semicontinuous decomposition of X.

Remark. If X is a nondegenerate continuum and Y is a 1-dimensional Peano continuum, then $BK(X,Y)$ and $BK_{\mathrm{s}}(X,Y)$ are dense G_δ-subsets of $C(X,Y)$ and $C_{\mathrm{s}}(X,Y)$ respectively (cf. [5], [14] and [16]). But for each $n\ge 2$ there exists a nondegenerate continuum X and an n-dimensional

Peano continuum Y such that $B(X,Y)$ and $B_{\mathrm{s}}(X,Y)$ are not dense subsets of $C(X,Y)$ and $C_{\mathrm{s}}(X,Y)$ respectively (cf. [16]).

3. Whitney preserving maps. A subcontinuum $A \subset X$ is *terminal in* X if whenever $B \in C(X)$ satisfies $A \cap B \neq \emptyset$, then either $A \subset B$ or $B \subset A$.

In [2] Espinoza proved that every Whitney preserving map from a continuum containing a dense arc component onto I is a homeomorphism. Also, in [3] he proved the following result.

THEOREM 3.1 (cf. Theorem 2.5 and Corollary 2.7 of [3]). *Let $f : X \to Y$ be a monotone open map such that $f^{-1}(y)$ is a nondegenerate terminal continuum in X for each $y \in Y$. Then f is Whitney preserving.*

In [3], as an application of Theorem 3.1, Espinoza proved that for each 1-dimensional continuum M there exists a 1-dimensional continuum M', different from M, such that there exists a Whitney preserving map from M' onto M. Hence there exist a lot of Whitney preserving maps which are not homeomorphisms.

In this section we generalize Theorem 3.1 using Krasinkiewicz maps.

If $f : X \to Y$ is a map, let $\mathcal{A}_f = \{f^{-1}(y) \mid y \in Y\}$ and $\mathcal{A}'_f = \{C \mid C$ is a component of a fiber of $f\}$.

Let $f : X \to Y$ be a Whitney preserving map. Then $\mathcal{A}_f$ need not be a continuous decomposition of X. For example let $f : [0, \pi] \to S^1$ be defined by $f(t) = e^{4ti}$. Then f is Whitney preserving (cf. Example 2 of [2]). But f is not an open map.

PROPOSITION 3.2. *Let $f : X \to Y$ be a μ,ν-Whitney preserving map. Then $\mathcal{A}'_f$ is a continuous decomposition of X and each element of $\mathcal{A}'_f$ is terminal in X.*

Proof. Let $s_0 = \max\{s \in I \mid f(\mu^{-1}(s)) = \nu^{-1}(0)\}$. Now we show that

$$(*)\qquad \mathcal{A}'_f = \mu^{-1}(s_0).$$

Let $A' \in \mathcal{A}'_f$. Since f is μ,ν-Whitney preserving and A' is a component of a fiber of f, it is easy to see that $\mu(A') \leq s_0$. If $s_0 = 0$, then $A' \in \mu^{-1}(s_0)$. If $s_0 > 0$ and $\mu(A') < s_0$, then we can take a subcontinuum $B' \subset X$ such that

(1) $A' \subset B'$, $A' \neq B'$,
(2) $\mu(B') < s_0$.

Since A' is a component of a fiber of f, by (1), $f(B')$ is a nondegenerate continuum. This is a contradiction by (2) and the assumption that f is Whitney preserving. So $\mu(A') = s_0$. This means $\mathcal{A}'_f \subset \mu^{-1}(s_0)$. To prove the converse, let $A'' \in \mu^{-1}(s_0)$. Then $f(A'')$ is a one-point set. So there exists $A_\star \in \mathcal{A}'_f$ such that $A'' \subset A_\star$. If $A'' \neq A_\star$, then $\mu(A_\star) > s_0$. Since $f(A_\star)$ is a one-point set and f is μ,ν-Whitney preserving, this is a contradiction. So $A'' = A_\star \in \mathcal{A}'_f$. This means $\mu^{-1}(s_0) \subset \mathcal{A}'_f$. So we have proved $(*)$.

By Theorem 2.3 of [3] and $(*)$, $\mathcal{A}'_f$ is continuous decomposition of X and each element of $\mathcal{A}'_f$ is terminal in X. This completes the proof. ■

COROLLARY 3.3. *Let $f : X \to Y$ be a monotone Whitney preserving map. Then f is an open map and each fiber of f is terminal in X.*

The next proposition is inspired by an idea of Lemma 2.4 of [3].

PROPOSITION 3.4. *Let $f : X \to Y$ be a map such that $\mathcal{A}'_f$ does not contain a one-point set. Then the following conditions are equivalent.*

(1) *$\mathcal{A}'_f$ is a continuous decomposition of X and each element of $\mathcal{A}'_f$ is terminal in X.*
(2) *$\mathcal{A}'_f$ is a continuous decomposition of X and f is a Krasinkiewicz map.*

Proof. (1)⇒(2) is obvious, so we only prove (2)⇒(1). To do this we prove that each element of $\mathcal{A}'_f$ is terminal in X. Let $0 < t < 1$ and let $m : \mathcal{A}'_f \to I$ be the constant function such that $m(A) = t$ for each $A \in \mathcal{A}'_f$. Since $\mathcal{A}'_f$ is closed in $C(X)$, by Theorem 16.10 of [4], m can be extended to a Whitney map $\mu : C(X) \to I$. Now we show that

$$(**) \qquad \mathcal{A}'_f = \mu^{-1}(t).$$

$\mathcal{A}'_f \subset \mu^{-1}(t)$ is obvious, so we only prove $\mu^{-1}(t) \subset \mathcal{A}'_f$. Let $A \in \mu^{-1}(t)$. Since f is a Krasinkiewicz map, A contains an element of $\mathcal{A}'_f$ or is contained in an element of $\mathcal{A}'_f$. Assume A contains $B \in \mathcal{A}'_f$. Since $\mu(A) = \mu(B) = t$, $A = B$. Assume A is contained in $C \in \mathcal{A}'_f$. Since $\mu(A) = \mu(C) = t$, $A = C$. In both cases, $A \in \mathcal{A}'_f$. So $(**)$ holds.

By Theorem 2.3 of [3], each element of $\mathcal{A}'_f$ is terminal in X. This completes the proof. ■

THEOREM 3.5. *Let X, Y be compacta and let $f : X \to Y$ be a monotone map such that $f^{-1}(y)$ is a nondegenerate continuum in X for each $y \in Y$. Then the following conditions are equivalent*:

(1) *f is an open map and each fiber of f is terminal in X.*
(2) *f is an open Krasinkiewicz map.*
(3) *f is a Whitney preserving map.*

Proof. (1)⇒(2) and (2)⇒(1) hold by Proposition 3.4; (1)⇒(3) by Theorem 3.1; and (3)⇒(1) by Corollary 3.3. ■

DEFINITION 3.6. A map $f : X \to Y$ is called *dimension raising* if $\dim X < \dim f(X)$.

It is clear that a dimension raising Whitney preserving map is not a homeomorphism. There does not always exist a dimension raising Whitney preserving map on each continuum X by Proposition 3.8.

A continuum X is *connected im kleinen at* $p \in X$, written *cik at* p, provided that every neighborhood of p contains a connected neighborhood of p. A continuum X is said to be *continuumwise accessible* if for every proper subcontinuum $A \subset X$ there exist a nondegenerate subcontinuum $B \subset X$ and a point $x \in A$ such that $A \cap B = \{x\}$ (cf. Definition 4 of [2]).

The next lemma is an immediate consequence of Corollary 6 of [2]. The proof is left to the reader.

LEMMA 3.7. *Let X be a continuum such that X is cik at some point or X is continuumwise accessible. If $f : X \to Y$ is Whitney preserving, then f is a light map.*

PROPOSITION 3.8. *Let X be a nondegenerate continuum such that*

(1) *X is cik at some point or X is continuumwise accessible*,

(2) *each nondegenerate subcontinuum of X contains an arc.*

If $f : X \to f(X)$ is a Whitney preserving map, *then* $\dim f(X) = 1$.

Proof. Assume that $\dim f(X) \geq 2$. By Theorem 5 of [1] there exists a nondegenerate hereditarily indecomposable continuum $Y \subset f(X)$. By Theorem 2 of [2], f is weakly confluent. So there exists a nondegenerate subcontinuum $A \subset X$ such that $f(A) = Y$. By (1) and Lemma 3.7, f is a light map. By (2), A contains an arc. Hence by Theorem 8.18 of [15], $f(A)$ contains a nondegenerate Peano continuum. Since Y is hereditarily indecomposable, this is a contradiction. This completes the proof. ∎

For example, if X is an arc (or a circle, or a $\sin(1/x)$-curve, etc.) and $f : X \to f(X)$ is a Whitney preserving map, then $\dim f(X) = 1$ by Proposition 3.8.

Now, as an application of Theorem 3.5 we prove Theorem 3.9. The proof, obtained by slightly modifying the proof of Theorem 3.1 of [12], uses 1-dimensional Bing–Krasinkiewicz–Lelek maps effectively.

THEOREM 3.9. *For each $n \geq 2$ and a continuum X with* $\dim X = n$ *there exists a 1-dimensional subcontinuum T and a monotone Whitney preserving map $q : T \to q(T)$ such that* $\dim q(T) \geq n$.

Proof. First we consider the case when $n \geq 3$. By Theorem 5 of [1], there exists a hereditarily indecomposable subcontinuum $Y \subset X$ such that $\dim Y \geq 2$. By Theorem 1.2 of [11], we can see that there exist a 1-dimensional continuum $T \subset Y$ and a monotone open map $q : T \to q(T)$ with nontrivial sufficiently small fibers such that $\dim q(T) = \infty$. It is easy to see that each subcontinuum in a hereditarily indecomposable continuum is terminal. Hence by Theorem 3.1, the assertion of Theorem 3.9 holds when $n \geq 3$.

Now we handle the case when $n = 2$. Let $w : C(X) \to I$ be a Whitney map for X. By Lemmas 2.1 and 2.3, there exists a 1-dimensional Bing–Krasinkiewicz–Lelek map $f : X \to I$. Let $f = h \circ g$ be the monotone-light decomposition of f with g monotone and h light. Then g is a 1-dimensional Bing–Krasinkiewicz–Lelek map to the 1-dimensional compactum $Z = g(X)$.

Since g is a Bing map, for each $a > 0$ we can define a decomposition $\mathcal{A}$ of X by

$$\begin{aligned}\mathcal{A} = &\{g^{-1}(z) \mid w(g^{-1}(z)) < a\}\\ &\cup \{A \in C(X) \mid w(A) = a \text{ and there exists } z \in Z \text{ such that } A \subset g^{-1}(z)\}.\end{aligned}$$

Let $q : X \to q(X)$ be the quotient map associated with $\mathcal{A}$ and $F = \bigcup\{g^{-1}(z) \mid w(g^{-1}(z)) \geq a\}$. Note that for each sufficiently small $a > 0$, F is 1-dimensional since g is a 1-dimensional Lelek map.

By arguments in the proof of Theorem 3.1 in [12],

(1) $\mathcal{A}$ is upper semicontinuous,

(2) the restriction $q|_F : F \to q(F)$ is an open map.

Choose $a > 0$ sufficiently small such that $\dim q(X) \geq \dim X = 2$ (cf. Corollary 9, p. 111 of [9]). As q coincides with g on $X \setminus F$, we have $\dim q(X \setminus F) = \dim g(X \setminus F) \leq \dim Z = 1$. Since F is closed in X, $q(F)$ is closed in $q(X)$. By the inequality $2 \leq \dim q(X) = \max\{\dim q(F), \dim q(X \setminus F)\}$, $\dim q(F) \geq 2$.

Pick a subcontinuum K of $q(F)$ with $\dim K \geq 2$ and define $T = q^{-1}(K)$. Since q is monotone, T is a continuum. Since $T \subset F$, we have $\dim T = 1$. The restriction $q|_T : T \to K$ is an open map by (2). It is easy to see that $q|_T$ is a Bing map such that $q_T^{-1}(y)$ is a nondegenerate continuum for each $y \in K$.

Now we show that $q|_T$ is a Krasinkiewicz map. Let $C \subset T$. We consider two cases.

(A) If $C \subset g^{-1}(z)$ for some $z \in Z$, then there exists $A \in \mathcal{A}$ such that $A \subset g^{-1}(z)$ and $A \cap C \neq \emptyset$. Note that A is a fiber of $q|_T$. Since g is a Bing map, $A \subset C$ or $C \subset A$. So C contains a fiber of $q|_T$ or is contained in such a fiber.

(B) If C is not contained in a fiber of g, then there exists $z \in Z$ such that $g^{-1}(z) \subset C$, because g is a monotone Krasinkiewicz map. Since $g^{-1}(z) \subset F$, $g^{-1}(z)$ contains an element A of $\mathcal{A}$. Hence $A \subset C$. Note that A is a fiber of $q|_T$. So C contains a fiber of $q|_T$.

By (A) and (B), $q|_T$ is a Krasinkiewicz map. Hence by Theorem 3.5, $q|_T$ is a Whitney preserving map. This completes the proof. ∎

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