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GENERAL TOPOLOGY

Homeomorphism Groups and the Topologist's Sine Curve

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Summary. It is shown that deleting a point from the topologist's sine curve results in a locally compact connected space whose autohomeomorphism group is not a topological group when equipped with the compact-open topology.

If X is a Hausdorff topological space then we let $\mathcal{H}(X)$ denote the group of autohomeomorphisms of X equipped with the compact-open topology. If $A, B \subset X$ then we define $[A, B] = \{h \in \mathcal{H}(X) \colon h(A) \subset B\}$ and we recall that the topology on $\mathcal{H}(X)$ is generated by the subbasis $\{[K, O]\colon K$ compact and O open in $X\}$. If X is locally compact, then composition is continuous on $\mathcal{H}(X)$ and if X is compact, then also the inverse operation is continuous, thus $\mathcal{H}(X)$ is a topological group; see Arens [1]. The Cantor set with a point removed is the standard example of a locally compact metric space such that the inverse operation on $\mathcal{H}(X)$ is discontinuous; see for instance [3]. It is a classic theorem of Arens [1] that if X is locally compact and locally connected then $\mathcal{H}(X)$ is a topological group. Dijkstra improved on this result in [3] with the following theorem. A *continuum* is a compact connected space.

Theorem 1. *If every point in a Hausdorff space X has a neighbourhood that is a continuum then $\mathcal{H}(X)$ is a topological group.*

It is an open problem to determine exactly for which locally compact (metric) spaces X the space $\mathcal{H}(X)$ is a topological group. Note that Theorem 1 states that it is sufficient that every point has a neighbourhood that is both compact and connected. A natural question that arises is whether this

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condition can be weakened to "every point has a compact neighbourhood and a connected neighbourhood". A variation on this question was posed to us by Fredric Ancel, namely whether local compactness together with connectedness of the whole space is sufficient. We will show that neither is the case by presenting a counterexample. Our counterexample is based on a classic space, namely the topologist's sine curve, which means that we reveal a new facet of this interesting space.

The *topologist's sine curve* S is a compact subspace of the plane $\mathbb{R}^2$ that is the union of the following two sets:

$$A = \{(0, y) : -1 \le y \le 1\} \quad \text{and} \quad B = \{(x, \sin(1/x)) : 0 < x \le 1/\pi\}.$$

Note that B is homeomorphic to a half-open interval and that the line segment A is contained in the closure of B, so that S is a connected space. However, it is not hard to see that there is no path in S that connects points of A with points of B, so S is not path connected. Also S is not locally connected at points of A. As such S is a popular counterexample in topology and can be found in many topology texts; see for instance [6, p. 137], [2, Examples 5.2.2 and 5.5.3], or [5, Example 24.7].

Let $p = (0, 1)$ be the upper endpoint of the arc A. The space we are interested in is $S_p = S \setminus \{p\}$. This space is obviously locally compact as an open subspace of S and it is connected by the same argument as used for S.

THEOREM 2. *The inverse operation on $\mathcal{H}(S_p)$ is not continuous.*

Proof. We will construct a sequence $h_1, h_2, \dots$ in $\mathcal{H}(S_p)$ that converges to the identity function e and such that the inverses $h_1^{-1}, h_2^{-1}, \dots$ do not converge to e. Let ρ denote the standard euclidean metric on $\mathbb{R}^2$.

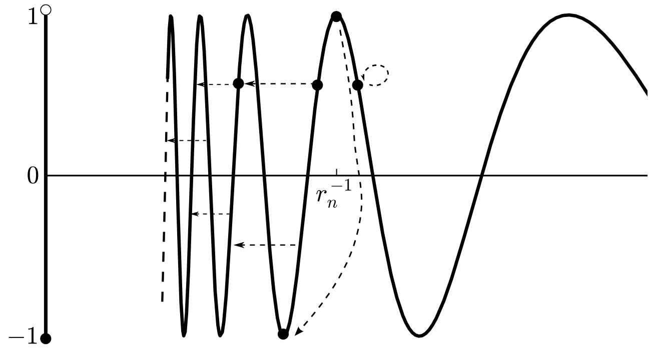
Let n be a natural number. Put $r_n = 2\pi n + \frac{1}{2}\pi$ and note that $\sin r_n = 1$ and $(1/r_n, 1) \in S$. Define $\alpha_n \in \mathcal{H}([\pi, \infty))$ by

(1) α_n is the identity on $[\pi, r_n - 2^{-n}]$;
(2) $\alpha_n(t) = t + 2\pi$ for $t \ge r_n + 2^{-n}$;
(3) α_n maps $[r_n - 2^{-n}, r_n + 2^{-n}]$ linearly onto $[r_n - 2^{-n}, r_n + 2\pi + 2^{-n}]$.

Note that it follows from (3) that the midpoint r_n of the first interval is mapped onto the midpoint $r_n + \pi$ of the second interval. We define the bijection $h_n \colon S \to S$ by letting h_n be the identity on A and on B,

$$h_n(1/t, \sin t) = (1/\alpha_n(t), \sin \alpha_n(t)) \quad \text{for } t \ge \pi.$$

Note that if $t \ge r_n + 2^{-n}$ then $h_n(1/t, \sin t) = (1/(t + 2\pi), \sin t)$ so the distance between $(1/t, \sin t)$ and its image $h_n(1/t, \sin t)$ is less than $1/t$, which is in turn less than $1/2\pi n$. Thus h_n is continuous at points of A. Since the restriction of h_n to the open set B is obviously continuous we see that h_n is continuous and thus a homeomorphism by compactness. Since $h_n(p) = p$ we can also regard h_n as an element of $\mathcal{H}(S_p)$.



Define the compactum $C = \{(x, y) \in S_p : y = -1\}$ and the open set $U = \{(x, y) \in S_p : y < 1\}$ and note that $e \in [C, U]$. For $n \in \mathbb{N}$ we have $h_n(1/r_n, 1) = (1/(r_n + \pi), -1)$ thus the inverse of h_n is not in $[C, U]$. We may conclude that $h_1^{-1}, h_2^{-1}, \ldots$ do not converge to e.

To show that $\lim_{n\to\infty} h_n = e$ in $\mathcal{H}(S_p)$ let K and O be compact respectively open in S_p with $e \in [K, O]$. Thus $K \subset O$ and the distance d between the compacta K and $S \setminus O$ is positive. Since $p \notin O$ and thus $p \notin K$ there is a positive ε so that $[0, \varepsilon] \times [1 - \varepsilon]$ is disjoint from K. Select a natural number N such that $\sin(\pi/2 - 2^{-N}) > 1 - \varepsilon$ and $1/2\pi N$ is less than both d and ε. Let $n \geq N$ and consider h_n and a point (x, y) in K. If $(x, y) \in A$ or $x \geq 1/(r_n - 2^{-n})$, then $h_n(x, y) = (x, y) \in O$. If $0 < x \leq 1/(r_n + 2^{-n})$ then $\rho((x, y), h_n(x, y)) < 1/2\pi n < d$, thus $h_n(x, y) \in O$. If $1/(r_n + 2^{-n}) < x < 1/(r_n - 2^{-n})$ then

$$y = \sin(1/x) \geq \sin(\pi/2 - 2^{-n}) \geq \sin(\pi/2 - 2^{-N}) > 1 - \varepsilon$$

and $x < 1/2\pi n \leq 1/2\pi N < \varepsilon$. Thus (x, y) cannot be in K in this case. We have shown that $h_n \in [K, O]$ for $n \geq N$ and hence that $\lim_{n\to\infty} h_n = e$ in $\mathcal{H}(S_p)$. ■

If we identify in S the endpoint $(0, -1)$ of A with the endpoint $(1/\pi, 0)$ of B then the resulting quotient space is path connected and known as the *Warsaw circle*; see [4, p. xiii]. If we also delete the point p from the Warsaw circle, then we get a path connected and locally compact space whose homeomorphism group is not a topological group.

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