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Singular Failures of GCH and Level by Level Equivalence

by

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Summary. We construct a model for the level by level equivalence between strong compactness and supercompactness in which below the least supercompact cardinal κ, there is an unbounded set of singular cardinals which witness the only failures of GCH in the universe. In this model, the structure of the class of supercompact cardinals can be arbitrary.

1. Introduction and preliminaries. In [1], the following theorem was proven.

THEOREM 1. *Suppose* $V \models$ *"ZFC + GCH +* $\mathcal{K} \neq \emptyset$ *is the class of supercompact cardinals + Level by level equivalence between strong compactness and supercompactness holds". There is then a partial ordering* $\mathbb{P} \in V$ *such that* $V^{\mathbb{P}} \models$ *"ZFC +* $\mathcal{K}$ *is the class of supercompact cardinals + Level by level equivalence between strong compactness and supercompactness holds". In* $V^{\mathbb{P}}$*, there is a stationary subset* S *of the least supercompact cardinal* κ *composed of singular strong limit cardinals of cofinality* ω *on which GCH fails.*

In any model V^* witnessing the conclusions of Theorem 1 constructed in [1], there are many regular cardinals at which GCH fails (and in particular, there are many inaccessible cardinals at which GCH fails). This is since V^* is built by forcing over either a model witnessing the conclusions of [3, Theorem 1] or a modification of this model, both of which contain many inaccessible cardinals at which GCH fails. This raises the following

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Question. Is it possible to construct a model for the level by level equivalence between strong compactness and supercompactness in which GCH fails precisely on a stationary subset of the least supercompact cardinal composed entirely of singular cardinals? More weakly, is it possible to construct a model for the level by level equivalence between strong compactness and supercompactness in which GCH fails precisely on an unbounded subset of the least supercompact cardinal composed entirely of singular cardinals?

The purpose of this paper is to answer the weaker version of the above Question in the affirmative. More specifically, we prove the following theorem.

Theorem 2. *Suppose* $V \models$ *"ZFC + GCH +* $\mathcal{K} \neq \emptyset$ *is the class of supercompact cardinals + Level by level equivalence between strong compactness and supercompactness holds". There is then a partial ordering* $\mathbb{P} \in V$ *such that* $V^{\mathbb{P}} \models$ *"ZFC +* $\mathcal{K}$ *is the class of supercompact cardinals + Level by level equivalence between strong compactness and supercompactness holds". In* $V^{\mathbb{P}}$, *there is an unbounded subset* A *of the least supercompact cardinal* κ *composed of singular strong limit cardinals of cofinality* ω *on which GCH fails precisely.*

In the model witnessing the conclusions of Theorem 2, it is of course the case that every limit cardinal is automatically a strong limit cardinal. Therefore, A is composed entirely of strong limit cardinals and consequently also witnesses failures of SCH.

We note that by Solovay's theorem of [17], GCH must hold at any singular strong limit cardinal above a strongly compact cardinal. Thus, as in [1], any failures of GCH that occur at singular strong limit cardinals must of necessity take place below the least strongly compact cardinal. Further, by Silver's theorem [16], if GCH fails at a singular strong limit cardinal δ of uncountable cofinality, then it fails at many singular strong limit cardinals below δ. In addition, any set having measure one with respect to a normal measure over a measurable cardinal must of course concentrate on regular cardinals. Therefore, one cannot improve Theorem 2 by having violations of GCH above the least supercompact cardinal, or by having A be composed entirely of singular cardinals of uncountable cofinality, or by changing "unbounded" to normal measure one.

We take this opportunity to point out that although the proofs of our new Theorem 2 and [1, Theorem 1] (Theorem 1 of this paper) are quite similar, there are stark differences both in the theorems proven and the design of the forcing conditions used in each case. In [1, Theorem 1], the goal is to create a stationary set of singular failures of GCH below the least supercompact cardinal κ in a model satisfying level by level equivalence between strong compactness and supercompactness, with no thought to regulating precisely the GCH pattern below κ. In the current situation, we are both seeking and obtaining just such an exact control. This requires that much greater care be taken in the construction of the partial orderings employed both in the proof of Theorem 2 and its generalization (Theorem 3) given at the end of the paper. Specifically, whereas the proof of [1, Theorem 1] only requires an iteration of Cohen forcing followed by an iteration of Prikry forcing, the forcing conditions used in this paper are much more intricate. In particular, for both Theorems 2 and 3, we must iterate very complicated partial orderings originally due to Gitik (see both [7] and [9, Section 2]) and make sure that the relevant definitions can in fact be presented correctly. This is especially true in the proof of Theorem 3, where each of the two cases found in the definition of the forcing conditions must be handled quite carefully.

We now very briefly give some preliminary information concerning notation and terminology. For anything left unexplained, readers are urged to consult [1]. When forcing, $q \geq p$ means that q *is stronger than* p, and $p \parallel \varphi$ means that p *decides* φ. For $\alpha < \beta$ ordinals, $(\alpha, \beta]$ and $[\alpha, \beta]$ are as in standard interval notation. If A is any set of ordinals, then A' is the *set of limit points* of A. If G is V-generic over $\mathbb{P}$, we will abuse notation slightly and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. We will, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$.

For κ a cardinal, the partial ordering $\mathbb{P}$ is κ*-closed* if for any $\delta < \kappa$, any increasing chain of conditions of length δ has an upper bound. As in [10], we will say that the partial ordering $\mathbb{P}$ is κ*-weakly closed and satisfies the Prikry condition* if it meets the following criteria:

1. $\mathbb{P}$ has two partial orderings $\leq$ and $\leq^*$ with $\leq^* \subseteq \leq$.
2. For every $p \in \mathbb{P}$ and every statement φ in the forcing language with respect to $\mathbb{P}$, there is some $q \in \mathbb{P}$ such that $p \leq^* q$ and $q \parallel \varphi$.
3. The partial ordering $\leq^*$ is κ-closed.

For more details on these definitions, readers are urged to consult [10] or [9].

Throughout the course of our discussion, we will refer to partial orderings $\mathbb{P}$ as being *Gitik iterations.* By this we will mean an Easton support iteration as first given by Gitik in [8] (and elaborated upon further in [10] and [9]), where at any stage δ at which a nontrivial forcing is done, we assume the partial ordering $\mathbb{Q}_\delta$ with which we force is η-weakly closed for some $\eta < \delta$ and satisfies the Prikry condition. For additional details and explanations, see [8] or [9]. By [8, Lemmas 1.4 and 1.2], if the first stage in the definition of $\mathbb{P}$ at which a nontrivial forcing is done is η_0-weakly closed, then forcing with $\mathbb{P}$ adds no bounded subsets to η_0.

Suppose V is a model of ZFC in which for all regular cardinals $\kappa < \lambda$, κ is λ strongly compact iff κ is λ supercompact, except possibly if κ is a measurable limit of cardinals δ which are λ supercompact. Such a model will be said to witness *level by level equivalence between strong compactness and supercompactness.* We will also say that κ *is a witness to level by level equivalence between strong compactness and supercompactness* iff for every regular cardinal $\lambda > \kappa$, κ is λ strongly compact iff κ is λ supercompact. Note that the exception is provided by a theorem of Menas [15], who showed that if κ is a measurable limit of cardinals δ which are λ strongly compact, then κ is λ strongly compact but need not be λ supercompact. When this situation occurs, the terminology we will henceforth use is that κ *is a witness to the Menas exception at* λ. Models in which level by level equivalence between strong compactness and supercompactness holds nontrivially were first constructed in [6].

We assume familiarity with the large cardinal notions of measurability, strongness, strong compactness, and supercompactness. Readers are urged to consult [11] and [12] for further details. We just mention that a cardinal κ will be said to be *supercompact up to a strong cardinal* λ if κ is δ supercompact for every $\delta < \lambda$.

2. The proof of Theorem 2. Suppose $V \vDash$ "ZFC + GCH + $\mathcal{K}$ is the class of supercompact cardinals + κ is the least supercompact cardinal + Level by level equivalence between strong compactness and supercompactness holds". Let $A_0 = \{\delta < \kappa \mid \delta$ is the limit of an ω sequence of strong cardinals$\}$. Then $A = A_0 - A_0'$ will be the unbounded subset of κ on which we will force failures of GCH. Our partial ordering $\mathbb{P}$ may therefore be informally described as the Gitik iteration of length κ which, for $\delta \in A$, does Gitik's forcing of [7] (see also [9, Section 2]) for forcing $2^\delta = \delta^{++}$ while preserving GCH elsewhere without either collapsing cardinals or adding bounded subsets of δ by using either long or short extenders. The iteration acts trivially otherwise, i.e., whenever $\delta \notin A$.

It is necessary to define $\mathbb{P}$ more formally, which we do as follows: $\mathbb{P} = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha < \kappa\rangle$ is the Gitik iteration of length κ such that $\mathbb{P}_0 = \{\emptyset\}$. $\dot{\mathbb{Q}}_\delta$ is a term for trivial forcing unless $\delta \in A$. In order to define $\dot{\mathbb{Q}}_\delta$ for $\delta \in A$, let $\langle \kappa_{n,\delta} \mid n < \omega\rangle$ be an increasing sequence of V-strong cardinals such that $\sup(\langle \kappa_{n,\delta} \mid n < \omega\rangle) = \delta$. Since $\delta \notin A_0'$, we may assume without loss of generality that $\sup(\{\kappa_{n,\gamma} \mid \gamma \in A,\ n < \omega, \text{ and } \gamma < \delta\}) < \kappa_{0,\delta}$. It will then inductively follow that $|\mathbb{P}_\delta| < \kappa_{0,\delta}$, which means that $\Vdash_{\mathbb{P}_\delta}$ "GCH holds for a final segment of cardinals which starts below $\kappa_{0,\delta}$" (and in fact, $\Vdash_{\mathbb{P}_\delta}$ "GCH holds for all cardinals greater than or equal to $|\mathbb{P}_\delta|^+$"). This means that there is (more than) enough GCH to allow the coding and Δ-system arguments of [7] or [9, Section 2] to be used so that $\dot{\mathbb{Q}}_\delta$ may be taken as a term for Gitik's forcing of [7] or [9, Section 2] for forcing $2^\delta = \delta^{++}$ while preserving GCH elsewhere without either collapsing cardinals or adding bounded subsets of δ by using either long or short extenders.

By the arguments of either [7] or [9, Section 2], for any $\delta \in A$, $\dot{\mathbb{Q}}_\delta$ is a term for a $\kappa_{0,\delta}$-weakly closed partial ordering satisfying the Prikry condition. Consequently, by [8, Lemmas 1.4 and 1.2] and the fact $\mathbb{P}$ is a Gitik iteration, $V^{\mathbb{P}} \models$ "Every $\delta \in A$ is a singular strong limit cardinal of cofinality ω such that $2^\delta = \delta^{++}$". Because $\mathbb{P}$ may be defined so that $|\mathbb{P}| = \kappa$, by the results of [14], $V^{\mathbb{P}} \models$ "$\mathcal{K} - \{\kappa\}$ is the class of supercompact cardinals above κ". In addition, the usual Easton arguments in tandem with the arguments of either [7] or [9, Section 2] show that $V^{\mathbb{P}} \models$ "GCH fails precisely on the members of A". Thus, the proof of Theorem 2 is completed by the following two lemmas.

LEMMA 2.1. *$V^{\mathbb{P}} \models$ "κ is the least supercompact cardinal".*

Proof. We combine the proofs of [1, Lemma 2.3] and [1, Lemma 2.1]. Since $V^{\mathbb{P}} \models$ "GCH fails on an unbounded set of singular strong limit cardinals below κ", by Solovay's theorem of [17], $V^{\mathbb{P}} \models$ "There are no strongly compact cardinals below κ". Thus, the proof of Lemma 2.1 will be complete once we have shown that $V^{\mathbb{P}} \models$ "κ is supercompact".

To do this, let $\lambda \geq \kappa^+$ be an arbitrary regular cardinal, and let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$ such that $M \models$ "κ is not λ supercompact". It is the case that $M \models$ "No cardinal $\delta \in (\kappa, \lambda]$ is strong". This is since otherwise, κ is supercompact up to a strong cardinal in M, and thus, by the proof of [5, Lemma 2.4], $M \models$ "κ is supercompact", a contradiction. This means that $j(\mathbb{P}) = \mathbb{P} * \dot{\mathbb{Q}}$, where the first nontrivial stage in $\dot{\mathbb{Q}}$ takes place well above λ.

We may now show that $V^{\mathbb{P}} \models$ "κ is λ supercompact" as in the proof of [1, Lemma 2.1]. Specifically, we apply the argument of [8, Lemma 1.5]. In particular, let G be V-generic over $\mathbb{P}$. Since $2^\lambda = \lambda^+$ in both V and $V[G]$, we may let $\langle \dot{x}_\alpha \mid \alpha < \lambda^+ \rangle$ be an enumeration in V of all of the canonical $\mathbb{P}$-names of subsets of $P_\kappa(\lambda)$. Because $\mathbb{P}$ is a Gitik iteration of length κ, $\mathbb{P}$ is κ-c.c. Consequently, $M[G]$ remains λ-closed with respect to $V[G]$. Therefore, by [8, Lemmas 1.4 and 1.2] and the fact $M[G]^\lambda \subseteq M[G]$, we may define in $V[G]$ an increasing sequence $\langle p_\alpha \mid \alpha < \lambda^+ \rangle$ of elements of $j(\mathbb{P})/G$ such that if $\alpha < \beta < \lambda^+$, p_β is an Easton extension of p_α (1), every initial segment of the sequence is in $M[G]$, and for every $\alpha < \lambda^+$, $p_{\alpha+1} \parallel$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{x}_\alpha)$". The remainder of the argument of [8, Lemma 1.5] remains valid and shows

(1) Roughly speaking, this means that p_β extends p_α as in a usual Easton support iteration, except that no stems of any components of p_α are extended. For a more precise definition, readers are urged to consult either [8] or [9].

that a supercompact ultrafilter $\mathcal{U}$ over $(P_\kappa(\lambda))^{V[G]}$ may be defined in $V[G]$ by $x \in \mathcal{U}$ iff $x \subseteq (P_\kappa(\lambda))^{V[G]}$ and for some $\alpha < \lambda^+$ and some $\mathbb{P}$-name $\dot{x}$ of x, in $M[G]$, $p_\alpha \Vdash_{j(\mathbb{P})/G}$ "$\langle j(\beta) \mid \beta < \lambda \rangle \in j(\dot{x})$". (The fact that $j''G = G$ tells us $\mathcal{U}$ is well-defined.) Thus, $\Vdash_{\mathbb{P}}$ "κ is λ supercompact". Since λ was arbitrary, this completes the proof of Lemma 2.1. ∎

Lemma 2.2. *$V^{\mathbb{P}} \vDash$ "Level by level equivalence between strong compactness and supercompactness holds".*

Proof. We modify the proof of [1, Lemma 2.4], quoting verbatim when appropriate. Since $\mathbb{P}$ may be defined so that $|\mathbb{P}| = \kappa$, and since $V \vDash$ "Level by level equivalence between strong compactness and supercompactness holds", by the results of [14], $V^{\mathbb{P}} \vDash$ "Level by level equivalence between strong compactness and supercompactness holds above κ". By Lemma 2.1, $V^{\mathbb{P}} \vDash$ "Level by level equivalence between strong compactness and supercompactness holds at κ". Thus, the proof of Lemma 2.2 will be complete once we have shown that $V^{\mathbb{P}} \vDash$ "Level by level equivalence between strong compactness and supercompactness holds below κ".

To do this, let $\delta < \kappa$ and $\lambda > \delta$ be such that $V^{\mathbb{P}} \vDash$ "δ is λ strongly compact and λ is regular". Let $\gamma = \sup(\{\alpha < \delta \mid \alpha$ is a nontrivial stage of forcing$\})$, and write $\mathbb{P} = \mathbb{P}_\gamma * \dot{\mathbb{P}}^\gamma$. By [8, Lemmas 1.4 and 1.2] and the definition of $\mathbb{P}$, $\Vdash_{\mathbb{P}_\gamma}$ "Forcing with $\dot{\mathbb{P}}^\gamma$ adds no bounded subsets to γ^*, the least V-strong cardinal above γ". We assume for the time being that $\lambda < \gamma^*$. Therefore, we may infer that $\Vdash_{\mathbb{P}_\gamma}$ "δ is λ strongly compact" iff $\Vdash_{\mathbb{P}}$ "δ is λ strongly compact", i.e., $V^{\mathbb{P}_\gamma} \vDash$ "δ is λ strongly compact".

We consider now two cases.

Case 1: $\gamma < \delta$. In this situation, by the definition of $\mathbb{P}$, $|\mathbb{P}_\gamma| < \delta$. Thus, by the results of [14], $V^{\mathbb{P}_\gamma} \vDash$ "δ is λ strongly compact" iff $V \vDash$ "δ is λ strongly compact". Since $V \vDash$ "Level by level equivalence between strong compactness and supercompactness holds", either $V \vDash$ "δ is λ supercompact", or $V \vDash$ "δ is a witness to the Menas exception at λ". Again by the results of [14], either $V^{\mathbb{P}_\gamma} \vDash$ "δ is λ supercompact", or $V^{\mathbb{P}_\gamma} \vDash$ "δ is a witness to the Menas exception at λ". Regardless of which of these occurs, δ does not witness a failure of level by level equivalence between strong compactness and supercompactness.

Case 2: $\gamma = \delta$. If this occurs, then by the definition of $\mathbb{P}$, it must be the case that $|\mathbb{P}_\delta| = \delta$. Note that since δ is measurable in $V^{\mathbb{P}_\delta}$, δ must be Mahlo in $V^{\mathbb{P}_\delta}$ and thus also Mahlo in V. Consequently, $\mathbb{P}_\delta$ is the direct limit of $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$, and $\mathbb{P}_\delta$ satisfies δ-c.c. in V. This means that since $\mathbb{P}_\delta$ satisfies δ-c.c. in $V^{\mathbb{P}_\delta}$ as well (this follows because δ is Mahlo in $V^{\mathbb{P}_\delta}$ and $\mathbb{P}_\delta$ is a subordering of the direct limit of $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$ as calculated in $V^{\mathbb{P}_\delta}$), (the proof of) [2, Lemma 8] (see in particular the argument found starting in [2, third paragraph of page 111]) or (the proof of) [4, Lemma 3] tells us that every δ-additive uniform ultrafilter over a cardinal $\beta \geq \delta$ present in $V^{\mathbb{P}_\delta}$ must be an extension of a δ-additive uniform ultrafilter over β in V. Therefore, since the λ strong compactness of δ in $V^{\mathbb{P}_\delta}$ implies that every $V^{\mathbb{P}_\delta}$-regular cardinal $\beta \in [\delta, \lambda]$ carries a δ-additive uniform ultrafilter in $V^{\mathbb{P}_\delta}$, and since the fact $\mathbb{P}_\delta$ is the direct limit of $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$ tells us the regular cardinals at or above δ in $V^{\mathbb{P}_\delta}$ are the same as those in V, the preceding sentence implies that every V-regular cardinal $\beta \in [\delta, \lambda]$ carries a δ-additive uniform ultrafilter in V. Ketonen's theorem of [13] then implies that δ is λ strongly compact in V.

Observe now that δ cannot witness in V the Menas exception at λ. The reason is that if this were the case, then δ would have to be a limit of cardinals which are λ supercompact in V. However, by the definition of $\mathbb{P}$, any such cardinal β would have to be in V supercompact up to a strong cardinal, which as we have already observed, implies that β is supercompact in V. This is a contradiction, since $\beta < \kappa$, and κ is the least supercompact cardinal in V. Thus, by the level by level equivalence between strong compactness and supercompactness in V, $V \vDash$ "δ is λ supercompact".

Let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of δ generated by a supercompact ultrafilter over $P_\delta(\lambda)$ such that $M \vDash$ "δ is not λ supercompact". Write $j(\mathbb{P}_\delta) = \mathbb{P}_\delta * \dot{\mathbb{Q}}$. As in Lemma 2.1, $M \vDash$ "No cardinal $\beta \in (\delta, \lambda]$ is strong". We may consequently infer that the first nontrivial stage in $\dot{\mathbb{Q}}$ is well above λ. Hence, since in analogy to the proof of Lemma 2.1, $2^\lambda = \lambda^+$ in both V and $V^{\mathbb{P}_\delta}$, we may apply the same argument as given in the proof of Lemma 2.1 to infer that $V^{\mathbb{P}_\delta} \vDash$ "δ is λ supercompact".

We have now shown that Lemma 2.2 is true if $\lambda < \gamma^*$. We consequently assume that $\lambda \geq \gamma^*$. In this situation, it is then the case that regardless of whether we are in Case 1 or Case 2, some cardinal below κ is supercompact in V up to a strong cardinal and hence is fully supercompact in V. This contradicts that κ is the least V-supercompact cardinal and therefore completes the proof of Lemma 2.2. ∎

Lemmas 2.1–2.2 complete the proof of Theorem 2.

3. Concluding remarks. In conclusion to this paper, we make several remarks. First, we note that Gitik's forcing of [7] or [9, Section 2] may be modified to produce failures of GCH different from $2^\lambda = \lambda^{++}$ on the set A from Theorem 2. For details, readers are referred to either of these papers. If the forcing is modified so that $2^\lambda > \lambda^{++}$ for $\lambda \in A$, however, GCH will not fail precisely on the members of A, since $2^{\lambda^+} > \lambda^{++}$ whenever

$\lambda \in A$. Because of the nature of our iteration, though, GCH will continue to hold at all (strongly) inaccessible cardinals, even with the modification just described.

As our construction shows, A contains none of its limit points. This raises the question of whether it is possible to prove a version of Theorem 2 where A contains some of its limit points. A modification of the construction just given shows that this is indeed the case. Specifically, we have the following theorem.

THEOREM 3. *Suppose* $V \models$ *"ZFC + GCH +* $\mathcal{K} \neq \emptyset$ *is the class of supercompact cardinals + Level by level equivalence between strong compactness and supercompactness holds". There is then a partial ordering* $\mathbb{P} \in V$ *such that* $V^{\mathbb{P}} \models$ *"ZFC +* $\mathcal{K}$ *is the class of supercompact cardinals + Level by level equivalence between strong compactness and supercompactness holds". In* $V^{\mathbb{P}}$*, there is an unbounded subset* A *of the least supercompact cardinal* κ *composed of singular strong limit cardinals of cofinality* ω *on which GCH fails precisely. In addition, the supremum of any* ω *sequence of consecutive members of* A *is a member of* A *as well.*

Sketch of proof. As before, suppose $V \models$ "ZFC + GCH + $\mathcal{K}$ is the class of supercompact cardinals + κ is the least supercompact cardinal + Level by level equivalence between strong compactness and supercompactness holds". Let $A_0 = \{\delta < \kappa \mid \delta$ is the limit of an ω sequence of strong cardinals$\}$. Then $A = A_0 - A_0''$ will be the unbounded subset of κ containing some of its limit points on which we will force failures of GCH. Our partial ordering $\mathbb{P}$ may therefore once again be informally described as the Gitik iteration of length κ which, for $\delta \in A$, does Gitik's forcing of [7] (once again, see also [9, Section 2]) for forcing $2^\delta = \delta^{++}$ while preserving GCH elsewhere without either collapsing cardinals or adding bounded subsets of δ by using either long or short extenders as appropriate. The iteration acts trivially otherwise, i.e., whenever $\delta \notin A$.

As we did earlier, it is necessary to define $\mathbb{P}$ more formally, which we do as follows: $\mathbb{P} = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha < \kappa\rangle$ is the Gitik iteration of length κ such that $\mathbb{P}_0 = \{\emptyset\}$. $\dot{\mathbb{Q}}_\delta$ is a term for trivial forcing unless $\delta \in A$. In order to define $\dot{\mathbb{Q}}_\delta$ for $\delta \in A$, we assume first that δ is not a limit point of A, i.e., that $\delta \notin A_0'$. The definition is then as given in the proof of Theorem 2. More specifically, let $\langle \kappa_{n,\delta} \mid n < \omega\rangle$ be an increasing sequence of strong cardinals such that $\sup(\langle \kappa_{n,\delta} \mid n < \omega\rangle) = \delta$. Since $\delta \notin A_0'$, we may assume without loss of generality that $\sup(\{\kappa_{n,\gamma} \mid \gamma \in A,\ n < \omega, \text{ and } \gamma < \delta\}) < \kappa_{0,\delta}$. It will then as before inductively follow that $|\mathbb{P}_\delta| < \kappa_{0,\delta}$, which means that $\Vdash_{\mathbb{P}_\delta}$ "GCH holds for a final segment of cardinals which starts below $\kappa_{0,\delta}$" (and in fact, $\Vdash_{\mathbb{P}_\delta}$ "GCH holds for all cardinals greater than or equal to $|\mathbb{P}_\delta|^+$"). This means that there is once again (more than) enough GCH to allow the coding and Δ-system arguments of [7] or [9, Section 2] to be used so that $\dot{\mathbb{Q}}_\delta$ may be taken as a term for Gitik's forcing of [7] or [9, Section 2] for forcing $2^\delta = \delta^{++}$ while preserving GCH elsewhere without either collapsing cardinals or adding bounded subsets of δ by using either long or short extenders.

If $\delta \in A$ is also a limit point of A, then since $\delta \notin A''_0$, $\sup(\{\gamma \in A \mid \gamma < \delta$ is a limit point of $A\}) = \eta < \delta$. If we let $\langle \delta_n \mid n < \omega \rangle$ be the first ω members of A greater than η, by the definition of A, it must now be the case that $\sup(\langle \delta_n \mid n < \omega \rangle) = \delta$. It must further be the case that $\langle \kappa_{0,\delta_n} \mid n < \omega \rangle$ is such that $\sup(\langle \kappa_{0,\delta_n} \mid n < \omega \rangle) = \delta$. It will then follow inductively that for each $n < \omega$, $|\mathbb{P}_{\delta_n}| < \kappa_{0,\delta_n}$, which means that $\Vdash_{\mathbb{P}_{\delta_n}}$ "GCH holds for a final segment of cardinals which starts below κ_{0,δ_n}" (and in fact, $\Vdash_{\mathbb{P}_{\delta_n}}$ "GCH holds for all cardinals greater than or equal to $|\mathbb{P}_{\delta_n}|^+$"). As before, by the arguments of either [7] or [9, Section 2], for any $n < \omega$, $\dot{\mathbb{Q}}_{\delta_n}$ is a term for a κ_{0,δ_n}-weakly closed partial ordering satisfying the Prikry condition. Consequently, by [8, Lemmas 1.4 and 1.2], the fact $\mathbb{P}$ is a Gitik iteration, the arguments of [14], the arguments of either [7] or [9, Section 2], and the usual Easton arguments, $V^{\mathbb{P}_\delta} \vDash$ "The only cardinals in the open interval (η, δ) at which GCH fails are the first ω members of A greater than η + For each $n < \omega$, $o(\kappa_{0,\delta_n})$ is (at least) $\kappa^{+\omega}_{0,\delta_n}$". This means that there is once again (more than) enough GCH to allow the coding and Δ-system arguments of [7] or [9, Section 2] to be used so that $\dot{\mathbb{Q}}_\delta$ may be taken as a term for Gitik's forcing of [7] or [9, Section 2] for forcing $2^\delta = \delta^{++}$ while preserving GCH elsewhere without either collapsing cardinals or adding bounded subsets of δ by using short extenders. Because forcing with $\mathbb{P}_\delta$ will have destroyed the strongness of any V-strong cardinal below δ, it will not be possible as before to use long extenders in the definition of $\dot{\mathbb{Q}}_\delta$. By its definition, $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{Q}}_\delta$ is κ_{0,δ_0}-weakly closed and satisfies the Prikry condition".

We may view the iteration $\mathbb{P}$ as being defined on consecutive "blocks" of cardinals $B_\gamma = \langle \gamma_i \mid i \le \omega \rangle$ of length $\omega + 1$, where each $\gamma_i \in A$, $\gamma_\omega = \sup_{i<\omega} \gamma_i$, and for $i < \omega$, $\gamma_i \notin A'_0$. For each $i < \omega$, it is the case that $\Vdash_{\mathbb{P}_{\gamma_i}}$ "$\dot{\mathbb{Q}}_{\gamma_i}$ is κ_{0,γ_i}-weakly closed and satisfies the Prikry condition", i.e., $\Vdash_{\mathbb{P}_{\gamma_i}}$ "$\dot{\mathbb{Q}}_{\gamma_i}$ is κ_{0,γ_0}-weakly closed and satisfies the Prikry condition". In addition, it is also true that $\Vdash_{\mathbb{P}_{\gamma_\omega}}$ "$\dot{\mathbb{Q}}_{\gamma_\omega}$ is κ_{0,γ_0}-weakly closed and satisfies the Prikry condition". If we now let $\dot{\mathbb{Q}}$ be a term for the portion of the iteration which acts on the members of B_γ, it is then the case that $\Vdash_{\mathbb{P}_{\gamma_0}}$ "$\dot{\mathbb{Q}}$ is κ_{0,γ_0}-weakly closed and satisfies the Prikry condition". This means that slight modifications of the arguments found in the paragraph immediately preceding the proof of Lemma 2.1 and in the proofs of Lemmas 2.1 and 2.2 remain valid and show that $V^{\mathbb{P}}$ is a model of ZFC containing exactly the same supercompact cardinals as V does in which GCH fails precisely on the members of A and in which level by level equivalence between strong compactness and supercompactness holds. This completes our sketch of the proof of Theorem 3. ∎

The construction just given may be modified further, so that A contains, e.g., limit points which are limits of limit points, limit points which are limits of limits of limit points, etc. However, our methods do not allow for A to be stationary, since we always seem to need to omit from A certain limit points of high order. We thus conclude by asking if it is possible for A to be stationary in Theorem 2, in analogy to [1, Theorem 1].

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