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NUMBER THEORY

Stern Polynomials as Numerators of Continued Fractions

by

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Summary. It is proved that the nth Stern polynomial $B_n(t)$ in the sense of Klavžar, Milutinović and Petr [Adv. Appl. Math. 39 (2007)] is the numerator of a continued fraction of n terms. This generalizes a result of Graham, Knuth and Patashnik concerning the Stern sequence $B_n(1)$. As an application, the degree of $B_n(t)$ is expressed in terms of the binary expansion of n.

The diatomic sequence b_n defined by the formula

$$b_1 = 1, \quad b_{2n} = b_n, \quad b_{2n+1} = b_n + b_{n+1} \quad (n = 1, 2, \dots)$$

has been studied by many authors (see [7]). In particular, Graham, Knuth and Patashnik [2, Exer. 6.50] have proved that if n has binary representation

(1) $$n = \overset{a_1}{1}\overset{a_2}{0} \dots \overset{a_k}{1} \quad (a_i > 0),$$

then b_n is the numerator of the continued fraction

$$a_1 + \frac{1|}{|a_2} + \dots + \frac{1|}{|a_k}.$$

The sequence b_n has been generalized to polynomials in two different ways [1], [3]. We shall follow the definition given by Klavžar, Milutinović and Petr [3]:

$$\begin{aligned} B_0(t) &= 0, \\ B_1(t) &= 1, \\ B_{2n}(t) &= tB_n(t), \\ B_{2n+1}(t) &= B_n(t) + B_{n+1}(t) \quad (n = 1, 2, \dots), \end{aligned}$$

and we shall prove the following generalization of the last result.

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THEOREM 1. *If* (1) *holds, then* $B_n(t)$ *is the numerator of the continued fraction*

$$T_{a_1} + \frac{t^{a_1}|}{|T_{a_2}} + \cdots + \frac{t^{a_{k-1}}|}{|T_{a_k}},$$

where

$$T_a = 1 + \cdots + t^{a-1} = \frac{t^a - 1}{t - 1}.$$

As an application we shall prove

THEOREM 2. *If* (1) *holds, then the degree of* $B_n(t)$ *equals*

$$a_1 + \cdots + a_k - k + \left\lfloor \frac{l_1+1}{2} \right\rfloor + \left\lfloor \frac{l_2+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{l_j+1}{2} \right\rfloor,$$

where $l_1, \ldots, l_j$ *are the lengths of blocks of* 1*'s occurring in this order in the sequence* $a_2, \ldots, a_k$.

For the proof of Theorem 1 we need the following:

DEFINITION. $K_0 = 1$, $K_1(x_1) = T_{x_1}$, and for $k = 2, 3, \ldots$,

$$K_k(x_1, \ldots, x_k) = T_{x_k} K_{k-1}(x_1, \ldots, x_{k-1}) + t^{x_{k-1}} K_{k-2}(x_1, \ldots, x_{k-2}).$$

LEMMA 1 ([6, Corollary 2.2]). *For* $\alpha \ge 0$,

$$B_{2^\alpha - 1}(t) = T_\alpha.$$

LEMMA 2 ([5, Lemma 1]). *For* $m \ge 0$ *and* $2^\alpha \ge r \ge 0$,

$$B_{2^\alpha m + r}(t) = B_{2^\alpha - r}(t) B_m(t) + B_r(t) B_{m+1}(t).$$

LEMMA 3. *For* $\beta \ge \alpha \ge 0$,

$$B_{2^\beta - 2^\alpha + 1}(t) = T_\alpha T_{\beta - \alpha} + t^{\beta - \alpha}.$$

Proof. Apply Lemma 2 with $m = 2^{\beta-\alpha} - 1$, $r = 1$. ∎

LEMMA 4. *For every integer* $k \ge 2$ *and all positive integers* x_i $(i < k)$,

$$K_k(x_1, \ldots, x_{k-2}, x_{k-1} - 1, 1) = K_{k-1}(x_1, \ldots, x_{k-1}).$$

Proof. For $k = 2$ we have

$$K_2(x_1 - 1, 1) = K_1(x_1 - 1) + t^{x_1 - 1} = T_{x_1} = K_1(x_1).$$

For $k \ge 3$, by the definition above,

$$\begin{aligned}
&K_k(x_1, \ldots, x_{k-2}, x_{k-1} - 1, 1)\\
&\quad = K_{k-1}(x_1, \ldots, x_{k-2}, x_{k-1} - 1) + t^{x_{k-1} - 1} K_{k-2}(x_1, \ldots, x_{k-2})\\
&\quad = K_{k-1}(x_1, \ldots, x_{k-2}, x_{k-1} - 1) + (T_{x_{k-1}} - T_{x_{k-1} - 1}) K_{k-2}(x_1, \ldots, x_{k-2})\\
&\quad = K_{k-1}(x_1, \ldots, x_{k-1}) + K_{k-1}(x_1, \ldots, x_{k-2}, x_{k-1} - 1)\\
&\qquad - T_{x_{k-1} - 1} K_{k-2}(x_1, \ldots, x_{k-2}) - t^{x_{k-2}} K_{k-3}(x_1, \ldots, x_{k-3})\\
&\quad = K_{k-1}(x_1, \ldots, x_{k-1}). \ ∎
\end{aligned}$$

Proof of Theorem 1. We shall prove by induction on k a slightly more general formula

$$B_n(t) = K_k(a_1, \dots, a_k) \tag{2}$$

provided k is odd and

$$n = \overset{a_1}{1}\overset{a_2}{0} \dots \overset{a_k}{1}, \tag{3}$$

where $a_i > 0$ $(1 \le i \le k,\ i \ne k-1)$, $a_{k-1} \ge 0$.

For $k = 1$ the formula (2) follows from Lemma 1. Assume now that $k \ge 3$ is odd, (3) holds and the formula (2) is true for $k-2$. Then

$$n = 2^{a_{k-1}+a_k} m + 2^{a_k} - 1, \qquad m = \overset{a_1}{1}\overset{a_2}{0} \dots \overset{a_{k-2}}{1}.$$

By Lemma 2,

$$B_n(t) = B_{2^{a_{k-1}+a_k} - 2^{a_k} + 1}(t) B_m(t) + B_{2^{a_k}-1}(t) B_{m+1}(t),$$

and by Lemmas 1 and 3,

$$B_n(t) = (T_{a_k} T_{a_{k-1}} + t^{a_{k-1}}) B_m(t) + T_{a_k} t^{a_{k-2}} B_{\frac{m+1}{2^{a_{k-2}}}}(t).$$

Now, by the inductive assumption and by Lemma 4,

$$\begin{aligned} B_m(t) &= K_{k-2}(a_1, \dots, a_{k-2}), \\ B_{\frac{m+1}{2^{a_{k-2}}}}(t) &= K_{k-2}(a_1, \dots, a_{k-3} - 1, 1) = K_{k-3}(a_1, \dots, a_{k-3}). \end{aligned}$$

Hence

$$B_n(t) = (T_{a_k} T_{a_{k-1}} + t^{a_{k-1}}) K_{k-2}(a_1, \dots, a_{k-2}) + T_{a_k} t^{a_{k-2}} K_{k-3}(a_1, \dots, a_{k-3}),$$

while by the definition

$$\begin{aligned} K_k(a_1, \dots, a_k) &= T_{a_k} K_{k-1}(a_1, \dots, a_{k-1}) + t^{a_{k-1}} K_{k-2}(a_1, \dots, a_{k-2}) \\ &= T_{a_k} T_{a_{k-1}} K_{k-2}(a_1, \dots, a_{k-2}) + T_{a_k} t^{a_{k-2}} K_{k-3}(a_1, \dots, a_{k-3}) \\ &\quad + t^{a_{k-1}} K_{k-2}(a_1, \dots, a_{k-2}). \end{aligned}$$

Therefore

$$B_n(t) = K_k(a_1, \dots, a_k)$$

and the inductive proof is complete. ∎

Now Theorem 1 follows in view of §5 of [4].

For the proof of Theorem 2 we need two lemmas.

Lemma 5. *If in the notation of* [4],

$$\frac{A_\nu}{B_\nu} = \beta_0 + \frac{\alpha_1|}{|\beta_1} + \dots + \frac{\alpha_\nu|}{|\beta_\nu}, \tag{4}$$

then

$$A_\nu = \beta_0\beta_1\cdots\beta_\nu\Bigg(1 + \sum_{\mu=1}^{\lfloor(\nu+1)/2\rfloor}\ \sum_{0\le i_1<\cdots<i_\mu<\nu}\frac{1}{\beta_{i_1}\cdots\beta_{i_\mu}}\prod_{\lambda=1}^{\mu}\frac{\alpha_{i_\lambda+1}}{\beta_{i_\lambda+1}}\Bigg),$$

where $i_{\lambda+1} \ge i_\lambda + 2$ $(1 \le \lambda \le \mu)$.

Proof. See [4, formula (13) on p. 9], where to avoid the collision of notation we have replaced a by α, b by β and where B_n (not $B_n(t)$) does not represent the Stern polynomial, but in accordance with the notation of [4] the denominator of the continued fraction (4). ∎

LEMMA 6. *If* $\alpha_i = t^{a_i}$ $(i = 1,\dots,k-1)$, $\beta_i = T_{a_{i+1}}$ $(i = 0,\dots,k-1)$ *and integers* i_λ $(1 \le \lambda \le \mu \le \lfloor k/2\rfloor)$ *satisfy the conditions*

$$0 \le i_1 < \cdots < i_\mu < k-1, \quad i_{\lambda+1} \ge i_\lambda + 2 \quad (\lambda < \mu), \tag{5}$$

then the polynomial

$$P = \frac{\beta_0\beta_1\cdots\beta_{k-1}}{\beta_{i_1}\cdots\beta_{i_\mu}}\prod_{\lambda=1}^{\mu}\frac{\alpha_{i_\lambda+1}}{\beta_{i_\lambda+1}}$$

is monic of degree

$$a_1 + \cdots + a_k - k + \sum_{\lambda=1}^{\mu}(2 - a_{i_\lambda+2}).$$

Proof. The polynomials $\alpha_i(t)$ and $\beta_i(t)$ are monic and

$$\begin{aligned}\deg P &= a_1 + \cdots + a_k - k - \sum_{\lambda=1}^{\mu}(a_{i_\lambda+1} - 1) + \sum_{\lambda=1}^{\mu}(a_{i_\lambda+1} - a_{i_\lambda+2} + 1)\\ &= a_1 + \cdots + a_k - k + \sum_{\lambda=1}^{\mu}(2 - a_{i_\lambda+2}). \ \blacksquare\end{aligned}$$

Proof of Theorem 2. In view of Theorem 1 and Lemmas 5 and 6, if (1) holds, then the degree of B_n equals $a_1 - 1$ for $k = 1$, while for $k \ge 3$ it is the maximum of

$$a_1 + \cdots + a_k - k + \sum_{\lambda=1}^{\mu}(2 - a_{i_\lambda+2}) \tag{6}$$

over all sequences of integers i_λ satisfying (5). If blocks of 1 occurring in the sequence $a_2,\dots,a_k$ start at positions $p_1,\dots,p_j$ and thus end at positions $p_1 + l_1 - 1,\dots,p_j + l_j - 1$ $(p_1 > 1,\ p_{i+1} > p_i + l_i)$, then the maximum of (6) is attained at

$$i_1 = p_1 - 2, \quad i_2 = p_1, \ \ldots, \ i_{\lfloor l_1+1/2 \rfloor} = p_1 + 2\left\lfloor \frac{l_1+1}{2} \right\rfloor - 4,$$

$$i_{\lfloor (l_1+1)/2 \rfloor + 1} = p_2 - 2, \ \ldots, \ i_{\lfloor (l_1+1)/2 \rfloor + \lfloor (l_2+1)/2 \rfloor} = p_2 + 2\left\lfloor \frac{l_2+1}{2} \right\rfloor - 4, \ \ldots,$$

$$i_{\lfloor (l_1+1)/2 \rfloor + \lfloor (l_2+1)/2 \rfloor + \cdots + \lfloor (l_j+1)/2 \rfloor} = p_j + 2\left\lfloor \frac{l_j+1}{2} \right\rfloor - 4.$$

The value of the maximum is that given in the theorem. ∎

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