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FUNCTIONS OF A COMPLEX VARIABLE

On a Problem of Best Uniform Approximation and a Polynomial Inequality of Visser

by

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Summary. In this paper, a generalization of a result on the uniform best approximation of $\alpha\cos nx + \beta\sin nx$ by trigonometric polynomials of degree less than n is considered and its relationship with a well-known polynomial inequality of C. Visser is indicated.

1. Introduction

1.1. A classical result on best approximation. Let us denote by $\mathcal{T}_m$ the class of all trigonometric polynomials $t(x) := \sum_{\mu=-m}^{m} c_\mu e^{i\mu x}$ of degree at most m with coefficients in $\mathbb{C}$. If t belongs to $\mathcal{T}_m$ and $t(x)$ is real for all real x then we say that t belongs to $\mathcal{T}_m^{(\mathbb{R})}$.

The following result [1, p. 66] gives the best uniform approximation of the function $\alpha\cos nx + \beta\sin nx$ by trigonometric polynomials in $\mathcal{T}_{n-1}^{(\mathbb{R})}$.

THEOREM A. *Let α and β be any real numbers. Then, for any trigonometric polynomial $t \in \mathcal{T}_{n-1}^{(\mathbb{R})}$, we have*

$$\max_{-\pi\le x\le\pi} |\alpha\cos nx + \beta\sin nx - t(x)| \ge \sqrt{\alpha^2+\beta^2}. \tag{1.1}$$

REMARK 1. If t belongs to $\mathcal{T}_{n-1}$ then $s(x) := (t(x) + \overline{t(x)})/2$ belongs to $\mathcal{T}_{n-1}^{(\mathbb{R})}$ and

$$|\alpha\cos nx + \beta\sin nx - t(x)| \ge |\alpha\cos nx + \beta\sin nx - s(x)| \quad (-\pi \le x \le \pi).$$

Hence, (1.1) also holds for any $t \in \mathcal{T}_{n-1}$.

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Remark 2. Clearly, we could have written

$$\sup_{-\infty<x<\infty} |\alpha \cos nx + \beta \sin nx - t(x)|$$

instead of $\max_{-\pi\le x\le\pi} |\alpha \cos nx + \beta \sin nx - t(x)|$ on the left-hand side of (1.1).

Note that $\sum_{\mu=-m}^{m} c_\mu e^{i\mu z}$ is well defined for any $z \in \mathbb{C}$ and is holomorphic throughout $\mathbb{C}$. Thus, a trigonometric polynomial $t(x) := \sum_{\mu=-m}^{m} c_\mu e^{ix}$ is the restriction of an entire function, to $\mathbb{R}$. It may be added that $t(z)$ is an entire function of exponential type $\tau \ge m$. In order to elaborate on this statement we recall some definitions.

1.2. Functions of exponential type. Let f be an entire function and let $M(r) := \max_{|z|=r} |f(z)|$. The function f is said to be of *order* ρ (see [3, p. 8]) if

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \rho \in [0, \infty].$$

A constant has order 0, by convention. An entire function f of finite positive order ρ is of *type* T if $\limsup_{r\to\infty} r^{-\rho} \log M(r) = T \in [0, \infty]$.

Let S be an unbounded subset of the complex plane, like the open angle $\mathcal{A}(\theta_1, \theta_2) := \{z = re^{i\theta} : \theta_1 < \theta < \theta_2\}$ or its closure $\overline{\mathcal{A}}(\theta_1, \theta_2)$. A function f is said to be of *exponential type* τ in S if it is differentiable at every interior point of S and, for each $\varepsilon > 0$, there exists a constant K depending on ε but not on z, such that $|f(z)| < Ke^{(\tau+\varepsilon)|z|}$ for all $z \in S$.

In view of the preceding definitions, an entire function of order less than 1 is of exponential type τ for any $\tau \ge 0$; functions of order 1 and type $T \le \tau$ are also of exponential type τ. As mentioned above, a trigonometric polynomial t of degree at most m is the restriction of an entire function of exponential type τ ($\ge m$) to $\mathbb{R}$. Trigonometric polynomials are bounded on the real axis and they are 2π-periodic. It is known (see [3, Theorem 6.10.1]) that if $f(z)$ is an entire function of exponential type τ which is periodic on the real axis with period Δ then it must be of the form $f(z) = \sum_{\nu=-n}^{n} a_\nu e^{2\pi i\nu z/\Delta}$ with $n \le \lfloor \Delta\tau/(2\pi) \rfloor$.

Let f be of exponential type in the angle $\mathcal{A}(\alpha, \beta)$. The dependence of its growth on the direction in which z tends to infinity is characterized by the function

$$h(\theta) = h_f(\theta) := \limsup_{r\to\infty} \frac{\log |f(re^{i\theta})|}{r} \qquad (\alpha < \theta < \beta),$$

called the *indicator function* of f. Unless $h_f(\theta) \equiv -\infty$, it is continuous. For this and other properties of the indicator function see [3, Chapter 5]. For an entire function f of exponential type, the indicator function $h_f(\theta)$ is defined for all θ. It is clear that if f is an entire function of exponential type τ then $h_f(\theta) \le \tau$ for $0 \le \theta < 2\pi$.

1.3. Statement of the main result. Returning to Theorem A we note that (1.1) can be written as

$$\max_{-\pi\le x\le\pi}\left|\frac{\alpha+i\beta}{2}e^{-inx}-g(x)+\frac{\alpha-i\beta}{2}e^{inx}\right|\ge\left|\frac{\alpha+i\beta}{2}\right|+\left|\frac{\alpha-i\beta}{2}\right|.$$

With this it should be clear that the following result says considerably more than Theorem A.

THEOREM 1. *Let $0<\sigma<\tau$. Then for any $A, B \in \mathbb{C}$ and any entire function g of exponential type σ, we have*

$$\sup_{-\infty<x<\infty}|Ae^{-i\tau x}-g(x)+Be^{i\tau x}|\ge|A|+|B|. \tag{1.2}$$

The following result is contained in Theorem 1.

COROLLARY 1. *Let $\{\lambda_\nu\}_{\nu=0}^n$ be an increasing sequence of $n+1$ numbers in $\mathbb{R}$ and $\{a_\nu\}_{\nu=0}^n$ a sequence of $n+1$ numbers in $\mathbb{C}$. Then*

$$|a_0|+|a_n|\le\sup_{-\infty<x<\infty}\left|a_0e^{i\lambda_0x}+\sum_{\nu=1}^{n-1}a_\nu e^{i\lambda_\nu x}+a_ne^{i\lambda_nx}\right|. \tag{1.3}$$

REMARK 3. It may be noted that $\sum_{\nu=1}^{n-1}a_\nu e^{i\lambda_\nu x}$ is in general not periodic; it is *uniformly almost periodic* in the sense of H. Bohr (see [2, p. 6]).

In the case where $\lambda_\nu=\nu$ for $\nu=0,1,\dots,n$, Corollary 1 says that for any sequence of $n+1$ numbers in $\mathbb{C}$, $|a_0|+|a_n|\le\max_{-\pi\le x\le\pi}|\sum_{\nu=0}^n a_\nu e^{i\nu x}|$. This may also be stated as follows.

COROLLARY 2. *Let $p(z):=\sum_{\nu=0}^n a_\nu z^\nu$ be a polynomial of degree n such that $|p(z)|\le M$ for $|z|=1$. Then $|a_0|+|a_n|\le M$.*

Corollary 2 is known as *Visser's inequality* [7, p. 84, Theorem 3]. In [4], [5] and [6, Chapter 16], the reader will find various generalizations of that inequality; Corollary 1 seems to be a new one. Visser's proof of Corollary 2 is based on certain properties of the nth roots of unity. We do not see how his approach would get us anywhere under the conditions of Corollary 1.

2. Some auxiliary results. The following result [3, Theorem 6.2.4], a consequence of the *Phragmén–Lindelöf principle*, plays an important role in the study of functions of exponential type. We need it too.

LEMMA 1. *Let f be a function of exponential type in the open upper half-plane such that $h_f(\pi/2)\le c$. Furthermore, let f be continuous in* *the closed upper half-plane and suppose that* $|f(x)| \le M$ *on the real axis. Then*

$$|f(x+iy)| \le Me^{cy} \quad (-\infty < x < \infty,\ y > 0). \tag{2.1}$$

For our proof of Theorem 1 we also need the following result [3, p. 129].

LEMMA 2. *Let* $\omega(z)$ *be an entire function of exponential type having no zeros in the open upper half-plane and having*

$$h_\omega(\alpha) := \limsup_{r\to\infty} \frac{\log|\omega(re^{i\alpha})|}{r} \ge h_\omega(-\alpha) := \limsup_{r\to\infty} \frac{\log|\omega(re^{-i\alpha})|}{r}$$

for some $\alpha \in (0,\pi)$. *Then* $|\omega(z)| \ge |\omega(\overline{z})|$ *for* $\Im z > 0$.

3. Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1. Let $f(z) := Ae^{-i\tau z} - g(z) + Be^{i\tau z}$. We have to prove that $|A| + |B| \le \sup_{x\in\mathbb{R}} |f(x)|$ if g is an entire function of exponential type $\sigma < \tau$.

There is nothing to prove if A and B are both zero or if $|f(x)|$ is unbounded. So, let at least one of the two numbers A and B be different from 0. By considering $f(-z)$ if necessary we may suppose that $A \ne 0$. Let $\sup_{x\in\mathbb{R}} |f(x)| = M$. Clearly, $h_f(\pi/2) = \tau$. Hence, by Lemma 1, $|f(z)| \le Me^{\tau y}$ for $y := \Im z > 0$. In particular,

$$|Ae^{\tau y} - g(iy) + Be^{-\tau y}| \le Me^{\tau y} \quad (y > 0). \tag{3.1}$$

Note that $|g(x)| \le M + |A| + |B|$ on the real axis. Since $g(z)$ is of exponential type σ, we not only have $h_g(\pi/2) \le \sigma$ but also $h_g(-\pi/2) \le \sigma$. So, Lemma 1 may be applied to $g(z)$ if $y > 0$, and to $\overline{g(\overline{z})}$ if $y < 0$, in order to see that

$$|g(iy)| \le (M + |A| + |B|)\, e^{\sigma|y|} \quad (-\infty < y < \infty). \tag{3.2}$$

Now, divide the two sides of (3.1) by $e^{\tau y}$ and let $y \to \infty$. Taking into consideration inequality (3.2) for $y > 0$, we obtain

$$|A| \le M. \tag{3.3}$$

This completes the proof if B is 0. So, hereafter we suppose that A and B are both different from 0.

For $\lambda := |\lambda|e^{i\gamma}$ with $|\lambda| > 1$, let

$$\omega(z) = \omega_\lambda(z) := \lambda Me^{-i\tau z} - f(z) = (\lambda M - A)e^{-i\tau z} + g(z) - B\, e^{i\tau z}.$$

Then ω_λ is an entire function of exponential type such that

$$h_\omega(\pi/2) = h_\omega(-\pi/2) = \tau$$

and $\omega(z) \neq 0$ for $y := \Im z \geq 0$. By Lemma 2, $|\omega(z)| \geq |\omega(\overline{z})|$ for $y := \Im z > 0$. In particular, for any $y > 0$, we have

$$\left|(|\lambda| M e^{i\gamma} - A)e^{\tau y} + g(iy) - Be^{-\tau y}\right| \\ \geq \left|(|\lambda| M e^{i\gamma} - A)e^{-\tau y} + g(-iy) - Be^{\tau y}\right|.$$

Because of (3.3) it is possible to choose γ such that

$$\left||\lambda| M e^{i\gamma} - A\right| = |\lambda| M - |A|.$$

Hence, for any $y > 0$, we have

$$(|\lambda| M - |A|)e^{\tau y} + |g(iy) - Be^{-\tau y}| \geq |B| e^{\tau y} - (|\lambda| M - |A|)e^{-\tau y} - |g(-iy)|,$$

which may also be written as

$$(|\lambda| M - |A|) + |g(iy) - Be^{-\tau y}| e^{-\tau y} \\ \geq |B| - (|\lambda| M - |A|)e^{-2\tau y} - |g(-iy)| e^{-\tau y}.$$

Now let $y \to \infty$. Clearly, $(|\lambda| M - |A|)e^{-2\tau y}$ tends to 0. Because of (3.2) and the fact that $\sigma < \tau$, so do $|g(iy) - Be^{-\tau y}| e^{-\tau y}$ and $|g(-iy)| e^{-\tau y}$. We thus see that $|\lambda| M \geq |A| + |B|$, where $|\lambda|$ can be any number greater than 1. This is possible only if (1.2) holds. ■

Proof of Corollary 1. Set

$$\phi(z) := a_0 e^{i\lambda_0 z} + \sum_{\nu=1}^{n-1} a_\nu e^{i\lambda_\nu z} + a_n e^{i\lambda_n z}.$$

We have to show that $\sup_{-\infty < x < \infty} |\phi(x)| \geq |a_0| + |a_n|$. This holds if and only if

$$\sup_{-\infty < x < \infty} |e^{-i(\lambda_n + \lambda_0)z} \phi(2x)| \geq |a_0| + |a_n|. \tag{3.4}$$

In order to prove (3.4), we note that

$$\begin{aligned} e^{-i(\lambda_n + \lambda_0)z} \phi(2z) &= a_0 e^{-i(\lambda_n - \lambda_0)z} + \sum_{\nu=1}^{n-1} a_\nu e^{-i(\lambda_n - 2\lambda_\nu + \lambda_0)z} + a_n e^{i(\lambda_n - \lambda_0)z} \\ &= a_0 e^{-i(\lambda_n - \lambda_0)z} - g(z) + a_n e^{i(\lambda_n - \lambda_0)z}, \end{aligned}$$

where

$$g(z) := -\sum_{\nu=1}^{n-1} a_\nu e^{-i(\lambda_n - 2\lambda_\nu + \lambda_0)z}. \tag{3.5}$$

Since $\lambda_n - 2\lambda_\nu + \lambda_0$ decreases as ν increases, $g(z)$ is an entire function of exponential type σ, where

$$\sigma := \max\{|\lambda_n - 2\lambda_1 + \lambda_0|, |\lambda_n - 2\lambda_{n-1} + \lambda_0|\} < \lambda_n - \lambda_0.$$

Applying Theorem 1 with $A := a_0$, $B := a_n$, $\tau := \lambda_n - \lambda_0$ and $g(z)$ as in (3.5), we obtain (1.3). ■

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