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ORDINARY DIFFERENTIAL EQUATIONS

Global Attractor for a Class of Parabolic Equations with Infinite Delay

by

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Summary. We prove the existence of a compact connected global attractor for a class of abstract semilinear parabolic equations with infinite delay.

1. Introduction. The study of functional differential equations is motivated by the fact that when one wants to model some evolution phenomena arising in physics, biology, engineering, etc., with hereditary characteristic aftereffects, time lag and time delay can appear in the variables. Typical examples arise from research of materials with thermal memory, biochemical and population models, etc. (see e.g. [9, 21]). Partial differential equations (PDEs) with delay are often considered in models such as maturation time for population dynamics in mathematical biology and other fields. Such equations are naturally more difficult than ordinary differential equations with delay since they are infinite-dimensional both in time and space variables.

In recent years, the existence and long-time behavior of solutions to PDEs with delay has attracted wide attention. The development was initiated for PDEs with finite delay by Travis and Webb [19, 20], and continued by many other authors (see e.g. [21] and references therein). The problem for PDEs with infinite delay has also been discussed recently. However, most of existing results are devoted to the existence of solutions and stability of equilibrium points or steady states (see e.g. [1, 2, 3, 5, 10, 11]). On the other hand, it is known that attractors are a very useful tool (valid in more general situations than for stability) in investigating the asymptotical be-

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havior of solutions. However, as far as we know, most of existing results deal with the existence of attractors in the case of finite delay (see, for example, [4, 13, 14, 15, 17, 22]); only very few papers [6, 7] deal with the case of infinite delay in some concrete phase spaces.

Motivated by this fact, in this paper we study the existence of a global attractor for partial functional differential equations with infinite delay

$$\begin{cases} u'(t) + Au(t) = F(u_t), \quad t > 0, \\ u_0 = \varphi \in L^1_g(D(A^\alpha)), \end{cases} \tag{1.1}$$

where the operator A and the nonlinearity F satisfy the following conditions:

(A) The operator A is a positive sectorial operator with compact resolvent on a Banach space $(E, \|\cdot\|)$. Hence we can define fractional power spaces $D(A^\alpha)$, and the semigroup e^{-tA} generated by $-A$ satisfies the following estimate for some $\lambda > 0$:

$$\|e^{-tA}x\|_{D(A^\alpha)} \le C_\alpha e^{-\lambda t} t^{-\alpha}\|x\| \quad \text{for all } t > 0,\ x \in E$$

(see Sect. 2.1 for more details).

(F) The nonlinear term $F : L^1_g(D(A^\alpha)) \to E$, for some $\alpha \in [0, 1)$, is a function satisfying

$$\|F(\varphi) - F(\psi)\| \le L\|\varphi - \psi\|_{L^1_g} \quad \text{for all } \varphi, \psi \in L^1_g(D(A^\alpha)).$$

Here $L^1_g(D(A^\alpha))$ is the Banach space of functions mapping $(-\infty, 0]$ into the fractional power space $D(A^\alpha)$, which is defined in Sect. 2.2 below; and for each $u : (-\infty, T] \to D(A^\alpha)$, $T > 0$, and $t \in [0, T]$, u_t denotes, as usual, the element of $L^1_g(D(A^\alpha))$ defined by $u_t(\theta) = u(t + \theta)$ for $\theta \in (-\infty, 0]$.

It is known that numerous technical difficulties arise in dealing with partial differential equations with infinite delay. Let us explain the method used in the paper. First, we use the fixed point method to prove the existence of a unique mild solution to problem (1.1). Then we prove the continuous dependence of solutions on initial data. Therefore, we can define a continuous semigroup $\{S(t)\}$ associated to the problem. Finally, we prove that this semigroup has a global attractor by showing the existence of a bounded absorbing set and the asymptotic compactness of the semigroup.

The paper is organized as follows. In Section 2, for the convenience of the readers, we recall some properties of fractional power spaces and fractional power operators generated by the operator A, and some properties of the phase space $L^1_g(D(A^\alpha))$. The existence, uniqueness and continuous dependence of a mild solution to problem (1.1) is proved in Section 3. In Section 4, we prove the existence of a global attractor.

2. Preliminaries

2.1. Operator. We now recall some results of [16].

Since A is a sectorial operator, $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$. Because A is a positive sectorial operator, for $0 < \alpha < 1$, one can define

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-tA}\, dt,$$

where $\Gamma(\cdot)$ is the Gamma function. Since $A^{-\alpha}$ is one-to-one, we can define

$$A^\alpha = (A^{-\alpha})^{-1}.$$

In particular, $A^0 = I$.

Proposition 2.1 ([16]).

(1) *The operator A^α is closed with domain $D(A^\alpha) = R(A^{-\alpha})$, the range of $A^{-\alpha}$.*

(2) *$D(A^\alpha)$ is a Banach space with the norm $\|x\|_\alpha := \|A^\alpha x\|$, $x \in D(A^\alpha)$, and $\overline{D(A^\alpha)} = E$ for every $\alpha \geq 0$.*

(3) *If $\alpha \geq \beta > 0$, then $D(A^\alpha) \subset D(A^\beta)$.*

(4) *$e^{-tA} : E \to D(A^\alpha)$ for every $t > 0$ and $\alpha \geq 0$, and there exists $\lambda > 0$ such that*

$$\|e^{-tA}x\|_\alpha \leq C_\alpha e^{-\lambda t} t^{-\alpha} \|x\| \quad \text{for } t > 0,\ x \in E.$$

We now give a typical example of the operator A, appearing in [8]: A is a self-adjoint positive linear operator with discrete spectrum in a separable Hilbert space E (for example, the negative Laplacian operator $-\Delta$ with the homogeneous Dirichlet boundary condition in a bounded domain Ω; then $E = H_0^1(\Omega)$).

2.2. Phase space. Let $g : (-\infty, 0] \to \mathbb{R}$ be a positive function. Let $L_g^1(D(A^\alpha))$ consist of all classes of Lebesgue measurable functions $\varphi : (-\infty, 0] \to D(A^\alpha)$ such that $g(\cdot)\|\varphi(\cdot)\|_\alpha$ is Lebesgue integrable on $(-\infty, 0]$. The norm in $L_g^1(D(A^\alpha))$ is defined by

$$\|\varphi\|_{L_g^1} := \int_{-\infty}^0 g(\theta) \|\varphi(\theta)\|_\alpha \, d\theta.$$

We suppose that

(G) g satisfies the following:

(g1) there exists a locally bounded function $G : (-\infty, 0] \to [0, \infty)$ such that

$$g(\xi + \theta) \leq G(\xi) g(\theta) \quad \text{for all } \xi \leq 0 \text{ and } \theta \in (-\infty, 0] \setminus N_\xi,$$

where $N_\xi \subseteq (-\infty, 0]$ is a set of Lebesgue measure 0;

(g2) $k_1 := \int_{-\infty}^{0} g(\theta)\, d\theta < \infty$;
(g3) $k_2 := \int_{-\infty}^{0} g(\theta) e^{-\lambda\theta}\, d\theta < \infty$ for $\lambda > 0$ as in Proposition 2.1(4);
(g4) $G(-t) \to 0$ as $t \to \infty$.

A concrete example is $g(\theta) := e^{\rho\theta}$, where $\rho > \lambda$. Theorem 1.3.8 in [12] asserts that $L_g^1(D(A^\alpha))$ has the following properties:

(A) If $u : (-\infty, a) \to D(A^\alpha)$, $a > 0$, is such that $u_0 \in L_g^1(D(A^\alpha))$ and $u(\cdot)$ is continuous on $[0, a)$, then for all $t \in [0, a)$:
(1) $u_t \in L_g^1(D(A^\alpha))$,
(2) $\|u(t)\|_\alpha \leq \|u_t\|_{L_g^1}$,
(3) $\|u_t\|_{L_g^1} \leq K(t) \sup_{0\leq s\leq t} \|u(s)\|_\alpha + M(t)\|u_0\|_{L_g^1}$, where

$$K(t) = 1 + \int_{-t}^{0} g(\theta)\, d\theta \quad \text{and} \quad M(t) = G(-t).$$

(A1) For the function $u(\cdot)$ in (A)(1), $t \mapsto u_t$ is an $L_g^1(D(A^\alpha))$-valued continuous function $[0, a)$.
(B) $L_g^1(D(A^\alpha))$ is a Banach space (if we identify functions that are equal almost everywhere).
(C1) If $\{\varphi_n\}$ is a Cauchy sequence in $L_g^1(D(A^\alpha))$ that converges compactly to φ on $(-\infty, 0]$, then $\varphi \in L_g^1(D(A^\alpha))$ and $\|\varphi_n - \varphi\|_{L_g^1} \to 0$ as $n \to \infty$.

Let C_{00} be the set of continuous functions from $(-\infty, 0]$ into $D(A^\alpha)$ with compact support; denote by $\operatorname{supp}(\varphi)$ the support of φ in C_{00}. From a result in [12, Chapter 1], we have

Remark 2.1. Any function $\varphi \in C_{00}$ belongs to $L_g^1(D(A^\alpha))$. If $\operatorname{supp}(\varphi) \subset [-r, -s]$ for some $0 \leq s \leq r < \infty$, then there exists a constant $\delta(r, s)$ such that

$$\|\varphi\|_{L_g^1} \leq \delta(r, s) \sup_{\theta\in[-r,-s]} \|\varphi(\theta)\|_\alpha.$$

2.3. Global attractors. For the convenience of the reader, we recall some notions and results concerning global attractors from [18]. Let $(X, \|\cdot\|)$ be a Banach space (which in our case will be $L_g^1(D(A^\alpha))$) and $B_X(a, r)$ be the (closed) ball in X centered at a with radius r. We use the Hausdorff semi-distance $\delta_X(\cdot, \cdot)$ defined by

$$\delta_X(A, B) := \sup_{a\in A} \inf_{b\in B} \|a - b\| \quad \text{for } A, B \subset X.$$

Definition 2.1. Let $\{S(t)\}_{t\geq 0}$ be a semigroup in the Banach space X. A compact set $\mathcal{A} \subset X$ is said to be a *global attractor* for $\{S(t)\}_{t\geq 0}$ if the following conditions are satisfied:

(1) $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$ (invariance), and
(2) $\lim_{t\to\infty} \delta_X(S(t)D, \mathcal{A}) = 0$ for all bounded subsets D of X.

DEFINITION 2.2. A bounded subset $\mathcal{B}$ of X is said to be an *absorbing set* for the semigroup $\{S(t)\}_{t\geq 0}$ if for any bounded subset B of X, there exists $T(B) \geq 0$ such that

$$S(t)B \subset \mathcal{B} \quad \text{for all } t \geq T(B).$$

The following theorem gives sufficient conditions for existence of a global attractor.

THEOREM 2.2 ([18]). *Let $\{S(t)\}_{t\geq 0}$ be a continuous semigroup on X such that for every t, $S(t) = S_1(t) + S_2(t)$, where the operator $S_1(t)$ is uniformly compact for t large, i.e., for every bounded set B there exists t_0 which may depend on B such that*

$$\bigcup_{t\geq t_0} S_1(t)B \text{ is relatively compact in } X, \tag{2.1}$$

and $S_2(t)$ is a continuous mapping from X into itself such that

$$r_C(t) = \sup_{\varphi\in C} \|S_2(t)\varphi\|_X \to 0 \quad \text{as } t \to \infty, \tag{2.2}$$

for every bounded set $C \subset X$. If there exists an absorbing set $\mathcal{B}$ in X for $\{S(t)\}_{t\geq 0}$, then there exists a global attractor $\mathcal{A}$ in X, and $\mathcal{A} = \omega(\mathcal{B})$.

3. Existence and continuous dependence of solutions

DEFINITION 3.1. We say that a function $u : (-\infty, T] \to D(A^\alpha)$, $T > 0$, is a mild (in $D(A^\alpha)$) *solution* of the Cauchy problem (1.1) on the interval $[0, T]$ if $u_0 = \varphi$ and the restriction $u : [0, T] \to D(A^\alpha)$ is continuous and satisfies the integral equation

$$u(t) = e^{-tA}\varphi(0) + \int_0^t e^{-(t-s)A} F(u_s)\, ds, \quad 0 \leq t \leq T. \tag{3.1}$$

In the rest of this work we will call mild solutions just *solutions*. The following result is a direct consequence of Theorem 3.2 in [3].

THEOREM 3.1. *Let hypotheses* **(A)**, **(F)** *and* **(G)** *hold. Then for any $\varphi \in L^1_g(D(A^\alpha))$ and $T > 0$, there exists a unique solution of* (1.1) *on the interval* $[0, T]$.

We now prove the continuous dependence of solutions on initial data.

PROPOSITION 3.2. *Let hypotheses* **(A)**, **(F)** *and* **(G)** *hold and let $\varphi, \psi \in L^1_g(D(A^\alpha))$. Denote by u, v the solutions of problem* (1.1) *with initial data φ and ψ, respectively. If*

$$(1 + k_1)LC_\alpha \Gamma(1 - \alpha)\lambda^{\alpha-1} < 1,$$

then there exists a positive locally bounded function $m : [0, \infty) \to [0, \infty)$ *such that*

$$\|u_t - v_t\|_{L^1_g} \le m(t)\|\varphi - \psi\|_{L^1_g} \quad \textit{for all } t \ge 0. \tag{3.2}$$

Moreover, the map $t \mapsto \|u_t\|_{L^1_g}$ *belongs to the space* $C_b([0, \infty))$.

Proof. Set $w(t) = u(t) - v(t)$ and $w_0 = \varphi - \psi$. By (3.1), for $t \ge 0$,

$$\|w(t)\|_\alpha \le Ce^{-\lambda t}\|w(0)\|_\alpha + LC_\alpha \int_0^t e^{-\lambda(t-s)}(t-s)^{-\alpha}\|w_s\|_{L^1_g}\, ds. \tag{3.3}$$

If $0 \le \tau \le t$, then (3.3) yields

$$\begin{aligned}\|w(\tau)\|_\alpha &\le Ce^{-\lambda \tau}\|w(0)\|_\alpha + LC_\alpha \int_0^\tau e^{-\lambda(\tau-s)}(\tau-s)^{-\alpha}\|w_s\|_{L^1_g}\, ds \\ &\le C\|w_0\|_{L^1_g} + LC_\alpha \Gamma(1-\alpha)\lambda^{\alpha-1} \sup_{0\le s\le t}\|w_s\|_{L^1_g}.\end{aligned}$$

This implies that for all $0 \le s \le t$,

$$\begin{aligned}\|w_s\|_{L^1_g} &\le K(s) \sup_{0\le\tau\le s}\|w(\tau)\|_\alpha + M(s)\|w_0\|_{L^1_g} \\ &\le K(s) \sup_{0\le\tau\le t}\|w(\tau)\|_\alpha + M(s)\|w_0\|_{L^1_g} \\ &\le K(s)\Big(C\|w_0\|_{L^1_g} + LC_\alpha \Gamma(1-\alpha)\lambda^{\alpha-1} \sup_{0\le s\le t}\|w_s\|_{L^1_g}\Big) + M(s)\|w_0\|_{L^1_g} \\ &= [K(s)C + M(s)]\|w_0\|_{L^1_g} + K(s)LC_\alpha \Gamma(1-\alpha)\lambda^{\alpha-1} \sup_{0\le s\le t}\|w_s\|_{L^1_g}.\end{aligned}$$

Taking supremum over $s \in [0, t]$, we have

$$\begin{aligned}\sup_{0\le s\le t}\|w_s\|_{L^1_g} &\le \Big[K(t)C + \sup_{0\le s\le t} M(s)\Big]\|w_0\|_{L^1_g} \\ &\quad + K(t)LC_\alpha \Gamma(1-\alpha)\lambda^{\alpha-1} \sup_{0\le s\le t}\|w_s\|_{L^1_g} \\ &\le \Big[K(t)C + \sup_{0\le s\le t} M(s)\Big]\|w_0\|_{L^1_g} \\ &\quad + (1+k_1)LC_\alpha \Gamma(1-\alpha)\lambda^{\alpha-1} \sup_{0\le s\le t}\|w_s\|_{L^1_g}.\end{aligned}$$

Since $(1+k_1)LC_\alpha \Gamma(1-\alpha)\lambda^{\alpha-1} < 1$, we can rewrite this as

$$\|w_t\|_{L^1_g} \le \sup_{0\le s\le t}\|w_s\|_{L^1_g} \le \frac{K(t)C + \sup_{0\le s\le t} M(s)}{1 - (1+k_1)LC_\alpha \Gamma(1-\alpha)\lambda^{\alpha-1}}\|w_0\|_{L^1_g}.$$

Define $m : [0, \infty) \to [0, \infty)$ by

$$m(t) = \frac{K(t)C + \sup_{0\le s\le t} M(s)}{1 - (1+k_1)LC_\alpha \Gamma(1-\alpha)\lambda^{\alpha-1}};$$

then (3.2) holds.

Finally, that $t \mapsto \|u_t\|_{L^1_g}$ belongs to $C_b([0,\infty))$ follows immediately from the assumption **(G)** on g, and properties (A1), (A)(3) in Section 2.2. ∎

Theorem 3.1 and Proposition 3.2 allow us to define a continuous (nonlinear) semigroup $S(t) : L^1_g(D(A^\alpha)) \to L^1_g(D(A^\alpha))$ by the formula

$$S(t)\varphi = u_t(\cdot,\varphi), \quad t \ge 0, \tag{3.4}$$

where $u(\cdot,\varphi)$ is the unique global solution of (1.1) with initial datum $\varphi \in L^1_g(D(A^\alpha))$. The continuity of the semigroup with respect to t follows from (A1), and with respect to initial data from (3.2). In the next section, we will prove that $\{S(t)\}_{t\ge0}$ has a global attractor $\mathcal{A}$ in $L^1_g(D(A^\alpha))$.

4. Existence of a global attractor. The aim of this section is to prove the following result.

THEOREM 4.1. *Assume that hypotheses* **(A)**, **(F)** *and* **(G)** *hold and*

$$(1+k_1)LC_\alpha\Gamma(1-\alpha)\lambda^{\alpha-1} < 1.$$

Then the semigroup $\{S(t)\}_{t\ge0}$ *associated to problem* (1.1) *has a compact connected global attractor* $\mathcal{A}$ *in* $L^1_g(D(A^\alpha))$.

By Theorem 2.2, this theorem is a direct consequence of Propositions 4.2 and 4.3 below.

PROPOSITION 4.2. *Under the assumptions of Theorem* 4.1, *there exists a bounded absorbing set* $\mathcal{B}$ *in* $L^1_g(D(A^\alpha))$ *for the semigroup* $\{S(t)\}_{t\ge0}$.

Proof. We use some ideas of [11]. By condition (**F**), for all $s \ge 0$ we have

$$\|F(u_s)\| \le L\|u_s\|_{L^1_g} + \|F(0)\| = L\|u_s\|_{L^1_g} + N.$$

Hence

$$\begin{aligned}\|u(t)\|_\alpha &\le \|e^{-tA}\varphi(0)\|_\alpha + \int_0^t \|e^{-(t-s)A}F(u_s)\|_\alpha\,ds \\ &\le Ce^{-\lambda t}\|\varphi(0)\|_\alpha + C_\alpha\int_0^t e^{-\lambda(t-s)}(t-s)^{-\alpha}\|F(u_s)\|\,ds \\ &\le Ce^{-\lambda t}\|\varphi\|_{L^1_g} + C_\alpha\int_0^t e^{-\lambda(t-s)}(t-s)^{-\alpha}(L\|u_s\|_{L^1_g}+N)\,ds \\ &= Ce^{-\lambda t}\|\varphi\|_{L^1_g} + C_{\alpha,N} + \int_0^t x(t-s)\|u_s\|_{L^1_g}\,ds,\end{aligned} \tag{4.1}$$

where $C_{\alpha,N} = C_\alpha N\Gamma(1-\alpha)\lambda^{\alpha-1}$, and the function $x : (0,\infty) \to (0,\infty)$ is defined by

$$x(s) := C_\alpha Le^{-\lambda s}s^{-\alpha}.$$

Now using the definition of the norm in $L^1_g(D(A^\alpha))$ we have

$$\begin{aligned}\|u_t\|_{L^1_g} &= \int_{-\infty}^{-t} g(\theta)\|\varphi(t+\theta)\|_\alpha\,d\theta + \int_{-t}^{0} g(\theta)\|u(t+\theta)\|_\alpha\,d\theta \\ &\le G(-t)\|\varphi\|_{L^1_g} + \int_0^t g(s-t)\|u(s)\|_\alpha\,ds.\end{aligned}$$

Substituting in this expression the upper bound obtained in (4.1), and applying conditions (g3) and (g4), we obtain

$$\begin{aligned}\|u_t\|_{L^1_g} &\le G(-t)\|\varphi\|_{L^1_g} + C_{\alpha,N} + \int_0^t x(t-s)\|u_s\|_{L^1_g}\,ds \\ &\quad + \int_0^t g(s-t)\Big[Ce^{-\lambda s}\|\varphi\|_{L^1_g} + C_{\alpha,N} + \int_0^s x(s-\tau)\|u_\tau\|_{L^1_g}\,d\tau\Big]\,ds \\ &\le G(-t)\|\varphi\|_{L^1_g} + C_{\alpha,N} + \int_0^t x(t-s)\|u_s\|_{L^1_g}\,ds + Ck_2G(-t)\|\varphi\|_{L^1_g} \\ &\quad + C_{\alpha,N}k_1 + \int_0^t g(s-t)\Big[\int_0^s x(s-\tau)\|u_\tau\|_{L^1_g}\,d\tau\Big]\,ds \\ &\le (1+Ck_2)G(-t)\|\varphi\|_{L^1_g} + C_{\alpha,N}(1+k_1) + (x * \|u_s\|_{L^1_g})(t) \\ &\quad + (\widetilde{g} * x * \|u_s\|_{L^1_g})(t),\end{aligned}$$

where we have employed the function $\widetilde{g}(s) := g(-s)$ for $s \ge 0$. From the previous inequality we can write

$$\|u_t\|_{L^1_g} \le f_0(t)\|\varphi\|_{L^1_g} + C + \mathcal{K}(\|u_s\|_{L^1_g})(t), \tag{4.2}$$

where f_0 is a continuous function that vanishes at infinity, C is a constant, and $\mathcal{K}$ is the operator defined by

$$\mathcal{K}(f) := (x + \widetilde{g} * x) * f.$$

Since $\widetilde{g}$ and x are positive integrable functions on $[0,\infty)$, it is not difficult to see that $\mathcal{K}$ is a positive bounded linear operator on the space $C_b([0,\infty))$ of continuous bounded functions, endowed with the norm of uniform convergence, and the subspace $C_0([0,\infty))$ formed by the functions that vanish at infinity is invariant under $\mathcal{K}$. Furthermore, it is easy to see from the definition of $x(\cdot)$ that $\mathcal{K}$ is a contraction. We know from Proposition 3.2 that the function $t \mapsto \|u_t\|_{L^1_g}$ belongs to $C_b([0,\infty))$, so from (4.2) we infer that

$$\|u_t\|_{L^1_g} \le (I-\mathcal{K})^{-1}[f_0(t)\|\varphi\|_{L^1_g} + C] \le f_1(t)\|\varphi\|_{L^1_g} + C_1$$

for some $f_1 \in C_0([0,\infty))$ and a certain positive real number C_1.

If B is a bounded set in $L^1_g(D(A^\alpha))$, there exists $d > 0$ such that

$$\|\varphi\|_{L^1_g} \le d \quad \text{for all } \varphi \in B.$$

Since $f_1(t)$ vanishes at infinity, there exists a time $T = T(B) > 0$ such that

$$f_1(t)d \le C_1 \quad \text{for all } t \ge T.$$

Letting $R = 2C_1$, we deduce that the closed ball $\mathcal{B} = B_{L^1_g}(0, R)$ is a bounded absorbing set for $\{S(t)\}$ in $L^1_g(D(A^\alpha))$. ∎

PROPOSITION 4.3. *Under the assumptions of Theorem* 4.1*, the semigroup* $\{S(t)\}_{t\ge0}$ *satisfies conditions* (2.1) *and* (2.2)*, that is,* $\{S(t)\}_{t\ge0}$ *is asymptotically compact.*

Proof. Let $S(t) = S_1(t) + S_2(t)$, $t \ge 0$, where $\{S_1(t)\}_{t\ge0}$ is the solution semigroup of the equation

$$u(t) = \begin{cases} \int_0^t e^{-(t-s)A} F(u_s)\, ds, & t \ge 0, \\ 0, & t < 0, \end{cases}$$

and $\{S_2(t)\}_{t\ge0}$ is the solution semigroup of the equation

$$\begin{cases} u(t) = e^{-tA}\varphi(0), & t \ge 0, \\ u_0 = \varphi \in L^1_g(D(A^\alpha)). \end{cases}$$

It is easy to see that $\|S_2(t)\varphi\|_{L^1_g} \to 0$ as $t \to \infty$ whenever $\|\varphi\|_{L^1_g} \le r$. We now prove that $S_1(t)$ is compact for $t > 0$.

Let $(\psi_n)_{n\ge0}$ be a bounded sequence in $L^1_g(D(A^\alpha))$. First, we prove that

$$\{(S_1(t)\psi_n)(\theta)\}_{n\ge0}, \quad \theta \in (-\infty, 0],$$

is a totally bounded sequence in $D(A^\alpha)$, and for any $t > 0$ the sequence $(S_1(t)\psi_n)_{n\ge0}$ is equicontinuous in $(-\infty, 0]$. To this end, let $\theta \in (-\infty, 0]$; then for $n \ge 0$, we have

$$(S_1(t)\psi_n)(\theta) = \begin{cases} [S_1(t+\theta)\psi_n](0), & \theta \in [-t, 0], \\ 0, & \theta \in (-\infty, -t). \end{cases}$$

Let $0 < \epsilon < t + \theta$. We have

$$\begin{aligned} (S_1(t)\psi_n)(\theta) &= \int_0^{t+\theta} e^{-(t+\theta-s)A} F(u_s(\cdot, \psi_n))\, ds \\ &= e^{-\epsilon A} \int_0^{t+\theta-\epsilon} e^{-(t+\theta-\epsilon-s)A} F(u_s(\cdot, \psi_n))\, ds \\ &\quad + \int_{t+\theta-\epsilon}^{t+\theta} e^{-(t+\theta-s)A} F(u_s(\cdot, \psi_n))\, ds. \end{aligned}$$

Moreover,

$$\begin{aligned}\|F(u_s(\cdot,\psi_n))\| &\le \|F(u_s(\cdot,\psi_n)) - F(0)\| + \|F(0)\| \\ &\le L\|u_s(\cdot,\psi_n)\|_{L^1_g} + \|F(0)\| \le Lm(s)\|\psi_n\|_{L^1_g} + \|F(0)\|,\end{aligned}$$

where we have used Proposition 3.2. Hence, we can put

$$\alpha_t = \sup_{s\in[0,t]} \|F(u_s(\cdot,\psi_n))\| < \infty.$$

Since $e^{-\epsilon A}$ is compact, there exists a compact set W_ϵ such that

$$e^{-\epsilon A}\Big\{ \int_0^{t+\theta-\epsilon} e^{-(t+\theta-\epsilon-s)A} F(u_s(\cdot,\psi_n))\,ds : n \ge 0\Big\} \subset W_\epsilon.$$

Furthermore, for all $n \ge 0$,

$$\begin{aligned}\Big\| \int_{t+\theta-\epsilon}^{t+\theta} e^{-(t+\theta-s)A} F(u_s(\cdot,\psi_n))\,ds\Big\|_\alpha &\le C_\alpha \alpha_t \int_{t+\theta-\epsilon}^{t+\theta} e^{-\lambda(t+\theta-s)} (t+\theta-s)^{-\alpha}\,ds \\ &\le C_\alpha\alpha_t \int_{t+\theta-\epsilon}^{t+\theta} (t+\theta-s)^{-\alpha}\,ds = C_\alpha\alpha_t \frac{\epsilon^{1-\alpha}}{1-\alpha}.\end{aligned}$$

This shows the first assertion.

To establish the second assertion, let $\theta_0 \in (-\infty,0]$. For $\theta \in (-\infty,0]$ close enough to θ_0 such that $\theta_0 < \theta$, we see that

$$\begin{aligned}&(S_1(t)\psi_n)(\theta) - (S_1(t)\psi_n)(\theta_0) \\ &\qquad = \begin{cases} [S_1(t+\theta)\psi_n](0) - [S_1(t+\theta_0)\psi_n](0), & \theta_0 > -t, \\ [S_1(t+\theta)\psi_n](0), & \theta_0 = -t, \\ 0, & \theta_0 < -t. \end{cases}\end{aligned}$$

For $-t < \theta_0 < \theta \le 0$, we have

$$\begin{aligned}&\|(S_1(t)\psi_n)(\theta) - (S_1(t)\psi_n)(\theta_0)\|_\alpha \\ &\qquad = \Big\| \int_0^{t+\theta_0} [e^{-(t+\theta-s)A} - e^{-(t+\theta_0-s)A}] F(u_s(\cdot,\psi_n))\,ds\Big\|_\alpha \\ &\qquad\quad + \Big\| \int_{t+\theta_0}^{t+\theta} e^{-(t+\theta-s)A} F(u_s(\cdot,\psi_n))\,ds\Big\|_\alpha \\ &\qquad \le \Big\| [e^{-(\theta-\theta_0)A} - I] \int_0^{t+\theta_0} e^{-(t+\theta_0-s)A} F(u_s(\cdot,\psi_n))\,ds\Big\|_\alpha \\ &\qquad\quad + C_\alpha\alpha_t \frac{(\theta-\theta_0)^{1-\alpha}}{1-\alpha}.\end{aligned}$$

Moreover, there exists a compact set W such that

$$[e^{-(\theta-\theta_0)A} - I]\Big\{ \int_0^{t+\theta_0} e^{-(t+\theta_0-s)A} F(u_s(\cdot,\psi_n))\, ds : n \geq 0 \Big\} \subset W,$$

and using the fact that $(e^{-\cdot A}x)_{x\in W}$ is equicontinuous on the right at 0, we obtain

$$\lim_{\theta\to\theta_0^+} \|(S_1(t)\psi_n)(\theta) - (S_1(t)\psi_n)(\theta_0)\|_\alpha = 0.$$

By a similar argument for $-\infty < \theta < \theta_0 \leq 0$, we deduce the claimed equicontinuity.

By Arzelà–Ascoli's theorem, there are a continuous function $\varphi : (-\infty, 0] \to D(A^\alpha)$ and a subsequence φ_n of $(S_1(t)\psi_n)_{n\geq 0}$ which converges compactly to φ in $(-\infty, 0]$. By Remark 2.1, $(\varphi_n)_{n\geq 0}$ is also a norm Cauchy sequence in $L^1_g(D(A^\alpha))$. Then from (C1), φ is in $L^1_g(D(A^\alpha))$ and $\|\varphi_n - \varphi\|_{L^1_g} \to 0$ as $n \to \infty$. This completes the proof. ∎

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