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FUNCTIONAL ANALYSIS

On Some Properties of Separately Increasing Functions from $[0,1]^n$ into a Banach Space

by

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Summary. We say that a function f from $[0,1]$ to a Banach space X is increasing with respect to $E \subset X^*$ if $x^* \circ f$ is increasing for every $x^* \in E$. A function $f : [0,1]^m \to X$ is separately increasing if it is increasing in each variable separately. We show that if X is a Banach space that does not contain any isomorphic copy of c_0 or such that X^* is separable, then for every separately increasing function $f : [0,1]^m \to X$ with respect to any norming subset there exists a separately increasing function $g : [0,1]^m \to \mathbb{R}$ such that the sets of points of discontinuity of f and g coincide.

Throughout the paper, X will be a real Banach space and X^* its topological dual, and m will be a natural number, $m \geq 2$. The Banach space of all real bounded functions on a given set M equipped with the supremum norm will be denoted by $\mathcal{L}_\infty(M)$. The set of points of discontinuity of a function f from a metric space into X will be denoted by $D(f)$. For other notations and terminology the reader is referred to [14] and [5].

We say that a function f from $[0,1]$ to a Banach space X is *increasing with respect to* $E \subset X^*$ if $x^* \circ f$ is increasing (= nondecreasing) for every $x^* \in E$. We say that a function f from $[0,1]^m$ into X is *separately increasing* if it is increasing in each variable separately. Throughout the paper, we will assume that E is a *norming subset* of X^*, i.e., there exist constants $C \geq c > 0$ such that $c\|x\| \leq \sup\{|x^*(x)| : x^* \in E\} \leq C\|x\|$ for every $x \in X$. Every increasing function from the unit interval into a Banach lattice F is increasing with respect to the positive part of the unit ball of F^*. Consequently, the definitions cover all natural examples of increasing and separately increasing vector functions.

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Every real valued separately increasing function on $\mathbb{R}^m$ is continuous and differentiable almost everywhere with respect to the Lebesgue measure (see [2]). For every separately increasing function $f = (f_1, \dots, f_n) : \mathbb{R}^m \to \mathbb{R}^n$ the set $D(f) = D(f_1 + \dots + f_n)$ in view of the result in [2] has Lebesgue measure zero. This result does not extend to separately increasing functions on $[0,1]^m$ with values in Banach spaces. To see this it is enough to compose the projection onto the first variable of $[0,1]^m$ with an increasing function $g : [0,1] \to X$ discontinuous at each point of $[0,1]$. Such a g exists if and only if the space X contains an isomorphic and positive copy of the Banach space $D(0,1)$ of all real functions on $[0,1]$ that are right continuous at each point of $[0,1)$ with left-hand limit at each point of $(0,1]$, equipped with the supremum norm (see [11, Thm. 4] and [5, Cor. 2.3]).

The paper is devoted to relations between the following properties of separately increasing (with respect to a norming set) functions $f : [0,1]^p \to X$:

(1) f has separable range,
(2) there exists a separately increasing function $g : [0,1]^p \to \mathbb{R}$ such that $D(f) = D(g)$,
(3) the set $\{x^* \circ f : x^* \in X^*\}$ is separable in $\mathcal{L}_\infty([0,1]^p)$,
(4) the closed linear hull of $f([0,1]^p)$ does not contain any isomorphic and positive copy of the Banach space $D(0,1)$.

In the case $p = 1$ these four conditions are equivalent (see [11], [4] and [8]). Then of course the notions of separately increasing and increasing functions coincide and the condition (2) means that $D(f)$ is countable. In the case $p \geq 2$ no two of the properties (1)–(4) are equivalent (see Example 2.1). Since $D(0,1)$ is a nonseparable Banach space, the implication (1)⇒(4) is clear. Proposition 2.7 shows the implication (3)⇒(2). We show in Theorem 2.6 that if the closed linear hull of the range of a separately increasing function f does not contain any isomorphic copy of c_0, then f has property (2). We show that property (3) is a necessary condition to represent a separately increasing function $f : [0,1]^m \to X$ in the form of a pointwise converging series $\sum_{n=1}^{\infty} x_n g_n$, where $x_n \geq 0$ and g_n is a real separately increasing function for every n. Moreover, we show that if a Banach space X has an unconditional basis, then every separately increasing function $f : [0,1]^m \to X$ with respect to the order generated by the basis has property (3). The questions *whether every separately increasing* (*with respect to a norming set* E) *function* $f : [0,1]^m \to X$ *with property* (4) *or with property* (1) *also has property* (2) remain open.

The paper is divided into three sections. The first section gathers fundamental properties of increasing vector functions on $[0,1]$. The second section concerns separately increasing functions with values in Banach spaces. We show in the third section that the set $D(f)$ for any real separately increasing function f on $[0,1]^m$ has Hausdorff dimension less than or equal to $m-1$.

1. Properties of increasing functions on $[0,1]$. A subset E of X^* generates a partial order relation $\leq$ on X: $u \leq v$ whenever $x^*(u) \leq x^*(v)$ for every $x^* \in E$. The relation $\leq$ is closed, i.e., the subset $\{(u,v) : u \leq v\}$ of $X \times X$ is closed. If E is norming and $x, y, u, v \in X$ with $x \leq u \leq v \leq y$, then

$$\|y-x\| \geq \frac{1}{C} \sup_{x^* \in E} \{x^*(y-x)\} \geq \frac{1}{C} \sup_{x^* \in E} \{x^*(v-u)\} \geq \frac{c}{C} \|v-u\|.$$

It is clear that the following subsets of X^*: E, $\overline{E}^{w^*}$ (the closure of E in the weak* topology), $\mathrm{conv}(E)$ (the convex hull of E) and aE for any $a > 0$, generate the same order relation.

For a function $f : [0,1] \to \mathbb{C}$ we define its variation $\mathrm{var}(f)$ in the usual way, i.e.,

$$\mathrm{var}(f) = \sup\Big\{\sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)| : 0 \leq t_0 < t_1 < \cdots < t_n \leq 1,\, n \in \mathbb{N}\Big\}.$$

We define the variation of a vector function $f : [0,1] \to X$ by the formula

$$\mathrm{Var}(f) = \sup\{\mathrm{var}(x^* f) : x^* \in X^*,\ \|x^*\| \leq 1\}.$$

A function f is said to be *of bounded weak variation* if $\mathrm{Var}(f) < \infty$. We gather properties of functions of bounded weak variation in the following

Proposition 1.1.

(a) *If X is a Banach space which does not contain an isomorphic copy of c_0, then for every function $f : [0,1] \to X$ with bounded weak variation, the limits*

$$f(t-) = \lim_{s \to t-} f(s) \quad \text{and} \quad f(u+) = \lim_{s \to u+} f(s)$$

exist in the norm topology of X for every $t \in (0,1]$ and for every $u \in [0,1)$.

(b) *If $f : [0,1] \to X$ is an increasing function with respect to a norming subset E of X^*, then f has bounded weak variation and*

$$\mathrm{Var}(f) \leq \frac{C}{c} \|f(1) - f(0)\|$$

where $C = \sup_{\|x\| \leq 1} \{|x^*(x)| : x^* \in E\}$ and $c = \inf_{\|x\|=1} \sup\{|x^*(x)| : x^* \in E\}$.

These facts are known (part (a) can be deduced from [3, Thm. 6]), but for the sake of completeness we present their proofs.

Proof. (a) We only show the first case. Suppose that $\lim_{s\to t-} f(s)$ does not exist. Then there exist an $\varepsilon > 0$ and a strictly increasing sequence $\{t_n : n \in \mathbb{N}\} \subset [0,1]$ such that $\|f(t_{n+1}) - f(t_n)\| \geq \varepsilon$ for every n. Let $S : c_0 \to X$ be given by

$$S((\alpha_n)) = \sum_{n=1}^{\infty} \alpha_n (f(t_{n+1}) - f(t_n)).$$

Since f has bounded weak variation, for every $(\alpha_n) \in c_0$ we have

$$\|S((\alpha_k))\| \leq \sup_k\{|\alpha_k|\} \sup_{\|x^*\|\leq 1} \Big\{\sum_{n=1}^{\infty} |x^*(f(t_{k+1}) - f(t_k))|\Big\} \leq \|(\alpha_k)\|_{c_0} \mathrm{Var}(f).$$

This shows that the linear operator S is well defined and continuous. By the Bessaga–Pełczyński theorem (see [1]) there exists a subspace Y of c_0 isomorphic to c_0 such that the operator $S|_Y$ is an isomorphism. We arrive at a contradiction.

(b) For any increasing function $f : [0,1] \to X$ with respect to a norming subset E of X^* and any $\alpha_0, \alpha_1, \ldots, \alpha_n \in [0,1]$ such that $0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_n \leq 1$ we have

$$\begin{aligned}
&\sup_{\|x^*\|\leq 1} \Big\{\sum_{k=0}^{n-1} |x^*(f(\alpha_{k+1}) - f(\alpha_k))|\Big\} \\
&\qquad = \sup_{(\epsilon_1,\ldots,\epsilon_n)\in\{-1,1\}^n} \Big\{\Big\|\sum_{k=0}^{n-1} \epsilon_{k+1}(f(\alpha_{k+1}) - f(\alpha_k))\Big\|\Big\} \\
&\qquad \leq \frac{1}{c} \sup_{x^*\in E} \Big\{\sum_{k=0}^{n-1} x^*(f(\alpha_{k+1}) - f(\alpha_k))\Big\} \\
&\qquad = \frac{1}{c} \sup_{x^*\in E} \{x^*(f(1) - f(0))\} \leq \frac{C}{c}\|f(1) - f(0)\|.
\end{aligned}$$

Part (b) is a straightforward consequence of the above estimates. ∎

As a straightforward consequence of the proposition above we get

COROLLARY 1.2. *If X is a Banach space which does not contain an isomorphic copy of c_0, then every function $f : [0,1] \to X$ of bounded weak variation has relatively compact range.*

Let (M, ρ) be a metric space. For any function $f : M \to X$ the *oscillation function* $d_f : M \to \mathbb{R} \cup \{\infty\}$ is defined by

$$d_f(t) = \inf_{\delta>0} \sup\{\|f(s) - f(u)\| : s, u \in M,\ \rho(s,t) \leq \delta,\ \rho(u,t) \leq \delta\}.$$

It is clear that f is continuous at t if and only if $d_f(t) = 0$. The function d_f is upper semicontinuous (see [10]). For a given function $f : M \to X$ and for every $\varepsilon > 0$ we set

$$D(f) = \{t \in M : d_f(t) > 0\} \quad \text{and} \quad D(f, \varepsilon) = \{t \in M : d_f(t) \geq \varepsilon\}.$$

For every function $f : M \to X$ and every $\varepsilon > 0$ the set $D(f, \varepsilon)$ is closed in M and $D(f)$ is an F_σ subset of M.

It was shown in the proof of [11, Thm. 1] (see also [5, Prop. 3.1]) that for every increasing function $f : [0,1] \to X$ with respect to a norming set E we have

$$D(f) = \bigcup_{x^* \in \overline{E}^{w^*}} D(x^* \circ f).$$

This equality does not hold for functions of bounded weak variation (see [12, p. 233]). We will need the following quantitative version of the fact.

Theorem 1.3. *If $f : [0,1] \to X$ is an increasing function with respect to a norming subset E of X^*, then*

$$cd_f \leq \sup\{d_{x^* \circ f} : x^* \in \overline{E}^{w*}\} \leq Cd_f$$

where $C = \sup_{\|x\| \leq 1}\{|x^*(x)| : x^* \in E\}$ and $c = \inf_{\|x\|=1} \sup\{|x^*(x)| : x^* \in E\}$.

Proof. It is clear that $d_{x^* \circ f}(t) \leq Cd_f(t)$ for every $t \in [0,1]$ and $x^* \in E$.

It is enough to consider a function f with $f(0) = 0$ and $\|f(1)\| > 0$. Let H be the space of all increasing functions from $[0,1]$ into $[0, C\|f(1)\|]$ equipped with the pointwise convergence topology. It is clear that this space is homeomorphic to the Helly space (see [7, p. 164]). Therefore H is a Hausdorff, first countable, compact and sequentially compact space.

Let $\Phi_f : (\overline{E}^{w^*}, w^*) \to H$ be given by $\Phi_f(x^*) = x^* \circ f$. It is clear that Φ_f is continuous. Hence $H_f = \Phi_f(\overline{E}^{w^*})$ is a closed subset of H. Let t be in $[0,1]$. If $d_f(t) = 0$, the inequalities are clear. Suppose that $t \in (0,1)$ and $d_f(t) > 0$. Let $\min\{t, 1-t\} > 1/k$. Then for every $n \geq k$ there exist $s_n, u_n \in [t - 1/n, t + 1/n]$ such that $\|f(u_n) - f(s_n)\| > d_f(t) - 1/n$. Since E is a norming subset of X^*, we find $x_n^* \in E$ with

$$c(d_f(t) - 1/n) \leq |x_n^*(f(u_n) - f(s_n))| \leq x_n^*(f(t + 1/n) - f(t - 1/n))).$$

Let (j_n) be a sequence such that the sequences $(x_{j_n}^*(f(t + 1/j_n)))$ and $(x_{j_n}^*(f(t - 1/j_n)))$ are convergent. Then

$$\lim_{n\to\infty} x_{j_n}^*(f(t + 1/j_n) - f(t - 1/j_n)) \geq cd_f(t).$$

Let g be a cluster point of $\{x_{j_n}^* \circ f : n \in \mathbb{N}\}$. Since H_f is compact, g is a member of H_f. Consequently, there exists $x^* \in \overline{E}^{w^*}$ such that $\Phi_f(x^*) = g$.

It is easy to see that for every $s < t < u$ we have

$$g(s) = \lim_{n\to\infty} x^*_{j_n}(f(s)) \le \lim_{n\to\infty}(x^*_{j_n}(f(t-1/j_n))) < \lim_{n\to\infty}(x^*_{j_n}(f(t+1/j_n)))$$
$$\le \lim_{n\to\infty} x^*_{j_n}(f(u)) = g(u).$$

Hence

$$\lim_{s\to t+} g(s) - \lim_{s\to t-} g(s) \ge c d_f(t).$$

Thus we have shown that $d_{x^*\circ f}(t) \ge c d_f(t)$.

For $t = 0$ and $t = 1$ the considerations are similar. ∎

2. Properties of separately increasing functions on $[0,1]^m$**.** We start with the following two examples.

EXAMPLE 2.1. (a) Let M be an infinite set. For every $t \in [0,1]$ let $g_t : M \to \mathbb{R}$ be a function such that $0 \le g_t \le 1$. Let $f : [0,1]^2 \to \mathcal{L}_\infty(M)$ and $h : [0,1]^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} 1 & \text{if } y > 1-x, \\ 0 & \text{if } y < 1-x, \\ g_x & \text{if } y = 1-x, \end{cases} \quad \text{and} \quad h(s,t) = \begin{cases} 1 & \text{if } y \ge 1-x, \\ 0 & \text{if } y < 1-x. \end{cases}$$

It is clear that f and h are separately increasing functions. If $\{g_t : t \in [0,1]\}$ is a nonseparable subset of $\mathcal{L}_\infty(M)$, then f has nonseparable range but $D(f) = D(h)$.

If we put $M = [0,1]$ and $g_t = \chi_{\{t\}}$ for every $t \in [0,1]$, then $f([0,1]^2)$ is contained in a closed subspace of $\mathcal{L}_\infty([0,1])$ isomorphic to $c_0([0,1])$.

(b) Let M be a subset of $[0,1]^{[0,1]}$, compact and metrizable in the pointwise convergence topology but not separable in $\mathcal{L}_\infty([0,1])$ (for example: $M = \{g \in [0,1]^{[0,1]} : g^{-1}(\mathbb{R}\setminus\{0\}) \subset \{1/n : n \in \mathbb{N}\}\}$). For every $g \in M$ let $f_g : [0,1]^2 \to \mathbb{R}$ be given by

$$f_g(x,y) = \begin{cases} 1 & \text{if } y > 1-x, \\ 0 & \text{if } y < 1-x, \\ g(x) & \text{if } y = 1-x. \end{cases}$$

Let $L = \{f_g : g \in M\}$. It is clear that L is a compact and metrizable in the pointwise convergence topology but nonseparable (in $L_\infty([0,1]^2)$) subset of $[0,1]^{[0,1]^2}$. For every $x \in [0,1]^2$ let $p_x : L \to \mathbb{R}$ be given by $p_x(h) = h(x)$. We define $f : [0,1]^2 \to C(L)$ by the formula $f(x) = p_x$. It is clear that f is separately increasing function with respect to the set $\{\delta_l : l \in L\}$ of all Dirac measures on L and $D(f) = D(h)$ where h is defined in part (a). The set $\{\delta_l \circ f : l \in L\} = \{f_g : g \in M\}$ is not separable in $\mathcal{L}_\infty([0,1]^2)$. ∎

In what follows, we will apply the following definitions and notations. For every $x \in \mathbb{R}^m$ we put

$$\Gamma_x^- = x - \mathbb{R}_+^m \quad \text{and} \quad \Gamma_x^+ = x + \mathbb{R}_+^m$$

where as usual $\mathbb{R}_+$ denotes the set of positive real numbers. For every $f : [0,1]^m \to X$ and $x = (x_1, \ldots, x_m) \in [0,1]^m$ we define

$$f_x : [-\min\{x_1, \ldots, x_m\}, \min\{1 - x_1, \ldots, 1 - x_m\}] \to X$$

by the formula

$$f_x(t) = f(x + t(1, \ldots, 1)).$$

It is clear that if f is a separately increasing function with respect to E, then for every $x = (x_1, \ldots, x_m) \in [0,1]^m$ such that $-\min\{x_1, \ldots, x_m\} \neq \min\{1 - x_1, \ldots, 1 - x_m\}$ the function f_x is increasing with respect to E. Let $h : \mathbb{R} \to [0,1]$ be given by

$$h(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x \geq 0. \end{cases}$$

For every $f : [0,1]^m \to X$ we define $\tilde{f} : \mathbb{R}^m \to X$ by

$$\tilde{f}(x_1, \ldots, x_m) = f(h(x_1), \ldots, h(x_m)).$$

It is clear that $\tilde{f}$ is a separately increasing function on $\mathbb{R}^m$ if f is, and $\tilde{f}(x) = f(x)$ for every $x \in [0,1]^m$. For every $f : [0,1]^m \to X$ and $x \in \mathbb{R}^m$ we define $\tilde{f}_x : \mathbb{R} \to X$ by

$$\tilde{f}_x(t) = \tilde{f}(x + t(1, \ldots, 1)).$$

A crucial rule in our considerations is played by the following

THEOREM 2.2. *Let X be a Banach space with a norming subset E of X^* such that $c\|x\| \leq \sup\{|x^*(x)| : x^* \in E\} \leq C\|x\|$ for every $x \in X$. If $f : [0,1]^m \to X$ is a separately increasing function with respect to E, then*

(a) *for each $x \in (0,1)^m$,*

$$d_{f_x}(0) \leq d_f(x) \leq \frac{C}{c} d_{f_x}(0),$$

and consequently f is continuous at $x \in (0,1)^m$ if and only if f_x is continuous at 0,

(b) *for each $x \in [0,1]^m$,*

$$d_{\tilde{f}_x}(0) \leq d_{\tilde{f}}(x) \leq d_f(x) \leq \frac{C}{c} d_{\tilde{f}_x}(0),$$

and consequently f is continuous at $x \in [0,1]^m$ if and only if $\tilde{f}_x$ is continuous at 0,

(c) *for each* $x = (x_1, \dots, x_m) \in [0,1]^m$ *with* $\min\{x_1, \dots, x_m\} > 0$ *the limit* $f(x-) = \lim_{[0,1]^m \cap \Gamma_x^- \ni y \to x} f(y)$ *exists in the norm topology of* X *if and only if the limit* $f_x(0-) = \lim_{t \to 0-} f_x(t)$ *exists in the norm topology of* X*; moreover* $f(x-) = f_x(0-)$ *if the limits exist,*

(d) *for each* $x = (x_1, \dots, x_m) \in [0,1)^m$ *with* $\max\{x_1, \dots, x_m\} < 1$ *the limit* $f(x+) = \lim_{[0,1]^m \cap \Gamma_x^+ \ni y \to x} f(y)$ *exists in the norm topology of* X *if and only if the limit* $f_x(0+) = \lim_{t \to 0+} f_x(t)$ *exists in the norm topology of* X*; moreover* $f(x+) = f_x(0+)$ *if the limits exist.*

Proof. First note that f is bounded. This follows from the estimate

$$c\|f((1,\dots,1)) - f(x)\| \le \sup_{x^* \in E} x^*(f(1,\dots,1) - f(0)) \le C\|f(1,\dots,1) - f(0)\|$$

for every $x \in [0,1]^m$. Therefore the functions $d_f, d_{\tilde{f}}, d_{f_x}, d_{\tilde{f}_y}$ for each $x \in [0,1]^m$ and $y \in \mathbb{R}^m$ take their values in $\mathbb{R}$.

(a) Let $x = (x_1, \dots, x_m) \in (0,1)^m$. It is clear that $d_{f_x}(0) \le d_f(x)$. Since f is a separately increasing function with respect to E, for every $\varepsilon > 0$ there exist $0 < s \le \min\{\min\{x_1, \dots, x_m\}, \min\{1 - x_1, \dots, 1 - x_m\}\}$ and $y_1, y_2 \in x + [-s, s]^m$ such that

$$\begin{aligned} c(d_f(x) - \varepsilon) &\le c\|f(y_1) - f(y_2)\| \le \sup_{x^* \in E} |x^*(f(y_1) - f(y_2))| \\ &\le \sup_{x^* \in E} x^*(f(x + s(1,\dots,1)) - f(x - s(1,\dots,1))) \\ &\le C\|f_x(s) - f_x(-s)\| \le C(d_{f_x}(0) + \varepsilon). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $c d_f(x) \le C d_{f_x}(0)$.

(b) For every $x = (x_1, \dots, x_m) \in (0,1)^m$ we have $f(x) = \tilde{f}(x)$ and $f_x(t) = \tilde{f}_x(t)$ for $|t| \le \min\{\min\{x_1, \dots, x_m\}, \min\{1-x_1, \dots, 1-x_m\}\}$. Hence for every $x \in (0,1)^m$ we have

$$d_{\tilde{f}_x}(0) = d_{f_x}(0) \le d_f(x) = d_{\tilde{f}}(x) \le \frac{C}{c} d_{f_x}(0) = \frac{C}{c} d_{\tilde{f}_x}(0).$$

It is clear that part (a) is valid for any separately increasing function defined on any product $[a_1, b_1] \times \dots \times [a_m, b_m]$ where $a_1, \dots, a_m, b_1, \dots, b_m \in \mathbb{R}$ and $a_1 < b_1, \dots, a_m < b_m$. Consequently, for every $x \in \mathbb{R}^m$ we have $d_{\tilde{f}_x}(0) \le d_{\tilde{f}}(x) \le (C/c) d_{\tilde{f}_x}(0)$. Since h is a continuous function, for every $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ we have $d_f(h(x_1), \dots, h(x_m)) \ge d_{\tilde{f}}(x)$. Consequently, for every $x \in [0,1]^m$ we have $d_f(x) \ge d_{\tilde{f}}(x)$.

Let $x = (x_1, \dots, x_m) \in [0,1]^m \setminus (0,1)^m$. For any permutation π of the set $\{1, \dots, m\}$ let $p_\pi : \mathbb{R}^m \to \mathbb{R}^m$ be given by $p_\pi(y_1, \dots, y_m) = (y_{\pi(1)}, \dots, y_{\pi(m)})$. It is clear that p_π is a homeomorphism of $[0,1]^m$. In addition for any separately increasing function g on $[0,1]^m$ and each permutation π the function $g \circ p_\pi$ is also separately increasing and $\tilde{g} \circ p_\pi = \widetilde{g \circ p_\pi}$. Therefore it is enough to consider the following cases:

(1) $x = (x_1, \ldots, x_k, 0, \ldots, 0)$ for some $k \in \{1, \ldots, m-1\}$ and $u_k > 0$,
(2) $x = (x_1, \ldots, x_k, 1, \ldots, 1)$ for some $k \in \{1, \ldots, m-1\}$ and $u_k > 0$,
(3) $x = (x_1, \ldots, x_k, \overbrace{0, \ldots, 0}^{j}, 1, \ldots, 1)$ for some $k \in \{1, \ldots, m-2\}$ and $j \in \{1, \ldots, m-k-1\}$ and $u_k > 0$,
(4) $x = 0$ or $x = (1, \ldots, 1)$ or $x = (\overbrace{0, \ldots, 0}^{k}, 1, \ldots, 1)$ for some k in $\{1, \ldots, m-1\}$,

where $u_k = \min\{\min\{x_1, \ldots, x_k\}, \min\{1 - x_1, \ldots, 1 - x_k\}\}$.

(1) For every $0 < s < u_k$ and $y_1, y_2 \in [-s, s]^k \times [0, s]^{n-k}$ we have

$$\begin{aligned} c\|f(x+y_1) - f(x+y_2)\| \le \sup_{x^* \in E} \big\{x^*\big(&f(x_1+s, \ldots, x_k+s, s, \ldots, s) \\ &- f(x_1-s, \ldots, x_k-s, 0, \ldots, 0)\big)\big\} \\ &\le C\|\tilde{f}_x(s) - \tilde{f}_x(-s)\|. \end{aligned}$$

The definition of d_f and $d_{\tilde{f}_x}$ gives $cd_f(x) \le Cd_{\tilde{f}_x}(0)$.

(2) For every $0 < s < u_k$ and $y_1, y_2 \in [-s, s]^k \times [1-s, 1]^{m-k}$ we have

$$\begin{aligned} c\|f(x+y_1) - f(x+y_2))\| \le \sup_{x^* \in E} \big\{x^*\big(&f(x_1+s, \ldots, x_k+s, 1, \ldots, 1) \\ &- f(x_1-s, \ldots, x_k-s, 1-s \ldots, 1-s)\big)\big\} \\ &\le C\|\tilde{f}_x(s) - \tilde{f}_x(-s)\|. \end{aligned}$$

As above, this shows that $cd_f(x) \le Cd_{\tilde{f}_x}(0)$.

(3) For every $0 < s < u_k$ and $y_1, y_2 \in [-s, s]^k \times [0, s]^{n-k} \times [1-s, 1]^{m-k-l}$ we have

$$\begin{aligned} c\|f(x+y_1) - f(x+y_2)\| \le \sup_{x^* \in E} \big\{x^*\big(&f(x_1+s, \ldots, x_k+s, s, \ldots, s, 1, \ldots, 1) \\ &- f(x_1-s, \ldots, x_k-s, 0, \ldots, 0, 1-s, \ldots, 1-s)\big)\big\} \\ &\le C\|\tilde{f}_x(s) - \tilde{f}_x(-s)\|. \end{aligned}$$

As above, this shows that $cd_f(x) \le Cd_{\tilde{f}_x}(0)$.

For points in (4) the considerations are similar.

(c) If the first limit exists, then it is clear that the second exists too and they are equal. Suppose now that $\lim_{t \to 0-} f_x(t)$ exists in the norm topology of X. Take $\varepsilon > 0$. Then there exists $s > 0$ such that $\|f_x(-s) - f_x(0-)\| < \varepsilon$. For every $y \in x + [-s, 0)^m$ there exists $0 < t < s$ such that $y \in x + [-s, -t]^m$.

As f is separately increasing, we have

$$\begin{aligned} c\|f(y)-f_x(-s)\| &\le \sup_{x^*\in E} x^*\big(f(x-t(1,\dots,1))-f(x-s(1,\dots,1))\big)\\ &\le \sup_{x^*\in E} x^*(f_x(0-)-f_x(-s))\\ &\le C\|f_x(0-)-f_x(-s)\|\le C\varepsilon. \end{aligned}$$

Consequently, $\|f(y)-f_x(0-)\|\le (C/c+1)\varepsilon$.

The proof of (d) is similar. ■

As a straightforward consequence of [11, Thm. 4(c)] and the result above we obtain

COROLLARY 2.3. *If a Banach space X does not contain any isomorphic copy of $D(0,1)$ and $f:[0,1]^m\to X$ is separately increasing with respect to any norming subset E of X^*, then for any $x\in[0,1]^m$ the set $D(f)\cap\{x+t(1,\dots,1):t\in\mathbb{R}\}$ is countable.*

For separately increasing functions Theorem 1.3 takes the following form.

COROLLARY 2.4. *If $f:[0,1]^m\to X$ is a separately increasing function with respect to a norming subset E of X^*, then*

$$\frac{c^2}{C}d_f\le \sup\{d_{x^*\circ f}:x^*\in \overline{E}^{w^*}\}\le Cd_f,$$

where $C=\sup_{\|x\|\le 1}\{|x^*(x)|:x^*\in E\}$, $c=\inf_{\|x\|=1}\sup\{|x^*(x)|:x^*\in E\}$.

Proof. It is clear that for every $x^*\in E$ and $x\in[0,1]^m$ we have $d_{x^*\circ f}(x)\le Cd_f(x)$.

Let $x\in[0,1]^m$. Since $\tilde f_x$ is increasing with respect to E, by Theorem 1.3 there exists $x^*\in\overline{E}^{w^*}$ such that $d_{x^*\circ\tilde f_x}(0)\ge cd_{\tilde f_x}(0)$. In view of Theorem 2.2(b) we have $cd_{\tilde f_x}(0)\ge (c^2/C)d_f(x)$. ■

As a straightforward consequence of Proposition 1.1 and Theorem 2.2 we get

COROLLARY 2.5. *If a Banach space X does not contain any isomorphic copy of c_0, then for every function $f:[0,1]^m\to X$ separately increasing with respect to a norming subset E of X^* the limits*

$$f(x-)=\lim_{[0,1]^m\cap\Gamma_x^-\ni z\to x} f(z)\quad \textit{and}\quad f(y+)=\lim_{[0,1]^m\cap\Gamma_x^+\ni z\to y} f(z)$$

exist in the norm topology of X for every $x=(x_1,\dots,x_m)\in[0,1]^m$ with $\min\{x_1,\dots,x_m\}>0$ and for every $y=(y_1,\dots,y_m)\in[0,1]^m$ with $\max\{y_1,\dots,y_m\}<1$.

THEOREM 2.6. *If a Banach space X does not contain any isomorphic copy of c_0, then for every separately increasing function $f:[0,1]^m\to X$ with* *respect to any norming subset E of X^* there exists a separately increasing function $g : [0,1]^m \to \mathbb{R}$ such that $D(f) = D(g)$.*

Proof. Let A be a countable dense subset of $\mathbb{R}^m$. Let $\underline{f}, \overline{f} : \mathbb{R}^m \to X$ be given by

$$\underline{f}(x) = \lim_{A\cap\Gamma_x^- \ni z \to x} \tilde{f}(z) \quad \text{and} \quad \overline{f}(x) = \lim_{A\cap\Gamma_x^+ \ni z \to x} \tilde{f}(z).$$

According to Corollary 2.5 the functions $\underline{f}$ and $\overline{f}$ are well defined. Let Y be the closed linear hull of $\tilde{f}(A)$. It is clear that $\underline{f}$ and $\overline{f}$ take their values in Y and are separately increasing with respect to E. According to Corollary 2.5 and Theorem 2.2 for every $x \in \mathbb{R}^m$ we have

$$\underline{f}(x) = \lim_{t\to 0-} \tilde{f}(x + t(1,\dots,1)) \le \tilde{f}(x) \le \lim_{t\to 0+} \tilde{f}(x + t(1,\dots,1)) = \overline{f}(x).$$

It is easy to see that for every $x \in \mathbb{R}^m$ and $x^* \in \overline{E}^{w^*}$ we have

$$d_{x^*\circ\tilde{f}}(x) = x^*(\overline{f}(x)) - x^*(\underline{f}(x)).$$

Let $F = \{x^*|_Y : x^* \in \overline{E}^{w^*}\}$. Since Y is separable, F is a compact and metrizable subset of Y^* in the weak* topology of Y^*. Let $\{y_n^* : n \in \mathbb{N}\} \subset E$ be such that $\{y_n^*|_Y : n \in \mathbb{N}\}$ is dense in F in the weak* topology. Let $x \in D(f)$. In view of Theorem 2.2(b), $D(\tilde{f}) \cap [0,1]^m = D(f)$. According to Corollary 2.4 there exists $x^* \in \overline{E}^{w^*}$ such that $d_{x^*\circ\tilde{f}}(x) > 0$. Since $\underline{f}$ and $\overline{f}$ take values in Y, there exists n such that

$$|y_n^*(\underline{f}(x)) - x^*(\underline{f}(x))| \le d_{x^*\circ\tilde{f}}(x)/4, \qquad |y_n^*(\overline{f}(x)) - x^*(\overline{f}(x))| \le d_{x^*\circ\tilde{f}}(x)/4.$$

Hence

$$\begin{aligned} d_{x^*\circ\tilde{f}}(x) &\le |x^*(\overline{f}(x)) - y_n^*(\overline{f}(x))| + |y_n^*(\underline{f}(x)) - x^*(\underline{f}(x))| \\ &\quad + y_n^*(\overline{f}(x)) - y_n^*(\underline{f}(x)) \\ &\le d_{x^*\circ\tilde{f}}(x)/2 + d_{y_n^*\circ\tilde{f}}(x). \end{aligned}$$

This shows that $d_{y_n^*\circ\tilde{f}}(x) > 0$. In view of Theorem 2.2 we have $d_{y_n^*\circ f}(x) \ge d_{y_n^*\circ\tilde{f}}(x) > 0$. Thus we show that $D(f) = \bigcup_{n=1}^{\infty} D(y_n^* \circ f)$. Let $g = \sum_{n=1}^{\infty} (y_n^* \circ f)/2^n$. It is clear that $D(g) = \bigcup_{n=1}^{\infty} D(y_n^* \circ f)$. ∎

We will need the following simple facts.

Proposition 2.7.

(a) *Let (M,ρ) be a metric space. If A is a separable subset of $\mathcal{L}_\infty(M)$, then for every dense subset B of A we have*

$$\bigcup_{f\in A} D(f) = \bigcup_{f\in B} D(f).$$

(b) *Let X be a Banach space. If $f : [0,1]^m \to X$ is a separately increasing function with respect to a norming subset E of X^* such that the set of functions $\{x^* \circ f : x^* \in \overline{E}^{w^*}\}$ is separable in $\mathcal{L}_\infty([0,1]^m)$, then there exists a separately increasing function $g : [0,1]^m \to \mathbb{R}$ such that $D(f) = D(g)$.*

Proof. (a) Let $f \in A$, and let $x \in D(f)$. We can find $g \in B$ such that $\sup_{z\in M} |f(z) - g(z)| < d_f(x)/3$. Then

$$d_f(x) \le d_g(x) + 2 \sup_{z\in M} |f(z) - g(z)| < d_g(x) + 2d_f(x)/3.$$

This shows that $d_g(x) > 0$.

(b) Let $\{x_n^* : n \in \mathbb{N}\} \subset E$ be such that $\{x_n^* \circ f : n \in \mathbb{N}\}$ is dense in $\{x^* \circ f : x^* \in \overline{E}^{w^*}\}$ in the uniform topology. According to Corollary 2.4 and part (a) we have $D(f) = \bigcup_{x^*\in\overline{E}^{w^*}} D(x^* \circ f) = \bigcup_{n=1}^\infty D(x_n^* \circ f)$. Let $g = \sum_{n=1}^\infty (x_n^* \circ f)/2^n$. It is clear that $D(g) = \bigcup_{n=1}^\infty D(x_n^* \circ f)$. ■

Corollary 2.8. *Let X be a Banach space such that X^* is separable. If $f : [0,1]^m \to X$ is a separately increasing function with respect to a norming subset E of X^*, then there exists a separately increasing function $g : [0,1]^m \to \mathbb{R}$ such that $D(f) = D(g)$.*

Proof. Since f is bounded, the linear operator $S : X^* \to \mathcal{L}_\infty([0,1]^m)$ given by $S(x^*) = x^* \circ f$ is continuous. Since X^* is separable, $\{x^* \circ f : x^* \in X^*\} = S(X^*)$ is a separable subset of $\mathcal{L}_\infty([0,1]^m)$. An appeal to Proposition 2.7 completes the proof. ■

Proposition 2.9. *Let X be a Banach space and E a norming subset of X^*. If (v_n) is a sequence of positive elements of X and (f_n) is a sequence of separately increasing real functions such that the series $\sum_{n=1}^\infty v_n f_n(x)$ converges in the norm topology of X for each $x \in [0,1]^m$, then*

(a) *the series $\sum_{n=1}^\infty v_n(f_n(1,\dots,1) - f_n(0))$ converges unconditionally,*
(b) *the function $\sum_{n=1}^\infty v_n f_n$ has relatively compact range,*
(c) *the set $\{x^*(\sum_{n=1}^\infty v_n f_n) : x^* \in X^*\}$ is a separable subset of $\mathcal{L}_\infty([0,1]^m)$.*

Proof. Let $P = \{n \in \mathbb{N} : f_n(1,\dots,1) - f_n(0) > 0\}$. Since the series $\sum_{n=1}^\infty v_n f_n(0)$ and $\sum_{n=1}^\infty v_n f_n(1,\dots,1)$ are convergent in the norm topology of X, for every $k,l \in \mathbb{N}$, $l > k$ we have

$$c \sup_{(\epsilon_k,\dots,\epsilon_l)\in\{-1,1\}^{l-k+1}} \Big\|\sum_{n=k}^{l} \epsilon_n v_n(f_n(1,\dots,1) - f_n(0))\Big\|$$

$$\le \sup_{x^*\in E} \Big\{\sum_{n=k}^{\infty} x^*(v_n)(f_n(1,\dots,1)-f_n(0))\Big\} \le C\Big\|\sum_{n=k}^{\infty} v_n(f_n(1,\dots,1)-f_n(0))\Big\|$$

where $C = \sup_{\|x\|\leq 1}\{|x^*(x)| : x^* \in E\}$, $c = \inf_{\|x\|=1}\sup\{|x^*(x)| : x^* \in E\}$. Hence the series $\sum_{n=1}^{\infty} v_n(f_n(1,\ldots,1) - f_n(0))$ converges unconditionally. Therefore the linear operator $S : l_\infty \to X$ given by

$$S((\alpha_n)) = \sum_{n=1}^{\infty} \alpha_n v_n(f_n(1,\ldots,1) - f_n(0))$$

is compact. Moreover, for every $x \in [0,1]^m$,

$$\begin{aligned}\sum_{n=1}^{\infty} v_n f_n(x) &= \sum_{n\in P} v_n(f_n(x) - f_n(0)) + \sum_{n=1}^{\infty} v_n f_n(0)\\ &= \sum_{n\in P} v_n(f_n(1,\ldots,1) - f_n(0))\frac{f_n(x) - f_n(0)}{f_n(1,\ldots,1) - f_n(0)}\\ &\quad + \sum_{n=1}^{\infty} v_n f_n(0)\end{aligned}$$

is a member of the relatively compact set $S(\{w \in l_\infty : \|w\| \leq 1\}) + \sum_{n=1}^{\infty} v_n f_n(0)$. Since for each $x^* \in X^*$ the series $\sum_{n=1}^{\infty} |x^*(v_n)(f_n(1,\ldots,1) - f_n(0))|$ converges and l_1 is a separable Banach space and

$$\begin{aligned}x^*\Big(\sum_{n=1}^{\infty} v_n f_n\Big) &= \sum_{n\in P} x^*(v_n)(f_n(1,\ldots,1) - f_n(0))\frac{f_n - f_n(0)}{f_n(1,\ldots,1) - f_n(0)}\\ &\quad + x^*\Big(\sum_{n=1}^{\infty} v_n f_n(0)\Big),\end{aligned}$$

the set $\{x^*\big(\sum_{n=1}^{\infty} v_n f_n\big) : x^* \in X^*\}$ is separable in $\mathcal{L}_\infty([0,1]^m)$. ∎

A Schauder basis $\{e_n : n \in \mathbb{N}\}$ of a Banach space X is said to be *unconditional* if for each v in X its basic extension $v = \sum_{n=1}^{\infty} a_n e_n$ converges unconditionally (see [9]).

Corollary 2.10. *If $\{e_n : n \in \mathbb{N}\}$ is an unconditional basis of a Banach space X, then for every separately increasing function $f : [0,1]^m \to X$ with respect to $E = \{x^* \in X^* : x^*(e_n) \geq 0,\ n \in \mathbb{N},\ \|x^*\| \leq 1\}$ the set $\{x^* \circ f : x^* \in X^*\}$ is separable in $\mathcal{L}_\infty([0,1]^m)$ and there exists a separately increasing function $g : [0,1]^m \to \mathbb{R}$ such that $D(f) = D(g)$.*

Proof. Let $\{e_n^* : n \in \mathbb{N}\}$ be the sequence of biorthogonal functionals associated to the basis $\{e_n : n \in \mathbb{N}\}$, i.e., $e_n^*(e_k) = \delta_n^k$ where δ_n^k is the Kronecker delta. First we show that E is a norming set. Since $\{e_n : n \in \mathbb{N}\}$ is an unconditional basis, there exists a constant K such that for any sequence $\{\epsilon_n : n \in \mathbb{N}\} \subset \{0,1\}$ the linear operator $M_{(\epsilon_n)} : X \to X$ given by $M_{(\epsilon_n)}(\sum_{n=1}^{\infty} \alpha_n e_n) = \sum_{n=1}^{\infty} \epsilon_n \alpha_n e_n$ is well defined, continuous, and $\|M_{(\epsilon_n)}\| \leq K$ (see [9, p. 18]). Therefore the adjoint operator $M_{(\epsilon_n)}^* : X^* \to X^*$

also satisfies $\|M^*_{(\epsilon_n)}\| \le K$. Let $v \in X$ and $v^* \in X^*$ be such that $\|v^*\| \le 1$ and $v^*(v) = \|v\|$. Let $P = \{n \in \mathbb{N} : v^*(e_n) \ge 0\}$ and

$$\eta_n = \begin{cases} 1 & \text{if } n \in P, \\ 0 & \text{if } n \notin P. \end{cases}$$

Then $M^*_{(\eta_n)}(v^*)(e_k) = v^*(\eta_k e_k) \ge 0$ and $M^*_{(1-\eta_n)}(v^*)(e_k) = v^*((1-\eta_k)e_k) \le 0$ for every k. Therefore $M^*_{(\eta_n)}(v^*)$ and $-M^*_{(1-\eta_n)}(v^*)$ are elements of KE. The inequality $|v^*(v)| \le |M_{(\eta_n)}(v^*)(v)| + |M_{(1-\eta_n)}(v^*)(v)|$ provides the estimates

$$\frac{1}{2K}\|v\| \le \sup\{|x^*(v)| : x^* \in E\} \le \|v\|.$$

Let $f_n = e_n^* \circ f$. Then $f = \sum_{n=1}^{\infty} e_n f_n$ and the series converges at each point of $[0,1]^m$ in the norm topology of X. Since e_n^* is a member of $\|e_n^*\|E$, the function f_n is separately increasing. An appeal to Propositions 2.9 and 2.7 completes the proof. ∎

3. Geometric properties of $D(f)$. For a given $d \ge 0$ and $\eta > 0$ the measure $\mu_{d,\eta}$ for any subset A of $\mathbb{R}^m$ is defined by

$$\mu_{d,\eta}(A) = \inf\Big\{\sum_{n=1}^{\infty}(\operatorname{diam}(B_n))^d : A \subset \bigcup_{n=1}^{\infty} B_n,\ \operatorname{diam}(B_k) \le \eta,\ k \in \mathbb{N}\Big\}$$

where $\operatorname{diam}(C) = \sup\{\|x-y\|_2 : x, y \in C\}$ and $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^m$. The Hausdorff measure $\mu_d(A)$ and the Hausdorff dimension $d_{\mathrm{H}}(A)$ of a Borel subset A of $\mathbb{R}^m$ are defined by

$$\mu_d(A) = \lim_{\eta\to 0} \mu_{d,\eta}(A) \quad \text{and} \quad d_{\mathrm{H}}(A) = \inf\{d \ge 0 : \mu_d(A) = 0\}.$$

More on Hausdorff measures and Hausdorff dimension can be found in [6].

Proposition 3.1. *If $f : [0,1]^m \to \mathbb{R}$ is a separately increasing function, then*

(a) *for every $\varepsilon > 0$,*

$$\mu_{m-1}(D(f,\varepsilon)) \le (6m)^{m-1}\frac{f(1,\dots,1) - f(0)}{\varepsilon},$$

(b) $d_{\mathrm{H}}(D(f)) \le m-1$.

Proof. Let $V = \{(x_1,\dots,x_m) \in \mathbb{R}^m : x_1 + \dots + x_m = 0\}$. Let A be the orthogonal projection of $[0,1]^m$ onto V. For every $x \in A$ we consider the function $\tilde{f}_x$ on $[0,1]$. Let $B(r) = \{x \in \mathbb{R}^m : \|x\|_2 < r\}$. Let $\eta > 0$. For every $y \in A$ the set

$$D(f,\varepsilon) \cap \{y + t(1,\dots,1) : t \in [0,1]\} \subset y + D(\tilde{f}_y,\varepsilon)(1,\dots,1)$$

has at most

$$\frac{\tilde{f}_y(1) - \tilde{f}_y(0)}{\varepsilon} \le \frac{f(1,\dots,1) - f(0)}{\varepsilon} = N_\varepsilon$$

elements. For every $t \notin D(\tilde{f}_y, \varepsilon)$ there exists $\delta_1 > 0$ such that

$$\begin{aligned}\sup\{|\tilde{f}_y(s) - \tilde{f}_y(u)| : s, u \in [t - \delta_1, t + \delta_1]\} \\ = \sup\{|\tilde{f}(x) - \tilde{f}(z)| : x, z \in y + t(1,\dots,1) + [-\delta_1, \delta_1]^m\} < \varepsilon.\end{aligned}$$

For every $t \in D(\tilde{f}_y, \varepsilon)$ the function $\tilde{f}_y$ has left and right limits. Consequently, there exists $\delta_2 > 0$ such that

$$\begin{aligned}\sup\{|\tilde{f}(x) - \tilde{f}(z)| : x, z \in y + t(1,\dots,1) + (0, \delta_2]^m\} < \varepsilon, \\ \sup\{|\tilde{f}(x) - \tilde{f}(z)| : x, z \in y + t(1,\dots,1) + [-\delta_2, 0)^m\} < \varepsilon.\end{aligned}$$

Therefore for every $y \in A$ there exists δ_y such that $\eta > 4\delta_y\sqrt{m} > 0$ and

$$\begin{aligned}D(f, \varepsilon) \cap \big(\{y + t(1,\dots,1) : t \in [0,1]\} + [-\delta_y, \delta_y]^m\big) \\ \subset y + D(\tilde{f}_y, \varepsilon)(1,\dots,1) + [-2\delta_y, 2\delta_y]^m.\end{aligned}$$

Since the set A is compact, there exists a finite subset $\{y_1, \dots, y_p\}$ of A such that $A \subset \bigcup_{n=1}^p y_n + B(\delta_{y_n})$. The Hausdorff measure μ_{m-1} coincides on V with the $m-1$-dimensional Lebesgue measure multiplied by a constant (see [6]). Applying a standard procedure (see [13, Lemma 8.4]), we find $\{z_1, \dots, z_k\} \subset \{y_1, \dots, y_p\}$ such that $A \subset \bigcup_{n=1}^k z_n + B(\delta_{z_n})$ and the sets $z_1 + B(\delta_{z_1}/3), \dots, z_k + B(\delta_{z_k}/3)$ are pairwise disjoint. Therefore

$$D(f, \varepsilon) \subset \bigcup_{n=1}^k z_n + D(\tilde{f}_{z_n}, \varepsilon)(1,\dots,1) + [-2\delta_{z_n}, 2\delta_{z_n}]^m$$

and

$$3^{m-1}\mu_{m-1}((A + B(\eta)) \cap V) \ge \sum_{n=1}^k 2^{m-1}\delta_{z_n}^{m-1}.$$

Hence

$$\begin{aligned}\mu_{m-1,\eta}(D(f,\varepsilon)) &\le N_\varepsilon \sum_{n=1}^k (4\delta_{z_n}\sqrt{m}\,)^{m-1} \\ &\le (6\sqrt{m}\,)^{m-1} N_\varepsilon \mu_{m-1}((A + B(\eta)) \cap V) \\ &\le (6\sqrt{m}(\sqrt{m} + 2\eta))^{m-1} N_\varepsilon.\end{aligned}$$

This shows (a). Part (b) is a straightforward consequence of (a). ■

The result above enables us to express the smallness of $D(f)$ in terms of Hausdorff dimension. The following question is connected with Corollary 2.3: *is it true that if a Banach space X does not contain any isomorphic copy* *of $D(0,1)$, then $d_{\mathrm{H}}(D(f)) \leq m-1$ for each separately increasing function $f:[0,1]^m \to X$ with respect to any norming subset?*

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