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Local Controllability around Closed Orbits

by

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Summary. We give a necessary and sufficient condition for local controllability around closed orbits for general smooth control systems. We also prove that any such system on a compact manifold has a closed orbit.

1. Introduction

1.1. Motivation. The aim of this note is to formulate and prove a necessary and sufficient condition for local controllability of general control systems around a closed orbit.

Let M be a smooth (or real analytic) manifold, and let $\mathcal{U}$ be a subset of $\mathbb{R}^k$. Consider a smooth (or real analytic) control system

$$(\Sigma) \qquad \dot{x} = f(x,u), \quad u(\cdot) \in \mathcal{U},$$

where controls $u : [0,T] \to \mathcal{U}$ are bounded measurable, and the final time $T = T(u) \geq 0$ is not fixed and depends on the control u. If $u : [0,T(u)] \to \mathcal{U}$ is a control then a solution of the ordinary differential equation $\dot{x}(t) = f(x(t),u(t))$ is called a *trajectory* (or an *admissible curve*, or an *orbit*) of (Σ) generated by u.

The system (Σ) is said to be *controllable* if for every $x, y \in M$ there exists a control u defined on $[0,T(u)]$ such that if γ is the trajectory of (Σ) generated by u and satisfying $\gamma(0) = x$, then $\gamma(T(u)) = y$. The system (Σ) is *locally controllable* at a point x if there exists a neighbourhood U of x such that the restriction of (Σ) to U is a controllable system. A neighbourhood U as above is called a *controllable neighbourhood*.

There are a lot of results devoted to controllability question for control systems in connection with the existence of closed or 'almost closed' orbits:

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see, for instance, [2], [3], [5], [6], [10], [11]. Before we quote a few of them, we will fix some notation.

If $Z_1, \ldots, Z_l$ are vector fields on a manifold M then we denote by $\mathrm{Lie}\{Z_1, \ldots, Z_l\}$ the Lie algebra generated by $Z_1, \ldots, Z_l$. For $x \in M$, let $\mathrm{Lie}_x\{Z_1, \ldots, Z_l\}$ stand for the subspace in T_xM spanned by all vectors v of the form $v = W(x)$ where $W \in \mathrm{Lie}\{Z_1, \ldots, Z_l\}$. Recall that a point x is *Poisson stable* for a vector field X if for every neighbourhood V of x, and for every $T > 0$, there exist $t_1, t_2 > T$ such that $g_X^{t_1}(x) \in V$ and $g_X^{-t_2}(x) \in V$. Also, a vector field X defined on a Riemannian manifold is *conservative* if g_X^t preserves the natural measure on M. In both cases g_X^t stands for the flow of X.

Let us start by quoting two results on global controllability.

THEOREM (Bonnard [2]). *Consider an affine control system* $\dot{x} = X + \sum_{i=1}^k u_i Y_i$ *on an analytic manifold* M, *where* $\sum_{i=1}^k |u_i| \le 1$ *and the fields* X, Y_i, $i = 1, \ldots, k$, *are supposed to be analytic. Assume that the set of points which are Poisson stable for* X *is dense in* M. *Then the system in question is controllable if and only if* $\dim \mathrm{Lie}_x\{X, Y_1, \ldots, Y_k\} = \dim M$ *for every* $x \in M$.

Note that in the particular case when all orbits of X are closed, the set of points that are Poisson stable for X coincides with the whole of M.

THEOREM (Lobry [11]). *Consider an affine control system* $\dot{x} = X + \sum_{i=1}^k u_i Y_i$ *on a compact analytic manifold* M, *where* $\sum_{i=1}^k |u_i| \le 1$ *and the fields* X, Y_i, $i = 1, \ldots, k$, *are supposed to be analytic and conservative. Then the system in question is controllable if and only if* $\dim \mathrm{Lie}_x\{X, Y_1, \ldots, Y_k\} = \dim M$ *for every* $x \in M$.

The last two theorems are not exact quotations but can be deduced respectively from [2] and [11].

There are also results concerning local controllability. The result which is closest to our interests is as follows.

THEOREM (Nam and Arapostathis [12]). *Consider a smooth control system* $\dot{x} = X + \sum_{i=1}^k u_i Y_i$, $u \in \mathcal{U}$, *where* $\mathcal{U}$ *is a neighbourhood of* 0 *in* $\mathbb{R}^k$, *and let* Γ *be a closed orbit for* X. *Define* $\mathcal{G}_i = \{\mathrm{ad}^i X.Y_j : j = 1, \ldots, k\}$, *and suppose that there exists a point* $x \in \Gamma$ *such that*

$$\mathrm{rank}\{X, \mathcal{G}_0, \mathcal{G}_1, \ldots\}(x) = \dim M. \tag{1.1}$$

Then Γ *has a controllable neighbourhood.*

There are also other results (cf. for instance [5]), but they use stronger assumptions than those of [12]. As will be seen at the end of this paper, the assumptions in [12] can be weakened.

1.2. Statement of main results. The goal of this paper (which generalizes some ideas from sub-Lorentzian geometry that were developed by the author in [8]) is to prove two theorems: one concerns the existence of closed orbits, the other states necessary and sufficient conditions for local controllability around closed orbits. In order to state them, we first formulate our assumptions. Again, let

$$(\Sigma) \qquad \dot{x} = f(x,u) = f_u(x), \qquad u \in \mathcal{U},$$

be a control system, where M is a smooth manifold, $\mathcal{U}$ is (an arbitrary) subset of $\mathbb{R}^k$, f is a continuous mapping $M \times \mathcal{U} \to TM$, and f_u is a smooth vector field on M for every $u \in \mathcal{U}$. By a closed orbit of (Σ) we mean every trajectory $\Gamma : [a,b] \to M$ of (Σ) such that $a < b$, $\Gamma(a) = \Gamma(b)$ and $\Gamma_{|(a,b)}$ is not a constant curve. Our main assumption is

$$\dim \mathrm{Lie}_x\{f_u : \ u \in \mathcal{U}\} = n = \dim M \tag{1.2}$$

for every $x \in M$. As above, our controls are bounded measurable and the final time is not fixed. It follows from known results for ODEs with measurable right hand side (see e.g. [4]) that under such assumptions, to every control $u : [0,T] \to \mathcal{U}$ and every point $x_0 \in M$ there corresponds an admissible trajectory of (Σ) starting from x_0 (and defined maybe on a smaller interval).

The first result that we will prove is the following:

Theorem 1.1. *Consider the control system* (Σ) *for which* (1.2) *holds, and suppose that M is compact. Then the system* (Σ) *has closed orbits.*

Let $x \in M$ and take its neighbourhood U. Denote by $\mathcal{A}^+(x,U)$ the *reachable set from x in U* for the system (Σ), i.e. the set of endpoints of all trajectories of (Σ) that start from x, are generated by measurable controls (the final time is not fixed), and are contained in U. The sets $\mathcal{A}^+(x,M)$ will be denoted simply by $\mathcal{A}^+(x)$. Let us remark that controllability of (Σ) means that $\mathcal{A}^+(x) = M$ for every $x \in M$.

Suppose now that Γ is a closed orbit for (Σ). If a point x belongs to Γ then Γ_x will stand for the set $\Gamma \setminus \{x\}$.

Definition 1.1. We say that the closed orbit Γ is *regular* if there exists $x \in \Gamma$ and a neighbourhood U of x such that

$$\Gamma_x \cap \mathcal{A}^+(x,U) \subset \mathrm{int}\,\mathcal{A}^+(x,U). \tag{1.3}$$

Our second result can be stated as follows.

Theorem 1.2. *Suppose that Γ is a closed orbit for the system* (Σ) *for which* (1.2) *holds. Then the following conditions are equivalent:*

(i) *Γ is a regular closed orbit;*
(ii) (Σ) *is locally controllable at every point of Γ;*
(iii) *Γ has a controllable neighbourhood.*

Note that in Theorem 1.2, M is not supposed to be compact, and Γ need not be smooth. Theorem 1.2 slightly generalizes results from [12], as will be clarified at the end of the paper.

2. Proofs of theorems. Along with (Σ) we will consider the system

$$(\Sigma^-) \qquad \dot{x} = -f(x,u), \qquad u \in \mathcal{U}.$$

Let us record a simple observation which will be useful later.

LEMMA 2.1. *$\gamma(t)$ is a trajectory of the system (Σ) generated by a control $u(t)$ if and only if $\tilde{\gamma}(t) = \gamma(-t)$ is a trajectory of the system (Σ^-) generated by the control $\tilde{u}(t) = u(T(u) - t)$.*

Denote by $\mathcal{A}^-(x, U)$ the corresponding reachable set from x for (Σ^-). At the same time let $\mathcal{A}_0^+(x)$, $\mathcal{A}_0^-(x)$ be the reachable sets for (Σ) and (Σ^-), respectively, generated by piecewise constant controls. Recall now Krener's theorem [9] which states that under the assumption (1.2) we have the inclusion $\mathcal{A}_0^+(x) \subset \overline{\operatorname{int} \mathcal{A}^+(x)}$ (and the same for $\mathcal{A}_0^-(x)$). Therefore $\operatorname{int} \mathcal{A}^+(x)$ and $\operatorname{int} \mathcal{A}^-(x)$ are non-empty for every $x \in M$. Notice also that

$$x \in \overline{\operatorname{int} \mathcal{A}^+(x)} \cap \overline{\operatorname{int} \mathcal{A}^-(x)}$$

for any $x \in M$. Indeed, by Krener's theorem

$$x \in \mathcal{A}_0^+(x) \subset \overline{\operatorname{int} \mathcal{A}_0^+(x)} \subset \overline{\operatorname{int} \mathcal{A}^+(x)},$$

and the same for $\mathcal{A}^-(x)$. Now it is easy to show that

LEMMA 2.2. *$y \in \operatorname{int} \mathcal{A}^+(x)$ if and only if $x \in \operatorname{int} \mathcal{A}^-(y)$.*

Proof. Suppose that $y \in \operatorname{int} \mathcal{A}^+(x)$. Since $y \in \overline{\operatorname{int} \mathcal{A}^-(y)}$, it follows that $\operatorname{int} \mathcal{A}^+(x) \cap \operatorname{int} \mathcal{A}^-(y) \neq \emptyset$. Taking a $z \in \operatorname{int} \mathcal{A}^+(x) \cap \operatorname{int} \mathcal{A}^-(y)$ we see that there exist admissible curves for the system (Σ): σ_1 joining x to z, and (cf. Lemma 2.1) σ_2 joining z to y. Reversing time in $\sigma_1 \cup \sigma_2$ we obtain an admissible curve $\tilde{\sigma}$ for (Σ^-) that joins y to x, and which belongs to $\operatorname{int} \mathcal{A}^-(y)$ from a certain time $t_0 > 0$ on (for instance take t_0 corresponding to z). But this means that $\tilde{\sigma}$ stays in $\operatorname{int} \mathcal{A}^-(y)$ for all $t > t_0$, and therefore $x \in \operatorname{int} \mathcal{A}^-(y)$. ∎

To prove Theorem 1.1, we need to establish the following proposition.

PROPOSITION 2.1. *Under assumption* (1.2) *the family $\{\operatorname{int} \mathcal{A}^+(x)\}_{x \in M}$ forms an open covering of M.*

Proof. Fix $x \in M$ and consider a trajectory γ, $\gamma(0) = x$, of (Σ^-) such that $\gamma(t) \in \operatorname{int} \mathcal{A}^-(x)$ for a $t > 0$; by our assumptions such a curve exists. Now, the above lemmas imply that $x \in \operatorname{int} \mathcal{A}^+(\gamma(t))$, proving the assertion. ∎

Proof of Theorem 1.1. By Proposition 2.1 there are $x_1, \ldots, x_m \in M$ such that $M = \bigcup_{i=1}^m \operatorname{int} \mathcal{A}^+(x_i)$. Assume that $m = 1$, i.e. $M = \operatorname{int} \mathcal{A}^+(x_1)$.

Take a trajectory $\gamma : (-\varepsilon, \varepsilon) \to M$ of (Σ) such that $\gamma(0) = x_1$. Since $\gamma(-\varepsilon/2) \in \mathcal{A}^+(x_1)$, there exists a trajectory, say, σ of (Σ) connecting x_1 to $\gamma(-\varepsilon/e2)$. Then the concatenation $\gamma_{|[-\varepsilon/2]} \cup \sigma$ is a closed orbit for (Σ). Assume now that $m > 1$. Then we have $x_1 \in \operatorname{int} \mathcal{A}^+(x_{i_1})$ for an $i_1 \in \{1, \dots, m\}$, $x_{i_1} \in \operatorname{int} \mathcal{A}^+(x_{i_2})$ for $i_2 \in \{1, \dots, m\}$ etc. This yields an infinite sequence $\{x_{i_k}\}_{k=1}^{\infty}$ with $x_{i_k} \in \operatorname{int} \mathcal{A}^+(x_{i_{k+1}})$ and $i_k \in \{1, \dots, m\}$. Therefore we can find positive integers l and p such that $x_{i_l} \in \operatorname{int} \mathcal{A}^+(x_{i_{l+1}})$, $x_{i_{l+1}} \in \operatorname{int} \mathcal{A}^+(x_{i_{l+2}})$, $\dots$, $x_{i_{l+p}} \in \operatorname{int} \mathcal{A}^+(x_{i_l})$. ∎

Now we turn to the proof of Theorem 1.2. First of all let us list immediate properties of closed orbits. If Γ is a closed orbit for (Σ) then $\mathcal{A}^+(x_1)= \mathcal{A}^+(x_2)$ for every $x_1, x_2 \in \Gamma$. Moreover, $\mathcal{A}^+(x) = \mathcal{A}^+(\Gamma)$ for $x \in \Gamma$, where $\mathcal{A}^+(\Gamma) = \bigcup_{x\in\Gamma} \mathcal{A}^+(x)$. Since Γ, under a suitable parameterization, is a closed orbit also for (Σ^-), we have $\mathcal{A}^-(x_1) = \mathcal{A}^-(x_2) = \mathcal{A}^-(\Gamma)$ for any $x_1, x_2 \in \Gamma$. Let us also recall a standard fact from control theory asserting that the reachable set $\mathcal{A}^\pm(x)$ is open if and only if $x \in \operatorname{int} \mathcal{A}^\pm(x)$.

Next we prove

Lemma 2.3. *If Γ is a regular closed orbit for (Σ) then the set $\mathcal{A}^+(\Gamma)$ is open.*

Proof. Take an $x \in \Gamma$ and U such that (1.3) is satisfied, i.e. $\Gamma_x \cap \mathcal{A}^+(x, U) \subset \operatorname{int} \mathcal{A}^+(x, U)$. Clearly $\operatorname{int} \mathcal{A}^+(x, U) \subset \operatorname{int} \mathcal{A}^+(x)$. Choose y in $\Gamma_x \cap \mathcal{A}^+(x, U)$ and an open set V such that $y \in V \subset \mathcal{A}^+(x)$. For any $z \in V$ one can construct a trajectory of (Σ) joining y to z: we connect y to x by a suitable segment of Γ, and then x to z ($z \in \mathcal{A}^+(x)$). In this way we have proved that $V \subset \mathcal{A}^+(y)$, i.e. $y \in \operatorname{int} \mathcal{A}^+(y)$. This proves that $\mathcal{A}^+(y) = \mathcal{A}^+(\Gamma)$ is open, by the properties listed prior to the statement of the lemma. ∎

The last stage in proving Theorem 1.2 is the following observation.

Lemma 2.4. *Let Γ be a closed orbit for (Σ). Then Γ is regular for (Σ) if and only if it is regular for (Σ^-) (under a suitable parameterization).*

Proof. By symmetry, it is enough to prove one implication. Suppose that Γ is regular for (Σ) and choose x_1 and U such that $\Gamma_{x_1} \cap \mathcal{A}^+(x_1, U) \subset \operatorname{int} \mathcal{A}^+(x_1, U) \subset \operatorname{int} \mathcal{A}^+(x_1)$. Select $x_2 \in \Gamma_{x_1} \cap \mathcal{A}^+(x_1, U)$ and denote by $[x_1, x_2]$ the segment of Γ bounded by x_1 and x_2. By Lemma 2.3, for every $z \in [x_1, x_2]$ we have $x_2 \in \operatorname{int} \mathcal{A}^+(z)$ which, by Lemma 2.2, means that z is in $\operatorname{int} \mathcal{A}^-(x_2)$. Thus $[x_1, x_2] \subset \operatorname{int} \mathcal{A}^-(x_2)$, and consequently $\Gamma_{x_2} \cap \mathcal{A}^-(x_2, W) \subset \operatorname{int} \mathcal{A}^-(x_2, W)$ for a suitably chosen neighbourhood W of x_2, proving that Γ is regular for (Σ^-). ∎

Corollary 2.1. *If Γ is a regular closed orbit for (Σ) then the set $\mathcal{A}^-(\Gamma)$ is open.*

In order to finish the proof of Theorem 1.2 it is enough to notice that if Γ is a regular orbit for (Σ) then $U = \mathcal{A}^+(\Gamma) \cap \mathcal{A}^-(\Gamma)$ is a controllable neighbourhood. Indeed, take arbitrary $x, y \in U$. Since $x \in \mathcal{A}^-(\Gamma)$, there exists a trajectory σ_1 of (Σ) joining x to a point of Γ. Similarly, since $y \in \mathcal{A}^+(\Gamma)$, there exists a trajectory σ_2 of (Σ) joining a point of Γ to y. Finally, it is clear that any two points belonging to Γ can be joined by a trajectory of (Σ). The concatenation of σ_1, a suitable piece of Γ and σ_2 connects x to y and does not leave U. Indeed, take for instance σ_1. Obviously it is contained in $\mathcal{A}^-(\Gamma)$. But because $x \in \mathcal{A}^+(\Gamma)$, there exists an admissible curve joining a point of Γ to x, which implies that σ_1 is contained in $\mathcal{A}^+(\Gamma)$. Similarly we show that σ_2 does not leave U.

3. An example. Before we state our example let us recall the concept of geometric optimality and so-called singular extremals for the system (Σ). So fix a trajectory $\gamma : [0,T] \to U$ of (Σ), U being an open subset of M, which is generated by a control $\tilde{u} : [0,T] \to \mathcal{U}$. We say that γ (or $\tilde{u}$) is *geometrically optimal in* U if $\gamma([0,T]) \subset \partial_U \mathcal{A}^+(\gamma(0), U)$; here ∂_U denotes the boundary operator with respect to U. On the other hand, $\gamma : [0,T] \to M$ is called an *extremal* if there exists an absolutely continuous $p : [0,T] \to T^*M$ (called an *extremal lift*) such that $p(t) \in T^*_{\gamma(t)}M \setminus \{0\}$ for every t, and such that if we set $\mathcal{H}_u(x,p) = \langle p, f_u(x)\rangle$, then

(i) $(\dot{\gamma}(t), \dot{p}(t)) = \overrightarrow{\mathcal{H}_{\tilde{u}(t)}}(\gamma(t), p(t))$ a.e. on $[0,T]$, where $\overrightarrow{\mathcal{H}_u}$ is the Hamiltonian vector field on T^*M corresponding to the function $(x,p) \to \mathcal{H}_u(x,p)$,

(ii) $\mathcal{H}_{\tilde{u}(t)}(\gamma(t), p(t)) = 0$ on $[0,T]$, and

(iii) $\mathcal{H}_{\tilde{u}(t)}(\gamma(t), p(t)) = \max_{u \in \mathcal{U}} \mathcal{H}_u(\gamma(t), p(t))$ a.e. on $[0,T]$.

It follows from [1] that a necessary condition for geometric optimality in the above sense (corresponding to a free time problem) is that γ be an extremal. Now, an extremal $\gamma(t)$ generated by a control $\tilde{u}$ with values in $\operatorname{int}\mathcal{U}$ is called a *singular extremal* if there exists an extremal lift $p(t)$ such that additionally

(iv) $\frac{\partial \mathcal{H}_u(\gamma(t), p(t))}{\partial u}\big|_{u=\tilde{u}(t)} = 0$ for every t.

It is a standard fact that if γ is a geometrically optimal trajectory of (Σ) generated by a control $u : [0,T] \to \operatorname{int}\mathcal{U}$ with values in $\operatorname{int}\mathcal{U}$, then γ is a singular trajectory of (Σ).

Consider now a control affine system

$$\dot{x} = X + uY, \qquad |u| \le 1, \tag{3.1}$$

defined on a manifold M. Fix a point x_0 and a time interval $[0,T]$. Let γ be the trajectory of X starting at x_0; in other words γ is a trajectory of our control system generated by the control $u^0(t) \equiv 0$. Next, consider the so-called endpoint map Φ^{T,x_0}, i.e. the mapping which to each control $u : [0,T] \to [-1,1]$ assigns the point $\Phi^{T,x_0}(u) = \gamma_u(T)$, where γ_u is the trajectory of (3.1) that starts from x_0 and is generated by u. It can be proved (see e.g. [3]) that

$$\operatorname{im} d_{u^0}\Phi^{T,x_0} = \operatorname{span}\{Y(\gamma(T)), (\operatorname{ad}^k X.Y)(\gamma(T)) : k = 1,2,\ldots\},$$

where $\operatorname{ad} X.Y = [X,Y]$, and $\operatorname{ad}^{k+1} X.Y = [X, \operatorname{ad}^k X.Y]$, $k = 1,2,\ldots$. It is known (see again e.g. [3]) that γ is not a singular trajectory for (3.1) if and only if

$$\dim \operatorname{span}\{Y(\gamma(T)), (\operatorname{ad}^k X.Y)(\gamma(T)) : k = 1,2,\ldots\} = \dim M. \tag{3.2}$$

Now let us take a closer look at the result from [12] cited in the Introduction, applied to the system (3.1). Suppose that Γ is a closed orbit of X and fix an $x \in \Gamma$. If (1.1) is satisfied at x then (3.2) does not have to be satisfied, as explained in [12]. On the other hand assume that (3.2) is satisfied at x. Then of course (1.1) is also satisfied and, by the above remark, Γ is not a singular trajectory. Consequently, it is not geometrically optimal from x, and consequently it is a regular closed orbit for (3.1). Thus the satisfaction of (3.2) implies that Γ is a regular closed orbit.

Now, we are going to present a simple construction of a closed trajectory Γ which satisfies neither (3.2) nor (1.1), but anyway is a regular closed orbit.

To this end consider

$$W = \{(x_1, x_2, x_3) : x_2^2 + x_3^2 < 1,\ 0 \le x_1 \le 2\pi\} \subset \mathbb{R}^3.$$

Let us introduce the following equivalence relation on W: $(x_1,x_2,x_3) \sim (x_1', x_2', x_3')$ if and only if $x_2 = x_2'$, $x_3 = x_3'$, and either $x_1 = 0$, $x_1' = 2\pi$, or $x_1 = 2\pi$, $x_1' = 0$. Consider the factorization $p : W \to M = W/{\sim}$. The space M is a 3-dimensional manifold which in an obvious way can be embedded in $\mathbb{R}^3$. Let

$$\tilde{X} = \frac{\partial}{\partial x_1} + x_2^k \frac{\partial}{\partial x_3}, \qquad \tilde{Y} = \frac{\partial}{\partial x_2}, \qquad k \ge 3,$$

be vector fields on $\mathbb{R}^3$. After factorization they are transformed to vector fields

$$X = p_* \tilde{X}, \quad Y = p_* \tilde{Y} \tag{3.3}$$

on M. Now denote by (Σ) the control system (3.1) on M where X and Y are defined by (3.3). It is easily seen that the image under p of the x_1-axis, denoted by Γ, is a closed and singular trajectory for (Σ). Indeed, its extremal lift is given by $\lambda(t) = (t \bmod 2\pi, 0,0,0,0,1)$.

Define a rank 2 distribution H on M by letting $H = \operatorname{span}\{X, Y\}$. If x is a point in M and l is a positive integer, then we will write H_x^l for the span of all vectors of the form

$$[X_1,[X_2,\dots,[X_{i-1},X_i]\dots]](x),$$

where $X_1,\dots,X_i$ are smooth local sections of H defined near x, with $i\le l$. Now it is not difficult to see that if $S=\{x_2=0\}$, then H is a contact distribution on $M\setminus S$, i.e. $H_x^2=T_xM$ whenever $x\in M\setminus S$. It can also be seen that H has the following bracket properties on S: $H_x^l\subset H_x$, $1\le l\le k$, and $H_x^{k+1}=T_xM$ whenever $x\in S$.

All this permits us to conclude that, as is explained in [7], (Σ) is an affine control system induced by the generalized Martinet sub-Lorentzian structure of Hamiltonian type of order k. Suppose that k is odd. It follows from [7] that for every $x_0\in\Gamma$ there exists a neighbourhood U of x_0 and coordinates $\tilde x_1,\tilde x_2,\tilde x_3$ on U, with $\tilde x_1(x_0)=\tilde x_2(x_0)=\tilde x_3(x_0)=0$, such that $S\cap U=\{\tilde x_2=0\}$, $\Gamma\cap U=\{\tilde x_2=\tilde x_3=0\}$ and $\mathcal{A}^+(x_0,U)=A_1\cup A_2$, where

$$A_1=\{x\in U:\eta_1(\tilde x_1(x),\tilde x_2(x),\tilde x_3(x))\le 0,\ \tilde x_1(x)\ge 0,\ \tilde x_3(x)\ge 0\},$$
$$A_2=\{x\in U:\eta_2(\tilde x_1(x),\tilde x_2(x),\tilde x_3(x))\le 0,\ \tilde x_1(x)\ge 0,\ \tilde x_3(x)\le 0\},$$

with

$$\eta_1(\tilde x_1,\tilde x_2,\tilde x_3)=\tilde x_3+\frac{1}{2k}(\tilde x_1+\tilde x_2)\left(\tilde x_2^k-\frac{1}{2^k}(\tilde x_1+\tilde x_2)^k\right)+O(r^{k+2}),$$
$$\eta_2(\tilde x_1,\tilde x_2,\tilde x_3)=-\tilde x_3-\frac{1}{2k}(\tilde x_1-\tilde x_2)\left(\tilde x_2^k+\frac{1}{2^k}(\tilde x_1-\tilde x_2)^k\right)+O(r^{k+2});$$

here $r=(\tilde x_1^2+\tilde x_2^2+\tilde x_3^2)^{1/2}$.

Since $\eta_1(\tilde x_1,0,0)<0$ and $\eta_2(\tilde x_1,0,0)<0$ (we choose U to be sufficiently small), it is seen that $\Gamma_{x_0}\cap U\subset\operatorname{int}\mathcal{A}^+(x_0,U)$ and Γ is a regular closed orbit. At the same time one easily sees that

$$[\tilde X,\tilde Y]=-kx_2^{k-1}\frac{\partial}{\partial x_3},$$

which yields $\operatorname{ad}^l\tilde X.\tilde Y=0$ for all $l\ge 2$, meaning that (1.1) does not hold at any point of Γ.

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