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Construction of Non-MSF Non-MRA Wavelets for $L^2(\mathbb{R})$ and $H^2(\mathbb{R})$ from MSF Wavelets

by

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Summary. Considering symmetric wavelet sets consisting of four intervals, a class of non-MSF non-MRA wavelets for $L^2(\mathbb{R})$ and dilation 2 is obtained. In addition, we obtain a family of non-MSF non-MRA H^2-wavelets which includes the one given by Behera [Bull. Polish Acad. Sci. Math. 52 (2004), 169–178].

1. Introduction. In [4], Dai and Larson called a measurable subset W of the real line a *wavelet set* if the characteristic function χ_W of W is equal to $\sqrt{2\pi}$ times the modulus of the Fourier transform $\widehat{\psi}$ for some orthonormal wavelet ψ on $L^2(\mathbb{R})$. A function ψ in $L^2(\mathbb{R})$ whose successive dilates by a scalar d and all integral translates form an orthonormal basis for $L^2(\mathbb{R})$ is called an *orthonormal wavelet* for $L^2(\mathbb{R})$. An orthonormal wavelet whose Fourier transform has the support of smallest possible measure is called a *minimally supported frequency* (MSF) *wavelet*. In fact, an MSF wavelet ψ is a wavelet which is associated with a wavelet set W in the sense that the support of $\widehat{\psi}$ is W [1, 4–10]. One of the earliest wavelets, namely Shannon wavelet for dilation 2, has $W = [-2\pi, -\pi] \cup [\pi, 2\pi]$ as its wavelet set, which is a union of two disjoint intervals of $\mathbb{R}$. Wavelet sets in $\mathbb{R}$ which are unions of two disjoint intervals and also those which are unions of three disjoint intervals have been characterized by Ha, Kang, Lee and Seo [6]. In addition, they characterized those wavelet sets which are symmetric with respect to the origin and consist of four intervals. These are precisely $K_r = K_r^- \cup K_r^+$, where

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$$(1)\qquad K_r^+ = \left[\frac{2^r}{2^{r+1}-1}\,\pi, \pi\right] \cup \left[2^r\pi,\ \frac{2^{2r+1}}{2^{r+1}-1}\,\pi\right], \qquad K_r^- = -K_r^+,$$

and r is a positive integer. Further, they considered H^2-wavelet sets [1, 2, 6, 9] and characterized those H^2-wavelet sets which have just one interval and also those with two intervals. Indeed, H^2-wavelet sets consisting of two intervals are given by

$$(2)\qquad K_{r,k} = \left[\frac{2(k+1)}{2^{r+1}-1}\,\pi,\ \frac{2k}{2^r-1}\,\pi\right] \cup \left[\frac{2^{r+1}k}{2^r-1}\,\pi,\ \frac{2^{r+2}(k+1)}{2^{r+1}-1}\,\pi\right],$$

where $r \in \mathbb{N}$ and $1 \leq k < 2(2^r - 1)$.

Bownik and Speegle [3] characterized those dilations which admit non-MSF wavelets considering higher dimensional wavelets. Exploiting the structure of H^2-wavelet sets having two intervals, Behera [2] constructed a family of non-MSF Hardy wavelets for $H^2(\mathbb{R})$ which, in addition, turns out to be a family of non-MRA Hardy wavelets for $H^2(\mathbb{R})$ due to Theorem 4.2, established in this paper.

With the help of symmetric wavelet sets consisting of four intervals, we provide a class of non-MSF non-MRA wavelets for $L^2(\mathbb{R})$ and dilation 2 in Section 3. Also, considering H^2-wavelet sets with two intervals for $r \in \mathbb{N}$ and $k = 2^l - 1$, $1 \leq l \leq r$, we provide a family of non-MSF non-MRA H^2-wavelets and dilation 2 in Section 4, which includes the one given by Behera.

2. Prerequisites. A pair $(\{V_j\}_{j\in\mathbb{Z}}, \varphi)$ consisting of a family $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ together with a function $\varphi \in V_0$ is called a *multiresolution analysis* (MRA) if it satisfies the following conditions:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,
(b) $f \in V_j$ if and only if $f(2(\cdot)) \in V_{j+1}$ for all $j \in \mathbb{Z}$,
(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
(d) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$,
(e) $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

The function φ is called a *scaling function* for the given MRA. An MRA determines a function ψ lying in the orthogonal complement of V_0 in V_1 which is an orthonormal wavelet for $L^2(\mathbb{R})$. Such a ψ is called an MRA *wavelet* arising from the MRA $(\{V_j\}_{j\in\mathbb{Z}}, \varphi)$. The scaling function gives rise to a 2π-periodic function, known as the *low-pass filter* corresponding to φ, which satisfies

$$\widehat{\psi}(\xi) = e^{i\xi/2}\, \overline{m_0(\xi/2+\pi)}\, \widehat{\varphi}(\xi/2) \quad \text{for a.e. } \xi \in \mathbb{R}.$$

A multiresolution analysis for $H^2(\mathbb{R})$ and H^2-MRA wavelets can be described similarly.

For an orthonormal wavelet ψ, the formula

$$D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^j(\xi + 2k\pi))|^2$$

defines the *dimension function* D_ψ for ψ. We use the following characterization which works for both MRA wavelets and H^2-MRA wavelets [9].

RESULT 2.1. *A wavelet* $\psi \in L^2(\mathbb{R})$ (*resp.* $\psi \in H^2(\mathbb{R})$) *is an* MRA (resp. H^2-MRA) *wavelet iff* $D_\psi(\xi) = 1$ *for almost every* $\xi \in \mathbb{R}$.

Also, we use the following known characterization of orthonormal wavelets for $L^2(\mathbb{R})$ and $H^2(\mathbb{R})$ (see [9]).

RESULT 2.2. *A function* $\psi \in L^2(\mathbb{R})$ (*resp.* $\psi \in H^2(\mathbb{R})$) *is an orthonormal wavelet* (*resp.* H^2*-wavelet*) *iff*

(i) $\|\psi\|_2 = 1$,
(ii) $\rho(\xi) = \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = \chi_{\mathbb{R}}(\xi)$ (resp. $\chi_{\mathbb{R}^+}(\xi)$) *for a.e.* $\xi \in \mathbb{R}$,
(iii) $t_q(\xi) = \sum_{j\geq 0} \widehat{\psi}(2^j\xi)\,\overline{\widehat{\psi}(2^j(\xi + 2q\pi))} = 0$ *for a.e.* $\xi \in \mathbb{R}$ *and for* $q \in 2\mathbb{Z} + 1$.

3. Non-MSF non-MRA wavelets for $L^2(\mathbb{R})$. We write $K_r^+ = I_r^+ \cup J_r^+$, $r \in \mathbb{N}$, where

$$I_r^+ = \left[\frac{2^r}{2^{r+1} - 1}\,\pi, \pi\right] \quad \text{and} \quad J_r^+ = \left[2^r\pi, \frac{2^{2r+1}}{2^{r+1} - 1}\,\pi\right].$$

Recall that $I_r^- = -I_r^+$, $J_r^- = -J_r^+$, and $K_r = J_r^- \cup I_r^- \cup I_r^+ \cup J_r^+$. First, we have

LEMMA 3.1. *Under the above notation*, *for* $r, m \in \mathbb{N}$, *the following hold*:

(a) $2^{-m}I_r^+ + 2^r\pi \subset J_r^+$,
(b) $2^{-m}I_r^- - 2^r\pi \subset J_r^-$,
(c) $2^{-(m-1)}I_r^+ \cap 2^{-m}I_r^+ = \emptyset$,
(d) $2^{-m}I_r^- \cap 2^{-(m-1)}I_r^- = \emptyset$,
(e) $I_r^+ + 2^{r+m}\pi \subset 2^m J_r^+$,
(f) $I_r^- - 2^{r+m}\pi \subset 2^m J_r^-$.

Proof. This is straightforward.

From the characterization of wavelet sets stated below [4, 9], Lemma 3.3 can be easily obtained.

RESULT 3.2. *A measurable set* $W \subset \mathbb{R}$ *is a wavelet set if and only if*

(i) $\mathbb{R} = \dot{\bigcup}_{n\in\mathbb{Z}}(W + 2n\pi)$ *a.e.*,

(ii) $\mathbb{R} = \dot{\bigcup}_{n\in\mathbb{Z}}(2^n W)$ *a.e.*,

where $\dot{\bigcup}$ *denotes disjoint union.*

LEMMA 3.3. *Define* $\tau : \mathbb{R} \to [0, 2\pi)$ *by* $\tau(x) = x + 2p\pi$, *where* p *is an integer depending on* x. *Then*:

(a) $\tau(E) = \tau(E + 2k\pi)$ *for any* $k \in \mathbb{Z}$ *and* E *a measurable set in* $\mathbb{R}$,
(b) *for any disjoint measurable sets* E *and* F *in* $\mathbb{R}$ *contained in a wavelet set* W, $\tau(E) \cap \tau(F) = \emptyset$.

THEOREM 3.4. *For* $(r, m) \in \mathbb{N} \times \mathbb{N}$, *the function* $\psi_{r,m}$ *defined by*

$$\widehat{\psi}_{r,m}(\xi) = \begin{cases} 1/\sqrt{2} & \text{if } \xi \in I_r^+ \cup 2^{-m} I_r^+ \cup (2^{-m} I_r^+ + 2^r\pi) \cup I_r^- \\ & \qquad \cup\, 2^{-m} I_r^- \cup (2^{-m} I_r^- - 2^r\pi), \\ -1/\sqrt{2} & \text{if } \xi \in (I_r^+ + 2^{r+m}\pi) \cup (I_r^- - 2^{r+m}\pi), \\ 1 & \text{if } \xi \in (J_r^+ - (2^{-m} I_r^+ + 2^r\pi)) \cup (J_r^- - (2^{-m} I_r^- - 2^r\pi)), \\ 0 & \text{otherwise}, \end{cases}$$

is a non-MSF *non*-MRA *wavelet for* $L^2(\mathbb{R})$.

Proof. By Lemma 3.1, it is easily seen that the sets used to define $\widehat{\psi}_{r,m}(\xi)$ are pairwise disjoint. To illustrate, we have $2^{-m} I_r^- \cap (2^{-m} I_r^- - 2^r\pi) = \emptyset$ by Lemma 3.1(b) and (d). Now, we employ Result 2.2 to show that $\psi_{r,m}$ is a non-MSF wavelet for $L^2(\mathbb{R})$.

(i) Since

$$\begin{aligned} \|\widehat{\psi}_{r,m}\|_2^2 &= \int_{\mathbb{R}} |\widehat{\psi}_{r,m}(\xi)|^2\, d\xi \\ &= \frac{1}{2}\left(1 + \frac{1}{2^m} + \frac{1}{2^m} + 1\right)|I_r^+| + \frac{1}{2}\left(1 + \frac{1}{2^m} + \frac{1}{2^m} + 1\right)|I_r^-| \\ &\quad + |J_r^+| + |J_r^-| - \frac{1}{2^m}\,|I_r^+| - \frac{1}{2^m}\,|I_r^-| \\ &= |I_r^+| + |I_r^-| + |J_r^+| + |J_r^-| = 2\pi, \end{aligned}$$

it follows that $\|\psi_{r,m}\|_2 = 1$.

(ii) Since $\rho(2\xi) = \rho(\xi)$ for a.e. $\xi \in \mathbb{R}$, it suffices to show that $\rho(\xi) = 1$ on K_r. If $\xi \in I_r^+$, then by the definition of $\psi_{r,m}$, $2^j\xi \in \operatorname{supp} \widehat{\psi}_{r,m}$ if and only if $j = 0$ or $-m$. Hence

$$\rho(\xi) = |\widehat{\psi}_{r,m}(\xi)|^2 + |\widehat{\psi}_{r,m}(2^{-m}\xi)|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1.$$

Write

$$J_r^+ = (J_r^+ - (2^{-m} I_r^+ + 2^r\pi)) \cup (2^{-m} I_r^+ + 2^r\pi).$$

If $\xi \in J_r^+ - (2^{-m} I_r^+ + 2^r\pi)$, then $2^j\xi \in \operatorname{supp} \widehat{\psi}_{r,m}$ if and only if $j = 0$. Hence

$$\rho(\xi) = |\widehat{\psi}_{r,m}(\xi)|^2 = 1.$$

If $\xi \in 2^{-m} I_r^+ + 2^r\pi$, then $2^j\xi \in \operatorname{supp} \widehat{\psi}_{r,m}$ if and only if $j = 0$ or m. Hence

$$\rho(\xi) = |\widehat{\psi}_{r,m}(\xi)|^2 + |\widehat{\psi}_{r,m}(2^m\xi)|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 = 1.$$

If $\xi \in I_r^-$, then $2^j\xi \in \operatorname{supp} \widehat{\psi}_{r,m}$ if and only if $j = 0$ or $-m$. Hence

$$\rho(\xi) = |\widehat{\psi}_{r,m}(\xi)|^2 + |\widehat{\psi}_{r,m}(2^{-m}\xi)|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1.$$

Write

$$J_r^- = (J_r^- - (2^{-m} I_r^- - 2^r\pi)) \cup (2^{-m} I_r^- - 2^r\pi).$$

If $\xi \in J_r^- - (2^{-m} I_r^- - 2^r\pi)$, then $2^j\xi \in \operatorname{supp} \widehat{\psi}_{r,m}$ if and only if $j = 0$. Hence

$$\rho(\xi) = |\widehat{\psi}_{r,m}(\xi)|^2 = 1.$$

If $\xi \in 2^{-m} I_r^- - 2^r\pi$, then $2^j\xi \in \operatorname{supp} \widehat{\psi}_{r,m}$ if and only if $j = 0$ or m. Hence

$$\rho(\xi) = |\widehat{\psi}_{r,m}(\xi)|^2 + |\widehat{\psi}_{r,m}(2^m\xi)|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 = 1.$$

(iii) In view of $t_{-q}(\xi) = \overline{t_q(\xi - 2q\pi)}$, we will show that $t_q(\xi) = 0$ a.e., where q is a positive odd integer. The term $\widehat{\psi}_{r,m}(2^j\xi)\,\overline{\widehat{\psi}_{r,m}(2^j(\xi + 2q\pi))}$ is nonzero when both $2^j\xi$ and $2^j(\xi + 2q\pi)$ lie in the support of $\widehat{\psi}_{r,m}$. From the definition of $\psi_{r,m}$ and Lemma 3.3, we observe that this is possible if either $2^j q = 2^{r-1}$, or $2^j q = 2^{r+m-1}$. Since q is odd, either $j = r - 1$ and $q = 1$, or $j = r + m - 1$ and $q = 1$. In case $j = r - 1$ and $q = 1$, for $\xi > 0$, we have $2^j\xi \in 2^{-m} I_r^+$, so that $2^j(\xi + 2q\pi) \in 2^{-m} I_r^+ + 2^r\pi$ and hence $2^{j+m}\xi \in I_r^+$ and $2^{j+m}(\xi + 2q\pi) \in I_r^+ + 2^{r+m}\pi$. Thus

$$t_q(\xi) = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) = 0.$$

For $\xi < 0$, we have $2^j\xi \in 2^{-m} I_r^- - 2^r\pi$, so that $2^j(\xi + 2q\pi) \in 2^{-m} I_r^-$ and hence $2^{j+m}\xi \in I_r^- - 2^{r+m}\pi$, and $2^{j+m}(\xi + 2q\pi) \in I_r^-$. Thus

$$t_q(\xi) = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 0.$$

When $j = r + m - 1$ and $q = 1$, we prove that $t_q(\xi) = 0$ along similar lines.

Next, to show that $\psi_{r,m}$ is a non-MRA wavelet, we make use of Result 2.1. We will show that $D_{\psi_{r,m}} \neq 1$ on an interval of the real line.

In case $r \geq 2$, $D_{\psi_{r,m}} \geq 2$ on the interval $2^{-(m+1)} I_r^+$, where $m \in \mathbb{N}$. Indeed,

$$D_{\psi_{r,m}}(\xi) \geq |\widehat{\psi}_{r,m}(2\xi)|^2 + |\widehat{\psi}_{r,m}(2\xi + 2^r\pi)|^2 + |\widehat{\psi}_{r,m}(2^{m+1}\xi)|^2 \\ + |\widehat{\psi}_{r,m}(2^{m+1}\xi + 2^{r+m}\pi)|^2$$

and hence the assertion follows by noting that $2\xi \in 2^{-m}I_r^+$, $2(\xi + 2 \cdot 2^{r-2}\pi) \in 2^{-m}I_r^+ + 2^r\pi$, $2^{m+1}\xi \in I_r^+$ and $2^{m+1}(\xi + 2 \cdot 2^{r-2}\pi) \in I_r^+ + 2^{r+m}\pi$, where $\xi \in 2^{-(m+1)}I_r^+$.

In case $r = 1$, $D_{\psi_{1,m}} \geq 5/2$ on the interval $[5\pi/2, 8\pi/3] \subset L \equiv J_1^+ - (2^{-m}I_1^+ + 2\pi)$. Observe that

$$D_{\psi_{1,m}}(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\widehat{\psi}_{1,m}(2^j(\xi + 2k\pi))|^2 = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^-} |\widehat{\psi}_{1,m}(2^j(\xi + 2k\pi))|^2 \\ + \sum_{j=1}^{\infty} |\widehat{\psi}_{1,m}(2^j\xi)|^2 + \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^+} |\widehat{\psi}_{1,m}(2^j(\xi + 2k\pi))|^2.$$

For $k \in \mathbb{Z}^-$ and $j \geq 1$, we have

$$\sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^-} |\widehat{\psi}_{1,m}(2^j(\xi + 2k\pi))|^2 \geq |\widehat{\psi}_{1,m}(2^2(\xi - 2\pi))|^2 = \frac{5}{2}$$

for $\xi \in [5\pi/2, 8\pi/3] \subset L$, since $2^2(\xi - 2\pi) \in [2\pi, 8\pi/3] = L \cup (2^{-m}I_1^+ + 2\pi)$. If $k = 0$, then $2^j\xi \in \operatorname{supp} \widehat{\psi}_{1,m}$ iff $j = 0$, and we have

$$\sum_{j=-\infty}^{0} |\widehat{\psi}_{1,m}(2^j\xi)|^2 = 1.$$

Therefore, we obtain

$$D_{\psi_{1,m}}(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\widehat{\psi}_{1,m}(2^j(\xi + 2k\pi))|^2 = \sum_{j=1}^{\infty} |\widehat{\psi}_{1,m}(2^j\xi)|^2 \\ = 1 - \sum_{j=-\infty}^{0} |\widehat{\psi}_{1,m}(2^j\xi)|^2 = 0$$

for $\xi \in [5\pi/2, 8\pi/3] \subset L$.

If $k \in \mathbb{Z}^+$, then $2^j(\xi + 2k\pi) \notin \operatorname{supp} \widehat{\psi}_{1,m}$ for all $j \geq 1$, and we have

$$\sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^+} |\widehat{\psi}_{1,m}(2^j(\xi + 2k\pi))|^2 = 0$$

for $\xi \in [5\pi/2, 8\pi/3] \subset L$.

4. Non-MSF non-MRA wavelets for the Hardy space $H^2(\mathbb{R})$. Recall that H^2-wavelet sets with dilation 2 consisting of two intervals are given by (2) in the introduction. For $r \in \mathbb{N}$ and $k = 2^l - 1,\ 1 \le l \le r$, we have the following H^2-wavelet sets with two intervals:

$$K_r^l = \left[\frac{2^{l+1}}{2^{r+1}-1}\,\pi, \frac{2(2^l-1)}{2^r-1}\,\pi\right] \cup \left[\frac{2^{r+1}(2^l-1)}{2^r-1}\,\pi, \frac{2^{r+l+2}}{2^{r+1}-1}\,\pi\right] = I_r^l \cup J_r^l.$$

With the help of the following Lemma, we provide a class of non-MSF non-MRA H^2-wavelets which includes the one given by Behera.

LEMMA 4.1. *Under the above notation, for $r, m \in \mathbb{N}$ and an integer l satisfying $1 \le l \le r$, the following hold:*

(a) $2^{-m} I_r^l + 2^{l+1}\pi \subset J_r^l$,
(b) $2^{-m} I_r^l \cap 2^{-(m-1)} I_r^l = \emptyset$,
(c) $I_r^l + 2^{l+m+1}\pi \subset 2^m J_r^l$.

Proof. This is straightforward.

THEOREM 4.2. *For each $(r, m) \in \mathbb{N} \times \mathbb{N}$ and an integer l satisfying $1 \le l \le r$, the function $\psi_{r,m}^l$ defined by*

$$\widehat{\psi}_{r,m}^l(\xi) = \begin{cases} 1/\sqrt{2} & \text{if } \xi \in I_r^l \cup 2^{-m} I_r^l \cup (2^{-m} I_r^l + 2^{l+1}\pi), \\ -1/\sqrt{2} & \text{if } \xi \in I_r^l + 2^{l+m+1}\pi, \\ 1 & \text{if } \xi \in (J_r^l - (2^{-m} I_r^l + 2^{l+1}\pi)), \\ 0 & \text{otherwise}, \end{cases}$$

*is a non-*MSF *non-*MRA *wavelet for the Hardy space $H^2(\mathbb{R})$.*

Proof. The proof that $\psi_{r,m}^l$ is a non-MSF wavelet for $H^2(\mathbb{R})$ is similar to the proof that $\psi_{r,m}$ is a non-MSF wavelet for $L^2(\mathbb{R})$ in Theorem 3.4, employing Result 2.2.

To show that $\psi_{r,m}^l$ is a non-MRA wavelet for $H^2(\mathbb{R})$, we use Result 2.1. For $r \in \mathbb{N}$ and an integer l satisfying $1 \le l \le r$, $D_{\psi_{r,m}^l} \ge 2$ on the interval $2^{-(m+1)} I_r^l$, where $m \in \mathbb{N}$. Indeed,

$$\begin{aligned} D_{\psi_{r,m}^l}(\xi) \ge |\widehat{\psi}_{r,m}^l(2\xi)|^2 + |\widehat{\psi}_{r,m}^l(2\xi + 2^{l+1}\pi)|^2 + |\widehat{\psi}_{r,m}^l(2^{m+1}\xi)|^2 \\ + |\widehat{\psi}_{r,m}^l(2^{m+1}\xi + 2^{l+m+1}\pi)|^2 \end{aligned}$$

and hence the assertion follows by noting that $2\xi \in 2^{-m} I_r^l$, $2(\xi + 2 \cdot 2^{l-1}\pi) \in 2^{-m} I_r^l + 2^{l+1}\pi$, $2^{m+1}\xi \in I_r^l$ and $2^{m+1}(\xi + 2 \cdot 2^{l-1}\pi) \in I_r^l + 2^{l+m+1}\pi$, where $\xi \in 2^{-(m+1)} I_r^l$.

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