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PARTIAL DIFFERENTIAL EQUATIONS

A Dynamic Frictionless Contact Problem with Adhesion and Damage

by

Mohamed SELMANI and Lynda SELMANI

Presented by Bogdan BOJARSKI

Summary. We consider a dynamic frictionless contact problem for a viscoelastic material with damage. The contact is modeled with normal compliance condition. The adhesion of the contact surfaces is considered and is modeled with a surface variable, the bonding field, whose evolution is described by a first order differential equation. We establish a variational formulation for the problem and prove the existence and uniqueness of the solution. The proofs are based on the theory of evolution equations with monotone operators, a classical existence and uniqueness result for parabolic inequalities, and fixed point arguments.

1. Introduction. The adhesive contact between bodies, when a glue is added to keep the surfaces from relative motion, is receiving increasing attention in the mathematical literature. Analysis of models for adhesive contact can be found in [2, 3, 4, 6, 12, 14, 19], and recently in the monograph [20]. The novelty in all the above papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by β; it describes the pointwise fractional density of active bonds on the contact surface, and is sometimes referred to as the intensity of adhesion. Following [7, 8], the bonding field satisfies $0 \leq \beta \leq 1$; when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. We refer the reader to the extensive bibliography on the subject in [15, 17, 18].

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The subject of damage is extremely important in design engineering, since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General novel models for damage were derived in [9, 10] from the virtual power principle. Mathematical analysis of one-dimensional problems can be found in [11]. In all these papers the damage of the material is described by a damage function α, restricted to have values between zero and one. When $\alpha = 1$ there is no damage in the material, when $\alpha = 0$ the material is completely damaged, when $0 < \alpha < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [13, 16, 20].

In this paper, the inclusion describing the evolution of the damage field is

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K(\alpha) \ni \phi(\varepsilon(\mathbf{u}), \alpha),$$

where K denotes the set of admissible damage functions defined by

$$K = \{\xi \in H^1(\Omega) \mid 0 \le \xi \le 1 \text{ a.e. in } \Omega\},$$

k is a positive coefficient, $\partial\varphi_K$ represents the subdifferential of the indicator function of the set K and ϕ is a given constitutive function which describes the sources of the damage in the system. A general viscoelastic constitutive law with damage is given by

$$\boldsymbol{\sigma} = \mathcal{A}(\varepsilon(\dot{\mathbf{u}})) + \mathcal{G}(\varepsilon(\mathbf{u}), \alpha),$$

where $\mathcal{A}$ is a nonlinear viscosity function, $\mathcal{G}$ is a nonlinear elasticity function which depends on the internal state variable describing the damage of the material caused by elastic deformations, and the dot represents the time derivative, i.e., $\dot{\mathbf{u}} = \partial\mathbf{u}/\partial t$ and $\ddot{\mathbf{u}} = \partial^2\mathbf{u}/\partial t^2$. The essence of this paper is to couple a viscoelastic problem with damage and a frictionless contact problem with adhesion. We study a dynamic problem of frictionless adhesive contact. We model the material behavior with a viscoelastic constitutive law with damage and the contact with normal compliance with adhesion. We derive a variational formulation and prove the existence and uniqueness of a weak solution.

The paper is organized as follows. In Section 2 we introduce the notation and give some preliminaries. In Section 3 we present the mechanical problem, list the assumptions on the data, give the variational formulation of the problem and state our main existence and uniqueness result, Theorem 3.1. In Section 4 we give the proof of Theorem 3.1 based on the theory of evolution equations with monotone operators, a fixed point argument and a classical existence and uniqueness result for parabolic inequalities.

2. Notation and preliminaries. In this short section, we present the notation we shall use and some preliminary material. For more details, we refer the reader to [5].

We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while $(\cdot)$ and $|\cdot|$ represent the inner product and Euclidean norm on S_d and $\mathbb{R}^d$, respectively. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. We shall use the notation

$$\begin{aligned}
H &= L^2(\Omega)^d = \{\mathbf{u} = (u_i) \mid u_i \in L^2(\Omega)\},\\
\mathcal{H} &= \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},\\
H_1 &= \{\mathbf{u} = (u_i) \in H \mid \varepsilon(\mathbf{u}) \in \mathcal{H}\},\\
\mathcal{H}_1 &= \{\boldsymbol{\sigma} \in \mathcal{H} \mid \operatorname{Div} \boldsymbol{\sigma} \in H\},
\end{aligned}$$

where $\varepsilon : H_1 \to \mathcal{H}$ and $\operatorname{Div} : \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

Here and below, the indices i and j run from 1 to d, the summation convention over repeated indices is assumed, and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable.

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{aligned}
(\mathbf{u}, \mathbf{v})_H &= \int_\Omega u_i v_i \, dx && \forall \mathbf{u}, \mathbf{v} \in H,\\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} &= \int_\Omega \sigma_{ij} \tau_{ij} \, dx && \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H},\\
(\mathbf{u}, \mathbf{v})_{H_1} &= (\mathbf{u}, \mathbf{v})_H + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} && \forall \mathbf{u}, \mathbf{v} \in H_1,\\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \operatorname{Div} \boldsymbol{\tau})_H && \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}_1.
\end{aligned}$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$.

Let $H_\Gamma = H^{1/2}(\Gamma)^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $\mathbf{v} \in H_1$, we also write $\mathbf{v}$ for the trace $\gamma \mathbf{v}$ of $\mathbf{v}$ on Γ, and we denote by v_ν and $\mathbf{v}_\tau$ the normal and tangential components of $\mathbf{v}$ on Γ given by

$$v_\nu = \mathbf{v}.\boldsymbol{\nu}, \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}. \tag{2.1}$$

Similarly, for a regular (say C^1) tensor field $\boldsymbol{\sigma} : \Omega \to S_d$ we define its normal and tangential components by

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}).\boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}. \tag{2.2}$$

We recall that the following Green's formula holds:

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}.\mathbf{v}\, da \quad \forall \mathbf{v} \in H_1. \tag{2.3}$$

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \le p \le \infty$ and $k \ge 1$. We denote by $C(0,T;X)$ and $C^1(0,T;X)$ the spaces of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$|\mathbf{f}|_{C(0,T;X)} = \max_{t\in[0,T]} |\mathbf{f}(t)|_X,$$
$$|\mathbf{f}|_{C^1(0,T;X)} = \max_{t\in[0,T]} |\mathbf{f}(t)|_X + \max_{t\in[0,T]} |\dot{\mathbf{f}}(t)|_X.$$

Moreover, for a real number r, we use r_+ to represent its positive part, that is, $r_+ = \max\{0, r\}$. Finally, for the convenience of the reader, we recall the following version of the classical theorem of Cauchy–Lipschitz (see, e.g., [21, p. 60]).

THEOREM 2.1. *Assume that* $(X, |\cdot|_X)$ *is a real Banach space and* $T > 0$. *Let* $F(t,\cdot) : X \to X$ *be an operator defined a.e. on* $(0,T)$ *satisfying the following conditions*:

- *There exists* $L_F > 0$ *such that*
$$|F(t,x) - F(t,y)|_X \le L_F |x-y|_X \quad \forall x, y \in X,$$
a.e. $t \in (0,T)$.
- *There exists* $p \ge 1$ *such that* $t \mapsto F(t,x) \in L^p(0,T;X)$ *for all* $x \in X$.

Then for any $x_0 \in X$, *there exists a unique function* $x \in W^{1,p}(0,T;X)$ *such that*

$$\dot{x}(t) = F(t, x(t)), \quad a.e.\ t \in (0,T),$$
$$x(0) = x_0.$$

Theorem 2.1 will be used in Section 4 to prove the unique solvability of the intermediate problem involving the bonding field.

Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(\cdot,\cdot)_{X_1\times X_2}$.

3. Problem statement. We consider a viscoelastic body which occupies the domain $\Omega \subset \mathbb{R}^d$ with the boundary Γ divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $\operatorname{meas}(\Gamma_1) > 0$. The time interval of interest is $[0,T]$ where $T > 0$. The body is clamped on Γ_1 and so the displacement field vanishes there. A volume force of density $\mathbf{f}_0$ acts in $\Omega \times (0,T)$ and surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (0,T)$. We assume that the body is in adhesive frictionless contact with an obstacle, the so called foundation, over the potential contact surface Γ_3. Moreover, the process is dynamic, and thus the inertial terms are included in the equation of motion. We use a viscoelastic constitutive law with damage to model the material's behavior and an ordinary differential equation to describe the evolution of the bonding field. The mechanical formulation of the frictionless problem with normal compliance is as follows.

PROBLEM P. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to S_d$, *a damage field* $\alpha : \Omega \times [0,T] \to \mathbb{R}$ *and a bonding field* $\beta : \Gamma_3 \times [0,T] \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{G}(\varepsilon(\mathbf{u}), \alpha) \quad \text{in } \Omega \times (0,T), \tag{3.1}$$

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K(\alpha) \ni \phi(\varepsilon(\mathbf{u}), \alpha), \tag{3.2}$$

$$\varrho\ddot{\mathbf{u}} = \operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_0 \quad \text{in } \Omega \times (0,T), \tag{3.3}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{3.4}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{3.5}$$

$$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu\beta^2 R_\nu(u_\nu) \quad \text{on } \Gamma_3 \times (0,T), \tag{3.6}$$

$$-\boldsymbol{\sigma}_\tau = p_\tau(\beta)\mathbf{R}_\tau(\mathbf{u}_\tau) \quad \text{on } \Gamma_3 \times (0,T), \tag{3.7}$$

$$\dot{\beta} = -(\beta(\gamma_\nu(R_\nu(u_\nu))^2 + \gamma_\tau|\mathbf{R}_\tau(\mathbf{u}_\tau)|^2) - \varepsilon_a)_+ \quad \text{on } \Gamma_3 \times (0,T), \tag{3.8}$$

$$\frac{\partial\alpha}{\partial\nu} = 0 \quad \text{on } \Gamma \times (0,T), \tag{3.9}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad \alpha(0) = \alpha_0 \quad \text{in } \Omega, \tag{3.10}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{3.11}$$

The relation (3.1) represents the nonlinear viscoelastic constitutive law with damage; the evolution of the damage field is governed by the inclusion (3.2), where ϕ is the mechanical source of the damage growth, assumed to be a rather general function of the strains and damage itself, and $\partial\varphi_K$ is the subdifferential of the indicator function of the admissible damage functions set K. (3.3) represents the equation of motion where ϱ denotes the material mass density; (3.4) and (3.5) are the displacement and traction boundary conditions, respectively. Condition (3.6) represents the normal compliance conditions with adhesion where γ_ν is a given adhesion coefficient and p_ν is a given positive function which will be described below. In this condition the interpenetrability between the body and the foundation is allowed, that is, u_ν can be positive on Γ_3. The contribution of the adhesive traction to the normal traction is represented by the term $\gamma_\nu\beta^2 R_\nu(u_\nu)$, the adhesive traction is tensile and is proportional, with coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but only as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu L$. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The contact condition (3.6) was used in various papers (see e.g. [2, 3, 18, 20]). Condition (3.7) represents the adhesive contact condition on the tangential plane, in which p_τ is a given function and $\mathbf{R}_\tau$ is the truncation operator given by

$$\mathbf{R}_\tau(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{if } |\mathbf{v}| \le L, \\ L\mathbf{v}/|\mathbf{v}| & \text{if } |\mathbf{v}| > L. \end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but only as long as it does not exceed the bond length L. The frictional tangential traction is assumed to be much smaller than the adhesive one and is therefore omitted. The introduction of the operator R_ν, together with the operator $\mathbf{R}_\tau$ defined above, is motivated by mathematical arguments but it is not restrictive from the physical point of view, since no restriction on the size of the parameter L is made in what follows.

Next, (3.8) is an ordinary differential equation which describes the evolution of the bonding field and it was already used in [2]; see also [17, 18] for more details. Here, besides γ_ν, two new adhesion coefficients are involved, γ_τ and ε_a. Notice that in this model once debonding occurs, bonding cannot be reestablished, since (3.8) implies $\dot{\beta} \le 0$. (3.9) is a homogeneous Neumann boundary condition where $\partial\alpha/\partial\nu$ represents the normal derivative of α. In (3.10), we consider the initial conditions where $\mathbf{u}_0$ is the initial displacement, $\mathbf{v}_0$ the initial velocity and α_0 the initial damage. Finally, (3.11) is the initial condition, in which β_0 denotes the initial bonding.

To obtain the variational formulation of the problem (3.1)–(3.11), we introduce for the bonding field the set

$$Z = \{\theta : [0,T] \to L^2(\Gamma_3) \mid 0 \le \theta(t) \le 1 \ \forall t \in [0,T], \text{ a.e. on } \Gamma_3\},$$

and for the displacement field we need the closed subspace of H_1 defined by

$$V = \{\mathbf{v} \in H_1 \mid \mathbf{v} = 0 \text{ on } \Gamma_1\}.$$

Since $\operatorname{meas}(\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_k > 0$ which depends only on Ω and Γ_1 such that

$$|\varepsilon(\mathbf{v})|_{\mathcal{H}} \ge C_k |\mathbf{v}|_{H_1} \quad \forall \mathbf{v} \in V.$$

On V we consider the inner product and the associated norm given by

$$(\mathbf{u}, \mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad |\mathbf{v}|_V = |\varepsilon(\mathbf{v})|_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in V.$$

It follows from Korn's inequality that $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on

V and therefore $(V, |\cdot|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem there exists a constant C_0, depending only on Ω, Γ_1 and Γ_3, such that

$$|\mathbf{v}|_{L^2(\Gamma_3)^d} \leq C_0 |\mathbf{v}|_V \quad \forall \mathbf{v} \in V. \tag{3.12}$$

In the study of the mechanical problem (3.1)–(3.11), we make the following assumptions. The viscosity operator $\mathcal{A} : \Omega \times S_d \to S_d$ satisfies

(3.13) (a) There exists a constant $L_\mathcal{A} > 0$ such that

$$|\mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_2)| \leq L_\mathcal{A} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| \quad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in S_d, \text{ a.e. } \mathbf{x} \in \Omega.$$

(b) There exists $m_\mathcal{A} > 0$ such that

$$(\mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_2)).(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \geq m_\mathcal{A} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2 \quad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in S_d, \text{ a.e. } \mathbf{x} \in \Omega.$$

(c) The mapping $\mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\xi})$ is Lebesgue measurable on Ω for any $\boldsymbol{\xi} \in S_d$.

(d) The mapping $\mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \mathbf{0})$ is in $\mathcal{H}$.

The elasticity operator $\mathcal{G} : \Omega \times S_d \times \mathbb{R} \to S_d$ satisfies

(3.14) (a) There exists a constant $L_\mathcal{G} > 0$ such that

$$|\mathcal{G}(\mathbf{x}, \boldsymbol{\xi}_1, \alpha_1) - \mathcal{G}(\mathbf{x}, \boldsymbol{\xi}_2, \alpha_2)| \leq L_\mathcal{G}(|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + |\alpha_1 - \alpha_2|) \quad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in S_d, \forall \alpha_1, \alpha_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Omega.$$

(b) For any $\boldsymbol{\xi} \in S_d$ and $\alpha \in \mathbb{R}, \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \boldsymbol{\xi}, \alpha)$ is Lebesgue measurable on Ω.

(c) The mapping $\mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \mathbf{0}, 0)$ is in $\mathcal{H}$.

The damage source function $\phi : \Omega \times S_d \times \mathbb{R} \to \mathbb{R}$ satisfies

(3.15) (a) There exists a constant $L_\phi > 0$ such that

$$|\phi(\mathbf{x}, \boldsymbol{\xi}_1, \alpha_1) - \phi(\mathbf{x}, \boldsymbol{\xi}_2, \alpha_2)| \leq L_\phi(|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + |\alpha_1 - \alpha_2|) \quad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in S_d, \forall \alpha_1, \alpha_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Omega.$$

(b) For any $\boldsymbol{\xi} \in S_d$ and $\alpha \in \mathbb{R}, \mathbf{x} \mapsto \phi(\mathbf{x}, \boldsymbol{\xi}, \alpha)$ is Lebesgue measurable on Ω.

(c) The mapping $\mathbf{x} \mapsto \phi(\mathbf{x}, \mathbf{0}, 0) \in L^2(\Omega)$.

The normal compliance function $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

(3.16) (a) There exists a constant $L_\nu > 0$ such that

$$|p_\nu(\mathbf{x}, r_1) - p_\nu(\mathbf{x}, r_2)| \leq L_\nu |r_1 - r_2| \ \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3.$$

(b) The mapping $\mathbf{x} \mapsto p_\nu(\mathbf{x}, r)$ is measurable on Γ_3, for any $r \in \mathbb{R}$.

(c) $p_\nu(\mathbf{x}, r) = 0$ for all $r \leq 0$, a.e. $\mathbf{x} \in \Gamma_3$.

The tangential contact function $p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

(3.17) (a) There exists a constant $L_\tau > 0$ such that

$$|p_\tau(\mathbf{x}, d_1) - p_\tau(\mathbf{x}, d_2)| \leq L_\tau |d_1 - d_2| \ \forall d_1, d_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3.$$

(b) There exists $M_\tau > 0$ such that $|p_\tau(\mathbf{x}, d)| \leq M_\tau$ for all $d \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_3$.

(c) The mapping $\mathbf{x} \mapsto p_\tau(\mathbf{x}, d)$ is measurable on Γ_3, for any $d \in \mathbb{R}$.

(d) The mapping $\mathbf{x} \mapsto p_\tau(\mathbf{x}, 0) \in L^2(\Gamma_3)$.

We suppose that the mass density satisfies

$$\varrho \in L^\infty(\Omega), \text{ there exists } \varrho^* > 0 \text{ such that } \varrho(\mathbf{x}) \geq \varrho^*, \text{ a.e. } \mathbf{x} \in \Omega. \tag{3.18}$$

The adhesion coefficient and the limit bound satisfy

$$\gamma_\nu, \gamma_\tau, \varepsilon_a \in L^\infty(\Gamma_3), \qquad \gamma_\nu \geq 0, \qquad \gamma_\tau \geq 0, \qquad \varepsilon_a \geq 0. \tag{3.19}$$

We also suppose that the body forces and surface traction have the regularity

$$\mathbf{f}_0 \in L^2(0, T; H), \qquad \mathbf{f}_2 \in L^2(0, T; L^2(\Gamma_2)^d). \tag{3.20}$$

Finally, we assume that the initial data satisfy the following conditions:

$$\mathbf{u}_0 \in V, \qquad \mathbf{v}_0 \in L^2(\Omega)^d, \tag{3.21}$$

$$\alpha_0 \in K, \tag{3.22}$$

$$\beta_0 \in L^2(\Gamma_3), \qquad 0 \leq \beta_0 \leq 1, \qquad \text{a.e. on } \Gamma_3. \tag{3.23}$$

We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(\xi, \varphi) = k \int_\Omega \nabla \xi . \nabla \varphi \, dx. \tag{3.24}$$

We will use a modified inner product on $H = L^2(\Omega)^d$, given by

$$((\mathbf{u}, \mathbf{v}))_H = (\varrho \mathbf{u}, \mathbf{v})_H \qquad \forall \mathbf{u}, \mathbf{v} \in H,$$

that is, weighted with ϱ, and we let $\|\cdot\|_H$ be the associated norm, i.e.,

$$\|\mathbf{v}\|_H = (\varrho \mathbf{v}, \mathbf{v})_H^{1/2} \qquad \forall \mathbf{v} \in H.$$

It follows from assumptions (3.18) that $\|\cdot\|_H$ and $|\cdot|_H$ are equivalent norms on H, and also the inclusion mapping of $(V, |\cdot|_V)$ into $(H, \|\cdot\|_H)$ is continuous and dense. We denote by V' the dual of V. Identifying H with its dual, we can write the Gelfand triple

$$V \subset H \subset V'.$$

We use the notation $(\cdot, \cdot)_{V' \times V}$ to represent the duality pairing between V' and V; we have

$$(\mathbf{u}, \mathbf{v})_{V' \times V} = ((\mathbf{u}, \mathbf{v}))_H \qquad \forall \mathbf{u} \in H, \forall \mathbf{v} \in V.$$

Finally, we denote by $|\cdot|_{V'}$ the norm on V'. Assumptions (3.20) allow us, for a.e. $t \in (0,T)$, to define $\mathbf{f}(t) \in V'$ by

$$(3.25)\qquad (\mathbf{f}(t),\mathbf{v})_{V'\times V} = \int_{\Omega} \mathbf{f}_0(t).\mathbf{v}\,dx + \int_{\Gamma_2} \mathbf{f}_2(t).\mathbf{v}\,da \quad \forall \mathbf{v} \in V,$$

and

$$(3.26)\qquad \mathbf{f} \in L^2(0,T;V').$$

The adhesion functional $j : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ is defined by

$$(3.27)\qquad j(\beta,\mathbf{u},\mathbf{v}) = \int_{\Gamma_3} p_\nu(u_\nu)v_\nu\,da + \int_{\Gamma_3} (-\gamma_\nu\beta^2 R_\nu(u_\nu)v_\nu + p_\tau(\beta)\mathbf{R}_\tau(\mathbf{u}_\tau).\mathbf{v}_\tau)\,da.$$

Keeping in mind (3.16) and (3.17), we observe that the integrals in (3.27) are well defined. Using standard arguments based on Green's formula (2.3) we can derive the following variational formulation of the frictionless problem with normal compliance (3.1)–(3.11):

Problem PV. *Find a displacement field* $\mathbf{u} : [0,T] \to V$, *a stress field* $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$, *a damage field* $\alpha : [0,T] \to H^1(\Omega)$ *and a bonding field* $\beta : [0,T] \to L^\infty(\Gamma_3)$ *such that*

$$(3.28)\qquad \boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{G}(\varepsilon(\mathbf{u}(t)),\alpha(t)), \quad a.e.\ t \in (0,T),$$

$$(3.29)\qquad \alpha(t) \in K \text{ for all } t \in [0,T],$$
$$(\dot\alpha(t),\xi-\alpha(t))_{L^2(\Omega)} + a(\alpha(t),\xi-\alpha(t)) \geq (\phi(\varepsilon(\mathbf{u}(t)),\alpha(t)),\xi-\alpha(t))_{L^2(\Omega)} \quad \forall \xi \in K,$$

$$(3.30)\qquad (\ddot{\mathbf{u}}(t),\mathbf{v})_{V'\times V} + (\boldsymbol{\sigma}(t),\varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\beta(t),\mathbf{u}(t),\mathbf{v}) = (\mathbf{f}(t),\mathbf{v})_{V'\times V} \quad \forall \mathbf{v} \in V,\ \forall t \in [0,T],$$

$$(3.31)\qquad \dot\beta(t) = -(\beta(t)(\gamma_\nu(R_\nu(u_\nu(t)))^2 + \gamma_\tau|\mathbf{R}_\tau(\mathbf{u}_\tau(t))|^2) - \varepsilon_a)_+ \quad a.e.\ t \in (0,T),$$

$$(3.32)\qquad \mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad \alpha(0) = \alpha_0, \quad \beta(0) = \beta_0.$$

We notice that the variational problem PV is formulated in terms of the displacement, stress field, damage field and bonding field. The existence of a unique solution of problem PV is stated and proved in the next section. To this end, we consider the following remark which is used in different places of the paper.

Remark 3.1. We note that, in problem P and in problem PV, we do not need to impose explicitly the restriction $0 \leq \beta \leq 1$. Indeed, equations (3.31) guarantee that $\beta(\mathbf{x},t) \leq \beta_0(\mathbf{x})$, and therefore assumption (3.23) shows that $\beta(\mathbf{x},t) \leq 1$ for $t \geq 0$, a.e. $\mathbf{x} \in \Gamma_3$. On the other hand, if $\beta(\mathbf{x},t_0) = 0$ at time t_0, then it follows from (3.31) that $\dot\beta(\mathbf{x},t) = 0$ for all $t \geq t_0$, and therefore $\beta(\mathbf{x},t) = 0$ for all $t \geq t_0$, a.e. $\mathbf{x} \in \Gamma_3$. We conclude that $0 \leq \beta(\mathbf{x},t) \leq 1$ for all $t \in [0,T]$, a.e. $\mathbf{x} \in \Gamma_3$.

The main result in this section is the following existence and uniqueness result.

THEOREM 3.1. *Assume that* (3.13)–(3.23) *hold. Then problem* PV *has a unique solution* $(\mathbf{u},\boldsymbol{\sigma},\alpha,\beta)$ *which satisfies*

$$\mathbf{u} \in H^1(0,T;V) \cap C^1(0,T;H), \qquad \ddot{\mathbf{u}} \in L^2(0,T;V'), \tag{3.33}$$

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \operatorname{Div}\boldsymbol{\sigma} \in L^2(0,T;V'), \tag{3.34}$$

$$\alpha \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{3.35}$$

$$\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap Z. \tag{3.36}$$

A quadruplet $(\mathbf{u},\boldsymbol{\sigma},\alpha,\beta)$ which satisfies (3.28)–(3.32) is called a *weak solution* to the compliance contact problem P. We conclude that under the stated assumptions, problem (3.1)–(3.11) has a unique weak solution satisfying (3.33)–(3.36). The proof of Theorem 3.1 will be carried out in several steps and is based on the theory of evolution equations with monotone operators, a fixed point argument and a classical existence and uniqueness result for parabolic inequalities. To this end, we assume in the following that (3.13)–(3.23) hold. Below, C denotes a generic positive constant which may depend on Ω, Γ_1, Γ_2, $\Gamma_3, \mathcal{A}$, $\mathcal{G}, \phi, p_\nu$, $p_\tau, \gamma_\nu, \gamma_\tau, L$ and T but does not depend on t nor on the rest of the input data, and whose value may change from place to place. Moreover, for the sake of simplicity, we suppress, in what follows, the explicit dependence of various functions on $\mathbf{x} \in \Omega \cup \Gamma$. The proof of Theorem 3.1 will be provided in the next section.

4. Existence and uniqueness result. Let $\boldsymbol{\eta} \in L^2(0,T;V')$ be given. In the first step we consider the following variational problem.

PROBLEM PV_η. *Find a displacement field* $\mathbf{u}_\eta : [0,T] \to V$ *such that*

$$(\ddot{\mathbf{u}}_\eta(t),\mathbf{v})_{V'\times V} + (\mathcal{A}\varepsilon(\dot{\mathbf{u}}_\eta(t)),\varepsilon(\mathbf{v}))_{\mathcal{H}} + (\boldsymbol{\eta}(t),\mathbf{v})_{V'\times V} = (\mathbf{f}(t),\mathbf{v})_{V'\times V} \quad \forall \mathbf{v} \in V, \ \textit{a.e. } t \in (0,T), \tag{4.1}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0, \qquad \dot{\mathbf{u}}_\eta(0) = \mathbf{v}_0. \tag{4.2}$$

To solve problem PV_η, we apply an abstract existence and uniqueness result which we now recall for the convenience of the reader. Let V and H denote real Hilbert spaces such that V is dense in H and the inclusion map is continuous, H is identified with its dual and with a subspace of V', i.e., $V \subset H \subset V'$; we say that these inclusions define a *Gelfand triple*. The notations $|\cdot|_V, |\cdot|_{V'}$ and $(\cdot,\cdot)_{V'\times V}$ represent the norms on V and on V' and the duality pairing between them, respectively. The following abstract result may be found in [20, p. 48].

THEOREM 4.1. *Let V, H be as above, and let $A : V \to V'$ be a hemicontinuous and monotone operator which satisfies*

$$(Av, v)_{V' \times V} \geq \omega |v|_V^2 + \lambda \quad \forall v \in V, \tag{4.3}$$

$$|Av|_{V'} \leq C(|v|_V + 1) \quad \forall v \in V, \tag{4.4}$$

for some constants $\omega > 0$, $C > 0$ and $\lambda \in \mathbb{R}$. Then, given $\mathbf{u}_0 \in H$ and $\mathbf{f} \in L^2(0, T; V')$, there exists a unique function $\mathbf{u}$ which satisfies

$$\mathbf{u} \in L^2(0, T; V') \cap C(0, T; H), \quad \dot{\mathbf{u}} \in L^2(0, T; V'),$$
$$\dot{\mathbf{u}}(t) + A\mathbf{u}(t) = \mathbf{f}(t), \quad a.e.\ t \in (0, T), \quad \mathbf{u}(0) = \mathbf{u}_0.$$

We apply it to problem PV_η.

LEMMA 4.2. *There exists a unique solution to problem PV_η and it has the regularity expressed in* (3.33).

Proof. We define the operator $A : V \to V'$ by

$$(A\mathbf{u}, \mathbf{v})_{V' \times V} = (\mathcal{A}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in V. \tag{4.5}$$

It follows from (4.5) and (3.13)(a) that

$$|A\mathbf{u} - A\mathbf{v}|_{V'} \leq L_{\mathcal{A}} |\mathbf{u} - \mathbf{v}|_V \quad \forall \mathbf{u}, \mathbf{v} \in V, \tag{4.6}$$

which shows that $A : V \to V'$ is continuous, and so hemicontinuous. Now, by (4.5) and (3.13)(b), we find

$$(A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v})_{V' \times V} \geq m_{\mathcal{A}} |\mathbf{u} - \mathbf{v}|_V^2 \quad \forall \mathbf{u}, \mathbf{v} \in V, \tag{4.7}$$

i.e., $A : V \to V'$ is a monotone operator. Choosing $\mathbf{v} = \mathbf{0}_V$ in (4.7) we obtain

$$(A\mathbf{u}, \mathbf{u})_{V' \times V} \geq m_{\mathcal{A}} |\mathbf{u}|_V^2 - |A\mathbf{0}_V|_{V'} |\mathbf{u}|_V$$
$$\geq \frac{1}{2} m_{\mathcal{A}} |\mathbf{u}|_V^2 - \frac{1}{2m_{\mathcal{A}}} |A\mathbf{0}_V|_{V'}^2 \quad \forall \mathbf{u} \in V.$$

Thus, A satisfies condition (4.3) with $\omega = m_{\mathcal{A}}/2$ and $\lambda = -|A\mathbf{0}_V|_{V'}^2 / 2m_{\mathcal{A}}$. Next, by (4.6) we deduce that

$$|A\mathbf{u}|_{V'} \leq L_{\mathcal{A}} |\mathbf{u}|_V + |A\mathbf{0}_V|_{V'} \quad \forall \mathbf{u} \in V.$$

This inequality implies that A satisfies condition (4.4). Finally, we recall that by (3.26) and (3.21) we have $\mathbf{f} - \boldsymbol{\eta} \in L^2(0, T; V')$ and $\mathbf{v}_0 \in H$.

It now follows from Theorem 4.1 that there exists a unique function $\mathbf{v}_\eta$ which satisfies

$$\mathbf{v}_\eta \in L^2(0, T; V) \cap C(0, T; H), \quad \dot{\mathbf{v}}_\eta \in L^2(0, T; V'), \tag{4.8}$$

$$\dot{\mathbf{v}}_\eta(t) + A\mathbf{v}_\eta(t) + \boldsymbol{\eta}(t) = \mathbf{f}(t), \quad \text{a.e. } t \in (0, T), \tag{4.9}$$

$$\mathbf{v}_\eta(0) = \mathbf{v}_0. \tag{4.10}$$

Let $\mathbf{u}_\eta : [0, T] \to V$ be defined by

$$\mathbf{u}_\eta(t) = \int_0^t \mathbf{v}_\eta(s)\,ds + \mathbf{u}_0 \quad \forall t \in [0,T]. \tag{4.11}$$

It follows from (4.5) and (4.8)–(4.11) that $\mathbf{u}_\eta$ is a solution of the variational problem PV_η and it has the regularity expressed in (3.33). This concludes the proof of the existence part of Lemma 4.2. The uniqueness of the solution follows from the uniqueness of the solution to problem (4.9)–(4.10), guaranteed by Theorem 4.1. ∎

In the second step, we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.2 and consider the following initial-value problem.

PROBLEM PV_β. *Find the adhesion field $\beta_\eta : [0,T] \to L^2(\Gamma_3)$ such that*

$$\dot\beta_\eta(t) = -(\beta_\eta(t)(\gamma_\nu(R_\nu(u_{\eta\nu}(t)))^2 + \gamma_\tau|\mathbf{R}_\tau(\mathbf{u}_{\eta\tau}(t))|^2) - \varepsilon_a)_+, \quad a.e.\ t \in (0,T), \tag{4.12}$$

$$\beta_\eta(0) = \beta_0. \tag{4.13}$$

We have the following result.

LEMMA 4.3. *There exists a unique solution $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap Z$ to problem PV_β.*

Proof. For simplicity we suppress the dependence of various functions on Γ_3, and note that the equalities and inequalities below are valid a.e. on Γ_3. Define $F_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ by

$$F_\eta(t,\beta) = -(\beta(\gamma_\nu(R_\nu(u_{\eta\nu}(t)))^2 + \gamma_\tau|\mathbf{R}_\tau(\mathbf{u}_{\eta\tau}(t))|^2) - \varepsilon_a)_+ \tag{4.14}$$

for all $t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and $\mathbf{R}_\tau$ that F_η is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\beta \in L^2(\Gamma_3)$, the mapping $t \mapsto F_\eta(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using a version of the Cauchy–Lipschitz theorem given in Theorem 2.1 we deduce that there exists a unique function $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ which is a solution to problem PV_β. Also, the arguments used in Remark 3.1 show that $0 \le \beta_\eta(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, $\beta_\eta \in Z$ by the definition of Z, which concludes the proof. ∎

In the third step, we let $\theta \in L^2(0,T;L^2(\Omega))$ be given and consider the following variational problem for the damage field.

PROBLEM PV_θ. *Find a damage field $\alpha_\theta : [0,T] \to H^1(\Omega)$ such that*

$$\begin{aligned} &\alpha_\theta(t) \in K, \\ &(\dot\alpha_\theta(t), \xi - \alpha_\theta(t))_{L^2(\Omega)} + a(\alpha_\theta(t), \xi - \alpha_\theta(t)) \\ &\qquad \ge (\theta(t), \xi - \alpha_\theta(t))_{L^2(\Omega)} \quad \forall \xi \in K,\ a.e.\ t \in (0,T), \end{aligned} \tag{4.15}$$

$$\alpha_\theta(0) = \alpha_0. \tag{4.16}$$

To solve PV_θ, we recall the following standard result for parabolic variational inequalities (see, e.g., [1, p. 124]).

THEOREM 4.4. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed and convex set of V. Assume that $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\zeta > 0$ and c_0,*

$$a(v,v) + c_0|v|_H^2 \geq \zeta|v|_V^2 \quad \forall v \in V.$$

Then, for every $u_0 \in K$ and $f \in L^2(0,T;H)$, there exists a unique function $u \in H^1(0,T;H) \cap L^2(0,T;V)$ such that $u(0) = u_0$, $u(t) \in K$ for all $t \in [0,T]$, and for almost all $t \in (0,T)$,

$$(\dot{u}(t), v - u(t))_{V' \times V} + a(u(t), v - u(t)) \geq (f(t), v - u(t))_H \quad \forall v \in K.$$

We apply this theorem to problem PV_θ.

LEMMA 4.5. *Problem PV_θ has a unique solution α_θ such that*

$$\alpha_\theta \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{4.17}$$

Proof. The inclusion of $(H^1(\Omega), |\cdot|_{H^1(\Omega)})$ into $(L^2(\Omega), |\cdot|_{L^2(\Omega)})$ is continuous and its range is dense. We denote by $(H^1(\Omega))'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $L^2(\Omega)$ with itself, we can write the Gelfand triple

$$H^1(\Omega) \subset L^2(\Omega) \subset (H^1(\Omega))'.$$

We use the notation $(\cdot,\cdot)_{(H^1(\Omega))' \times H^1(\Omega)}$ for the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$. We have

$$(\alpha, \xi)_{(H^1(\Omega))' \times H^1(\Omega)} = (\alpha, \xi)_{L^2(\Omega)} \quad \forall \alpha \in L^2(\Omega), \xi \in H^1(\Omega),$$

and we note that K is a closed convex set in $H^1(\Omega)$. Then, using the definition (3.24) of the bilinear form a, and the fact that $\alpha_0 \in K$ in (3.22), it is easy to see that Lemma 4.5 is a straightforward consequence of Theorem 4.4. ∎

Finally, as a consequence of these results and using the properties of the operator $\mathcal{G}$, the functional j, and the function ϕ, for $t \in [0,T]$, we consider the element

$$\Lambda(\boldsymbol{\eta},\theta)(t) = (\Lambda^1(\boldsymbol{\eta},\theta)(t), \Lambda^2(\boldsymbol{\eta},\theta)(t)) \in V' \times L^2(\Omega), \tag{4.18}$$

defined by the equalities

$$(\Lambda^1(\boldsymbol{\eta},\theta)(t), \mathbf{v})_{V' \times V} = (\mathcal{G}(\varepsilon(\mathbf{u}_\eta(t)), \alpha_\theta(t)), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\beta_\eta(t), \mathbf{u}_\eta(t), \mathbf{v}) \quad \forall \mathbf{v} \in V, \tag{4.19}$$

$$\Lambda^2(\boldsymbol{\eta},\theta)(t) = \phi(\varepsilon(\mathbf{u}_\eta(t)), \alpha_\theta(t)). \tag{4.20}$$

We have the following result.

Lemma 4.6. *For* $(\boldsymbol{\eta},\theta) \in L^2(0,T;V' \times L^2(\Omega))$, *the function* $\Lambda(\boldsymbol{\eta},\theta) : [0,T] \to V' \times L^2(\Omega)$ *is continuous, and there is a unique element* $(\boldsymbol{\eta}^*,\theta^*) \in L^2(0,T;V' \times L^2(\Omega))$ *such that* $\Lambda(\boldsymbol{\eta}^*,\theta^*) = (\boldsymbol{\eta}^*,\theta^*)$.

Proof. Let $(\boldsymbol{\eta},\theta) \in L^2(0,T;V' \times L^2(\Omega))$ and $t_1,t_2 \in [0,T]$. Using (3.14), (3.16), (3.17), the definition of R_ν, $\mathbf{R}_\tau$ and Remark 3.1, we have

$$\begin{aligned}(4.21)\quad & |\Lambda^1(\boldsymbol{\eta},\theta)(t_1) - \Lambda^1(\boldsymbol{\eta},\theta)(t_2)|_{V'} \\ & \le |\mathcal{G}(\varepsilon(\mathbf{u}_\eta(t_1)),\alpha_\theta(t_1)) - \mathcal{G}(\varepsilon(\mathbf{u}_\eta(t_2)),\alpha_\theta(t_2))|_{\mathcal{H}} \\ & \quad + C|p_\nu(u_{\eta\nu}(t_1)) - p_\nu(u_{\eta\nu}(t_2))|_{L^2(\Gamma_3)} \\ & \quad + C|\beta_\eta^2(t_1)R_\nu(u_{\eta\nu}(t_1)) - \beta_\eta^2(t_2)R_\nu(u_{\eta\nu}(t_2))|_{L^2(\Gamma_3)} \\ & \quad + C|p_\tau(\beta_\eta(t_1))\mathbf{R}_\tau(\mathbf{u}_{\eta\tau}(t_1)) - p_\tau(\beta_\eta(t_2))\mathbf{R}_\tau(\mathbf{u}_{\eta\tau}(t_2))|_{L^2(\Gamma_3)} \\ & \le C(|\mathbf{u}_\eta(t_1) - \mathbf{u}_\eta(t_2)|_V + |\alpha_\theta(t_1) - \alpha_\theta(t_2)|_{L^2(\Omega)} + |\beta_\eta(t_1) - \beta_\eta(t_2)|_{L^2(\Gamma_3)}).\end{aligned}$$

Recall that above $u_{\eta\nu}$ and $\mathbf{u}_{\eta_\tau}$ denote the normal and tangential components of the function $\mathbf{u}_\eta$ respectively. Next, due to the regularities of $\mathbf{u}_\eta$, α_θ and β_η expressed in (3.33), (3.35) and (3.36), respectively, we deduce from (4.21) that $\Lambda^1(\boldsymbol{\eta},\theta) \in C(0,T;V')$. By a similar argument, from (4.20) and (3.15) it follows that

$$\begin{aligned}(4.22)\quad & |\Lambda^2(\boldsymbol{\eta},\theta)(t_1) - \Lambda^2(\boldsymbol{\eta},\theta)(t_2)|_{L^2(\Omega)} \\ & \le C(|\mathbf{u}_\eta(t_1) - \mathbf{u}_\eta(t_2)|_V + |\alpha_\theta(t_1) - \alpha_\theta(t_2)|_{L^2(\Omega)}).\end{aligned}$$

Therefore, $\Lambda^2(\boldsymbol{\eta},\theta) \in C(0,T;L^2(\Omega))$ and $\Lambda(\boldsymbol{\eta},\theta) \in C(0,T;V' \times L^2(\Omega))$. Let now $(\boldsymbol{\eta}_1,\theta_1),(\boldsymbol{\eta}_2,\theta_2) \in L^2(0,T;V' \times L^2(\Omega))$. We use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\dot{\mathbf{u}}_{\eta_i} = \mathbf{v}_{\eta_i} = \mathbf{v}_i, \alpha_{\theta_i} = \alpha_i$ and $\beta_{\eta_i} = \beta_i$ for $i = 1,2$. Arguments similar to those used in the proof of (4.21) and (4.22) yield

$$\begin{aligned}(4.23)\quad & |\Lambda(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda(\boldsymbol{\eta}_2,\theta_2)(t)|^2_{V' \times L^2(\Omega)} \\ & \le C(|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V^2 + |\alpha_1(t) - \alpha_2(t)|^2_{L^2(\Omega)} + |\beta_1(t) - \beta_2(t)|^2_{L^2(\Gamma_3)}).\end{aligned}$$

Since

$$\mathbf{u}_i(t) = \int_0^t \mathbf{v}_i(s)\,ds + \mathbf{u}_0, \qquad t \in [0,T],$$

we have

$$(4.24)\qquad |\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V^2 \le C\int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_V^2\,ds \qquad \forall t \in [0,T].$$

Moreover, from (4.1) we infer that a.e. on $(0,T)$,

$$\begin{aligned}(\dot{\mathbf{v}}_1 - \dot{\mathbf{v}}_2, \mathbf{v}_1 - \mathbf{v}_2)_{V' \times V} + (\mathcal{A}\varepsilon(\mathbf{v}_1) - \mathcal{A}\varepsilon(\mathbf{v}_2), \varepsilon(\mathbf{v}_1 - \mathbf{v}_2))_{\mathcal{H}} \\ + (\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \mathbf{v}_1 - \mathbf{v}_2)_{V' \times V} = 0.\end{aligned}$$

We integrate this equality with respect to time. We use the initial conditions $\mathbf{v}_1(0) = \mathbf{v}_2(0) = \mathbf{v}_0$ and (3.13) to find that

$$m_{\mathcal{A}} \int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_V^2 \, ds \leq -\int_0^t (\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s), \mathbf{v}_1(s) - \mathbf{v}_2(s))_{V' \times V} \, ds$$

for all $t \in [0, T]$. Then, using the inequality $2ab \leq a^2/\gamma + \gamma b^2$ we obtain

$$\int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_V^2 \, ds \leq C \int_0^t |\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)|_{V'}^2 \, ds \qquad \forall t \in [0, T]. \tag{4.25}$$

On the other hand, from the Cauchy problem (4.12)–(4.13) we can write

$$\beta_i(t) = \beta_0 - \int_0^t (\beta_i(s)(\gamma_\nu(R_\nu(u_{i\nu}(s)))^2 + \gamma_\tau |\mathbf{R}_\tau(\mathbf{u}_{i\tau}(s))|^2) - \varepsilon_a)_+ \, ds,$$

and then

$$\begin{aligned} &|\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)} \\ &\qquad \leq C \int_0^t |\beta_1(s)(R_\nu(u_{1\nu}(s)))^2 - \beta_2(s)(R_\nu(u_{2\nu}(s)))^2|_{L^2(\Gamma_3)} \, ds \\ &\qquad\quad + C \int_0^t |\beta_1(s)|\mathbf{R}_\tau(\mathbf{u}_{1\tau}(s))|^2 - \beta_2(s)|\mathbf{R}_\tau(\mathbf{u}_{2\tau}(s))|^2|_{L^2(\Gamma_3)} \, ds. \end{aligned}$$

Using the definition of R_ν and $\mathbf{R}_\tau$ and writing $\beta_1 = \beta_1 - \beta_2 + \beta_2$, we get

$$\begin{aligned} &|\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)} \\ &\qquad \leq C \Big(\int_0^t |\beta_1(s) - \beta_2(s)|_{L^2(\Gamma_3)} \, ds + \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|_{L^2(\Gamma_3)^d} \, ds \Big). \end{aligned} \tag{4.26}$$

Next, we apply Gronwall's inequality to deduce

$$|\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)} \leq C \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|_{L^2(\Gamma_3)^d} \, ds,$$

and from (3.12) we obtain

$$|\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)}^2 \leq C \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|_V^2 \, ds. \tag{4.27}$$

From (4.15) we deduce that

$$\begin{aligned} &(\dot{\alpha}_1 - \dot{\alpha}_2, \alpha_1 - \alpha_2)_{L^2(\Omega)} + a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \\ &\qquad\qquad \leq (\theta_1 - \theta_2, \alpha_1 - \alpha_2)_{L^2(\Omega)}, \qquad \text{a.e. } t \in (0, T). \end{aligned}$$

Integrating this inequality with respect to time, using the initial conditions $\alpha_1(0) = \alpha_2(0) = \alpha_0$ and the inequality $a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \geq 0$ we find

$$\frac{1}{2} |\alpha_1(t) - \alpha_2(t)|_{L^2(\Omega)}^2 \leq \int_0^t (\theta_1(s) - \theta_2(s), \alpha_1(s) - \alpha_2(s))_{L^2(\Omega)} \, ds,$$

which implies that

$$|\alpha_1(t)-\alpha_2(t)|^2_{L^2(\Omega)} \le \int_0^t |\theta_1(s)-\theta_2(s)|^2_{L^2(\Omega)}\,ds + \int_0^t |\alpha_1(s)-\alpha_2(s)|^2_{L^2(\Omega)}\,ds.$$

This inequality combined with Gronwall's inequality leads to

$$|\alpha_1(t)-\alpha_2(t)|^2_{L^2(\Omega)} \le C\int_0^t |\theta_1(s)-\theta_2(s)|^2_{L^2(\Omega)}\,ds \quad \forall t\in[0,T]. \tag{4.28}$$

We substitute (4.27) in (4.23) and we use (4.24) to obtain

$$\begin{aligned}
&|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)|^2_{V'\times L^2(\Omega)}\\
&\quad\le C\Big(|\mathbf{u}_1(t)-\mathbf{u}_2(t)|^2_V+\int_0^t |\mathbf{u}_1(s)-\mathbf{u}_2(s)|^2_V\,ds+|\alpha_1(t)-\alpha_2(t)|^2_{L^2(\Omega)}\Big)\\
&\quad\le C\Big(\int_0^t |\mathbf{v}_1(s)-\mathbf{v}_2(s)|^2_V\,ds+|\alpha_1(t)-\alpha_2(t)|^2_{L^2(\Omega)}\Big).
\end{aligned}$$

It now follows from the above and the estimates (4.25) and (4.28) that

$$\begin{aligned}
&|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\ |^2_{V'\times L^2(\Omega)}\\
&\qquad\qquad\le C\int_0^t |(\boldsymbol{\eta}_1,\theta_1)(s)-(\boldsymbol{\eta}_2,\theta_2)(s)|^2_{V'\times L^2(\Omega)}\,ds.
\end{aligned}$$

Reiterating this inequality m times leads to

$$\begin{aligned}
&|\Lambda^m(\boldsymbol{\eta}_1,\theta_1)-\Lambda^m(\boldsymbol{\eta}_2,\theta_2)|^2_{L^2(0,T;V'\times L^2(\Omega))}\\
&\qquad\qquad\le \frac{C^mT^m}{m!}\,|(\boldsymbol{\eta}_1,\theta_1)-(\boldsymbol{\eta}_2,\theta_2)|^2_{L^2(0,T;V'\times L^2(\Omega))}.
\end{aligned}$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $L^2(0,T;V'\times L^2(\Omega))$, and so Λ has a unique fixed point. ∎

Now, we have all the ingredients needed to prove Theorem 3.1.

Proof. Existence. Let $(\boldsymbol{\eta}^*,\theta^*)\in L^2(0,T;V'\times L^2(\Omega))$ be the fixed point of Λ given by (4.18). Denote by $\mathbf{u}_{\eta^*}$ the solution of problem PV_η for $\boldsymbol{\eta}=\boldsymbol{\eta}^*$, and let α_{θ^*} be the solution of problem PV_θ for $\theta=\theta^*$. Let β_{η^*} be the solution of problem PV_β for $\boldsymbol{\eta}=\boldsymbol{\eta}^*$. We denote by $\boldsymbol{\sigma}_{\eta^*}$ the function given by $\boldsymbol{\sigma}_{\eta^*}=\mathcal{A}\varepsilon(\dot{\mathbf{u}}_{\eta^*})+\mathcal{G}(\varepsilon(\mathbf{u}_{\eta^*}),\alpha_{\theta^*})$. Using (4.19), (4.20) and keeping in mind that $\Lambda^1(\boldsymbol{\eta}^*,\theta^*)=\boldsymbol{\eta}^*$, $\Lambda^2(\boldsymbol{\eta}^*,\theta^*)=\theta^*$, we find that the quadruplet $(\mathbf{u}_{\eta^*},\boldsymbol{\sigma}_{\eta^*},\alpha_{\theta^*},\beta_{\eta^*})$ is a solution of problem PV. This solution has the regularity expressed in (3.33)–(3.36): this follows from the regularities of the solutions of problems PV_η, PV_θ and PV_β. Moreover, it follows from (3.33), (3.13) and (3.14) that $\boldsymbol{\sigma}_{\eta^*}\in L^2(0,T;\mathcal{H})$. Choosing now $\mathbf{v}=\pm\varphi$ in (3.30), where $\varphi\in C_0^\infty(\Omega)^d$, and using (3.18) and (3.27) we find

$$\varrho\ddot{\mathbf{u}}(t)=\operatorname{Div}\boldsymbol{\sigma}(t)+\mathbf{f}_0(t),\qquad \text{a.e. } t\in(0,T).$$

Then assumptions (3.18) and (3.20), the regularity expressed in (3.33) and the above equality imply that $\mathrm{Div}\,\boldsymbol{\sigma} \in L^2(0,T;V')$, which shows that $\boldsymbol{\sigma}$ satisfies (3.34).

Uniqueness. Let $(\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*}, \alpha_{\theta^*}, \beta_{\eta^*})$ be the solution of PV obtained above and let $(\mathbf{u}, \boldsymbol{\sigma}, \alpha, \beta)$ be another solution which satisfies (3.33)–(3.36). We denote by $\boldsymbol{\eta} \in L^2(0,T;V')$ and $\theta \in L^2(0,T;L^2(\Omega))$ the functions

$$(\boldsymbol{\eta}(t), \mathbf{v})_{V'\times V} = (\mathcal{G}(\varepsilon(\mathbf{u}(t)), \alpha(t)), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\beta(t), \mathbf{u}(t), \mathbf{v}), \tag{4.29}$$

$$\theta(t) = \phi(\varepsilon(\mathbf{u}(t)), \alpha(t)). \tag{4.30}$$

Equalities (3.28), (3.30) and (4.29) with the initial condition $\mathbf{u}(0) = \mathbf{u}_0$ imply that $\mathbf{u}$ is a solution of PV_η, and since it follows from Lemma 4.2 that this problem has a unique solution, denoted $\mathbf{u}_\eta$, we conclude that

$$\mathbf{u} = \mathbf{u}_\eta. \tag{4.31}$$

Next, (3.31), (4.31) and the initial condition $\beta(0) = \beta_0$ imply that β is a solution of problem PV_β; since Lemma 4.3 shows that the problem has a unique solution, denoted β_η, we obtain

$$\beta = \beta_\eta. \tag{4.32}$$

Equalities (3.29), (4.30) and the initial condition $\alpha(0) = \alpha_0$ now imply that α is a solution of problem PV_θ; from Lemma 4.5 problem PV_θ has a unique solution, denoted α_θ, and it follows that

$$\alpha = \alpha_\theta. \tag{4.33}$$

Using (4.18) and (4.29)–(4.33), we conclude that $\Lambda(\boldsymbol{\eta}, \theta) = (\boldsymbol{\eta}, \theta)$ and by uniqueness of the fixed point of Λ it follows that

$$\boldsymbol{\eta} = \boldsymbol{\eta}^*, \qquad \theta = \theta^*. \tag{4.34}$$

The uniqueness of the solution is now a consequence of (4.31)–(4.34) together with the equality $\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{G}(\varepsilon(\mathbf{u}), \alpha)$. ∎

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Mohamed Selmani and Lynda Selmani
Department of Mathematics
University of Setif
19000 Setif, Algeria
E-mail: s_elmanih@yahoo.fr
maya91dz@yahoo.fr

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