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GEOMETRY

Extending Nearaffine Planes to Hyperbola Structures

by

Kinga CUDNA-SALMANOWICZ and Jan JAKÓBOWSKI

Presented by Jan RYCHLEWSKI

Summary. H. A. Wilbrink [Geom. Dedicata 12 (1982)] considered a class of Minkowski planes whose restrictions, called residual planes, are nearaffine planes. Our study goes in the opposite direction: what conditions on a nearaffine plane are necessary and sufficient to get an extension which is a hyperbola structure.

1. Basic concepts. Let Ω be a nonempty set, and Ξ some family of subsets of Ω. Elements of Ω are called *points*, elements of Ξ *lines*. Moreover let $\rhd : \ \Omega \times \Omega \setminus \{(Z,Z);\, Z \in \Omega\} \to \Xi$ be a surjection, called *join*, and let $\equiv \subset \Xi \times \Xi$ be an equivalence relation called *parallelism* of lines. The image of (X,Y) under the join map will be denoted by $X \rhd Y$. The point X is called a *base point* of the line $X \rhd Y$. In general join is not commutative. A line $X \rhd Y$ satisfying $X \rhd Y = Y \rhd X$ is called *straight*. All remaining lines are called *proper*. The set of all straight lines will be denoted by Υ [4, p. 345].

Definition 1.1 ([7, pp. 53–54]). A quadruple $\mathbf{NA} = (\Omega, \Xi, \rhd, \equiv)$ is a *nearaffine plane* if the following three groups of axioms hold:

I. *Axioms of lines*:

(**L1**) $X, Y \in X \rhd Y$ for all $X, Y \in \Omega$, $X \neq Y$.
(**L2**) $Z \in X \rhd Y \setminus \{X\} \Leftrightarrow X \rhd Y = X \rhd Z$ for all $X, Y, Z \in \Omega$, $X \neq Y$.
(**L3**) $X \rhd Y = Y \rhd X = X \rhd Z \Rightarrow X \rhd Z = Z \rhd X$ for all $X, Y, Z \in \Omega$, $Y \neq X \neq Z$.

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II. *Axioms of parallelism*:

(**P1**) For every line a and every point X there exists exactly one line with base point X parallel to a (we denote this line by $(X \equiv a)$).

(**P2**) $X \rhd Y \equiv Y \rhd X$ for all $X, Y \in \Omega$, $X \neq Y$.

(**P3**) $a \equiv b \wedge a \in \Upsilon \Rightarrow b \in \Upsilon$ for all $a, b \in \Xi$.

III. *Axioms of richness*:

(**R1**) There exist at least two nonparallel straight lines.

(**R2**) Every line a meets every straight line g with $g \not\equiv a$ in exactly one point.

Definition 1.2 ([7, p. 56]). A bijection $\varphi : \Omega \to \Omega$ is an *automorphism* of $\mathbf{NA} = (\Omega, \Xi, \rhd, \equiv)$ if $\varphi(P \rhd Q) = \varphi(P) \rhd \varphi(Q)$ and $a \equiv b \Leftrightarrow \varphi(a) \equiv \varphi(b)$ for any $P \neq Q$ and $a, b \in \Xi$.

Among nearaffine planes there is a distinguished class satisfying the following postulate [7, p. 55]:

(**V**) (*The Veblen condition*) Let a be a straight line containing points P, Q, R, and b a line different from a with base point P; moreover, let $S \in b \setminus \{P\}$. Then $(R \equiv Q \rhd S) \cap b \neq \emptyset$.

Now, let Π be some other set of *points*, provided with three pairwise disjoint families Σ_+, Σ_-, Λ of subsets, elements of which are called $(+)$*generators*, $(-)$*generators* and *circles*, respectively. Consider the following axioms:

(**M1**) For every point P there exists a unique $(+)$generator, denoted by $[P]_+$, and a unique $(-)$generator, denoted by $[P]_-$, containing P.

(**M2**) Every $(+)$generator meets every $(-)$generator in a unique point.

(**M3**) There is a circle containing at least three points.

(**M4**) Through three distinct points P, Q, R, no two of which are on a common generator, there is a unique circle, denoted by (P, Q, R).

(**M5**) Every circle intersects every generator in a unique point.

(**T**) Given a circle λ, a point $P \in \lambda$ and a point $Q \notin \lambda$ with P and Q not on a generator, there is one and only one circle μ through Q such that $\lambda \cap \mu = \{P\}$.

Definition 1.3 ([5, p. 269]). The quadruple $\mathbf{M} = (\Pi, \Sigma_+, \Sigma_-, \Lambda)$ is a *Minkowski plane* (resp. a *hyperbola structure*) if the axioms (**M1**)–(**M5**) and (**T**) (resp. (**M1**)–(**M5**)) hold.

2. New results. As H. A. Wilbrink has demonstrated in [6], every point Z of a Minkowski plane $\mathbf{M}$ satisfying two additional conditions induces a nearaffine (so-called residual) plane. For a given nearaffine plane $\mathbf{NA}$ we shall construct some hyperbola structure $\mathbf{H}(\mathbf{NA})$ such that $\mathbf{NA}$ is a residual plane with respect to some point Z of $\mathbf{H}(\mathbf{NA})$. A special case of this problem is solved in [4].

Such a construction is possible if **NA** satisfies a number of additional conditions. Since the straight lines of a residual plane are obtained from generators, it is obvious that $\mathbf{NA} = (\Omega, \Xi, \rhd, \equiv)$ must have exactly two classes Ψ_1, Ψ_2 of straight lines. Note that there exist nearaffine planes with more than two classes of straight lines, and two distinct lines may have three or more points in common (see e.g. [2, p. 207]). Let Ω_1, Ω_2, $\{Z\}$ be sets (elements of which are also called points) disjoint from each other and from Ω. We assume that there exist bijections $f_i : \Omega_i \to \Psi_i$ for $i = 1, 2$. Let $[P]_i$ denote the straight line through P belonging to Ψ_i, and set $P^i = f_i^{-1}([P]_i)$. All points marked with the superscript "i" belong to Ω_i (see Figure 1). Points of any structure will be denoted by capital Latin letters, lines of a nearaffine plane by small Latin letters, and circles of a hyperbola structure by small Greek letters.

Definition 2.1 (cf. [6, p. 123]).

(a) $\Gamma = \{(P, Q) \in \Omega \times \Omega;\ P \neq Q,\ P \rhd Q \notin \Psi_1 \cup \Psi_2\}$.

(b) For $(S, T) \in \Gamma$ we put

$$[S, T] = \{S, T, Z\} \cup \{R \in \Omega;\ (R, S), (R, T) \in \Gamma \wedge \neg(\exists_{P \rhd Q \in \Xi}\ S, T, R \in P \rhd Q \setminus \{P\})\}.$$

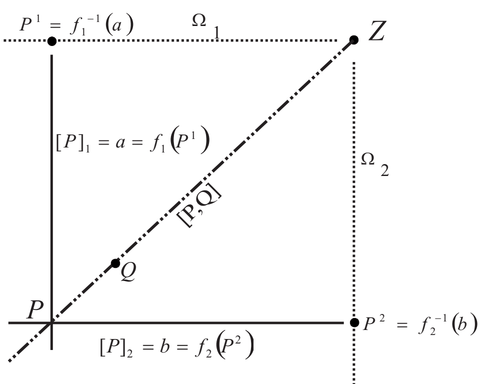

Fig. 1

Corollary 2.1. *$[S, T] = [T, S]$ and $U \in [S, T] \Leftrightarrow T \in [S, U]$. For every automorphism φ we have $\varphi([S, T] \setminus \{Z\}) = [\varphi(S), \varphi(T)] \setminus \{Z\}$.*

Definition 2.2.

$$\Pi = \Omega \cup \Omega_1 \cup \Omega_2 \cup \{Z\},$$
$$\Sigma_+ = \{[P]_1 \cup \{P^1\};\ P \in \Omega\} \cup \{\Omega_2 \cup \{Z\}\},$$

$$\Sigma_- = \{[P]_2 \cup \{P^2\};\ P \in \Omega\} \cup \{\Omega_1 \cup \{Z\}\},$$
$$\Lambda_1 = \{(P \rhd Q \setminus \{P\}) \cup \{P^1, P^2\};\ (P,Q) \in \Gamma)\},$$
$$\Lambda_2 = \{[S,T];\ (S,T) \in \Gamma\}, \qquad \Lambda_1 \cup \Lambda_2 = \Lambda.$$

LEMMA 2.1. *The structure* $(\Pi, \Sigma_+, \Sigma_-, \Lambda)$ *described in Definition* 2.2 *satisfies* (**M1**)–(**M3**) (see Figure 1).

THEOREM 2.1. *The quadruple* $\mathbf{H}(\mathbf{NA}) = (\Pi, \Sigma_+, \Sigma_-, \Lambda)$ *described in Definition* 2.2 *is a hyperbola structure if and only if the nearaffine plane* $\mathbf{NA} = (\Omega, \Xi, \rhd, \equiv)$ *satisfies the following conditions*:

(**H1**) *If* $U \in [S,T]$ *and* $U \neq S$ *then* $[S,T] = [S,U]$.
(**H2**) *Every set* $[S,T]$ *intersects every straight line in exactly one point.*
(**H3**) *For two proper lines* $P \rhd Q$ *and* $R \rhd S$ *with* $(P,R) \in \Gamma$, *the sets* $P \rhd Q \setminus \{P\}$ *and* $R \rhd S \setminus \{R\}$ *have at most two distinct points in common.*
(**H4**) *For every straight line* a *and two distinct points* $P, Q \notin a$ *with* $(P,Q) \in \Gamma$, *there exists a unique point* $S \in a$ *such that* $S \rhd P = S \rhd Q$ (see Figure 2).

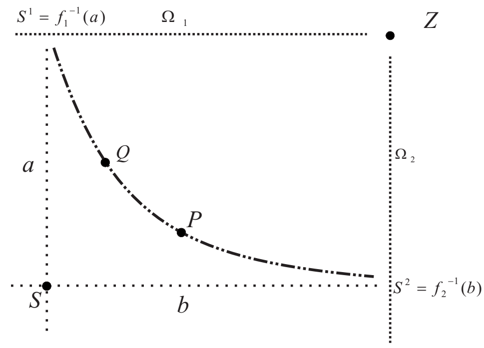

Fig. 2

Proof. Let (**M1**)–(**M5**) hold in $\mathbf{H}(\mathbf{NA})$.

For (**H1**), by Definitions 2.1, 2.2 and (**M4**), $U \in [S,T]$ means that $(U,S,T) = (S,T,Z) = (S,U,Z)$ in $\mathbf{H}(\mathbf{NA})$, i.e. $[S,T] = [S,U]$.

To prove (**H2**), consider any set $[S,T]$ and any straight line $a \in \Psi_1$. The set $[S,T]$ is a circle and $a \cup \{f_1^{-1}(a)\}$ is a generator in $\mathbf{H}(\mathbf{NA})$. Then by (**M5**), there exists a point R such that $\{R\} = [S,T] \cap (a \cup \{f_1^{-1}(a)\})$. Since $f_1^{-1}(a) \notin [S,T]$, we have $\{R\} = [S,T] \cap a$.

(**H3**) is immediate from (**M4**).

Finally, we prove (**H4**). Let a, P, Q satisfy the assumptions of (**H4**) and e.g. $a \in \Psi_1$. In $\mathbf{H}(\mathbf{NA})$, $a \cup \{f_1^{-1}(a)\}$, $\Omega_1 \cup \{Z\}$, $\Omega_2 \cup \{Z\}$ are generators and $P, Q, f_1^{-1}(a)$ are distinct points, no two of which are on the same generator (see Figure 2). By (**M2**) we have $\{f_1^{-1}(a)\} = (a \cup \{f_1^{-1}(a)\}) \cap (\Omega_1 \cup \{Z\})$ and by (**M4**), P, Q, $f_1^{-1}(a)$ determine exactly one circle λ. Define $\{S^2\} = (\Omega_2 \cup \{Z\}) \cap \lambda$ (cf. (**M5**)) and $\{S\} = [S^2]_- \cap [f_1^{-1}(a)]_+$, i.e. $f_1^{-1}(a) = S^1$. By Definition 2.2 and (**M4**) we have

$$S \rhd P \setminus \{S\} = \lambda \setminus \{S^1, S^2\} = (P, S^1, S^2) \setminus \{S^1, S^2\} \\ = (Q, S^1, S^2) \setminus \{S^1, S^2\} = S \rhd Q \setminus \{S\}.$$

Hence $S \rhd P = S \rhd Q$.

Conversely, let (**H1**)–(**H4**) hold in **NA**.

To prove (**M4**), let $P, Q, R \in \Pi$ be points such that no two are on a common generator. We have the following possibilities:

1. $P, Q, R \in \Omega$. Either there exists a proper line $X \rhd Y$ such that $P, Q, R \in X \rhd Y \setminus \{X\}$, or such a line does not exist. In the former case the line $X \rhd Y$ is uniquely determined by (**H3**) and we put $\lambda = (X \rhd Y \setminus \{X\}) \cup \{X^1, X^2\}$. In the latter case, by (**H1**) and Corollary 2.1, $[P, Q] = [P, R] = [Q, R]$ and $\lambda = [P, Q]$ is a unique circle containing P, Q, R.
2. $P, Q \in \Omega$, $R = Z$. No circle from Λ_1 contains Z. Then $[P, Q]$ is a unique circle through P, Q, R.
3. $P, Q \in \Omega$, $R^1 \in \Omega_1$. No circle from Λ_2 contains an element of Ω_1. Thus a circle through P, Q, R^1 must belong to Λ_1. There exists the straight line $f_1(R^1)$. Of course $P, Q \notin f_1(R^1)$. In view of (**H4**) there is a unique point $S \in f_1(R^1)$ such that $S \rhd P = S \rhd Q$. Therefore
$$\lambda = (S \rhd P \setminus \{S\}) \cup \{R^1, f_2^{-1}([S]_2)\} = (S \rhd P \setminus \{S\}) \cup \{S^1, S^2\}$$
is the only circle containing P, Q, R^1.
4. $P \in \Omega$, $Q^1 \in \Omega_1$, $R^2 \in \Omega_2$. Let $\{Y\} = f_1(Q^1) \cap f_2(R^2)$ (cf. (**R2**)). We obtain $\lambda = (Y \rhd P \setminus \{Y\}) \cup \{Q^1, R^2\} = (Y \rhd P \setminus \{Y\}) \cup \{Y^1, Y^2\}$.

For (**M5**), consider $\lambda \in \Lambda = \Lambda_1 \cup \Lambda_2$ and $\sigma \in \Sigma_+ \cup \Sigma_-$. The following cases are possible:

1. $\lambda = (P \rhd Q \setminus \{P\}) \cup \{P^1, P^2\}$, $\sigma = [R]_1 \cup \{R^1\}$ for some $P, Q, R \in \Omega$, $(P, Q) \in \Gamma$. By (**R2**) we have $P \rhd Q \cap [R]_1 = \{X\}$ for some $X \in \Omega$ and $X = P \Leftrightarrow P^1 = X^1 = R^1$. Then $\lambda \cap \sigma = \{X\}$ for $R^1 \neq P^1$ and $\lambda \cap \sigma = \{R^1\}$ for $R^1 = P^1$.
2. $\lambda = (P \rhd Q \setminus \{P\}) \cup \{P^1, P^2\}$, $\sigma = \Omega_1 \cup \{Z\}$. Then $\lambda \cap \sigma = \{P^1\}$.
3. $\lambda = [S, T]$, $\sigma = \Omega_1 \cup \{Z\}$. Then $\lambda \cap \sigma = \{Z\}$.
4. $\lambda = [S, T]$, $\sigma = [R]_1 \cup \{R^1\}$. By (**H2**) we have $[S, T] \cap [R]_1 = \{U\}$ for some $U \in \Omega$ and we obtain $\lambda \cap \sigma = \{U\}$.

PROPOSITION 2.1. *Every automorphism φ of a nearaffine plane* **NA** *satisfying the conditions* (**H1**)–(**H4**) *extends to an automorphism $\overline{\varphi}$ of the hyperbola structure* **H**(**NA**).

Proof. We define $\overline{\varphi}|_{\Omega}=\varphi$, $\overline{\varphi}(Z)=Z$, $\overline{\varphi}(X^i)=f_j^{-1}(\varphi(f_i(X^i)))$ for $X^i\in\Omega_i$, where $i,j\in\{1,2\}$, $j=i$ if $\varphi(\Psi_i)=\Psi_i$, and $j\neq i$ if $\varphi(\Psi_i)\neq\Psi_i$ (Figure 3). Thus $\overline{\varphi}$ is a bijection. By Definition 2.2, Definition 1.2 and Corollary 2.1, $\overline{\varphi}$ maps every circle of $\mathbf{H}(\mathbf{NA})$ onto a circle.

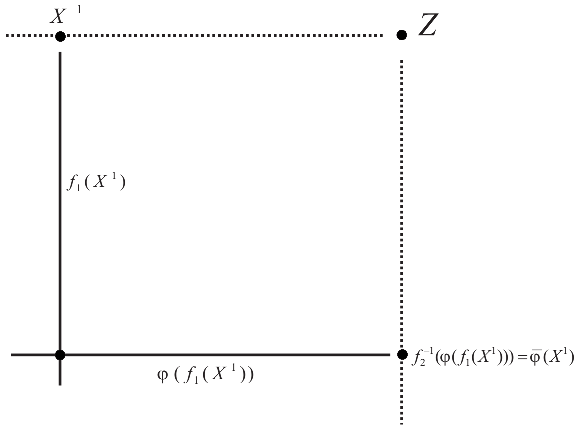

Fig. 3

We shall use the notation $P\rhd Q=\lambda^*$ for $\lambda\in\Lambda_1$, $\lambda=(P^1,P^2,Q)$.

Lemma 2.2 (see [2, Corollary 1, p. 215] for the finite case). *Let distinct lines a, b have base points on a common straight line.*

(a) *If* $a\equiv b$ *then* $a\cap b=\emptyset$.

(b) *If the Veblen condition holds and* $a\cap b=\emptyset$ *then* $a\equiv b$.

Proof. Let $g\in\Upsilon$ and let $P,Q\in g$ be the base points of a, b, respectively. If $a\equiv b$ and $S\in a\cap b$, then $a=P\rhd S$, $b=Q\rhd S$. By (**P2**) and (**P1**), we obtain $S\rhd P\equiv P\rhd S\equiv Q\rhd S\equiv S\rhd Q$, i.e. $S\rhd P=S\rhd Q$. This contradicts (**R2**). Now assume that $a\cap b=\emptyset$ and $a\not\equiv b$. Let $P\neq S\in a$ and $(S\equiv b)\cap g=\{R\}$. Then $b\equiv(S\equiv b)=S\rhd R\equiv R\rhd S$ and $a\cap R\rhd S\neq\emptyset$ but $a\cap b=\emptyset$, which contradicts (**V**).

The following generalizes Theorem 2.4 from [2, p. 217].

Proposition 2.2. *For any straight line g of a nearaffine plane* $\mathbf{NA}=(\Omega,\Xi,\rhd,\equiv)$ *let* $\mathcal{L}_g=\{a\in\Xi;\ a\equiv g\}\cup\{P\rhd Q\in\Xi;\ P\in g\}$. *Then* $\mathbf{A}(g)=(\Omega,\mathcal{L}_g)$ *is an affine plane if and only if* (**V**) *and* (**H4**) *hold in* **NA**.

Proof. $\Leftarrow$: Let $X,Y\in\Omega$, $X\neq Y$. If $X\rhd Y$ is straight in **NA** then it is a unique line through X, Y in $\mathbf{A}(g)$. Let $(X,Y)\in\Gamma$. If $X\in g$ then $X\rhd Y\in\mathcal{L}_g$ and of course, it is a unique line through X, Y with base point on g. Similarly for $Y\in g$. If $X,Y\notin g$ then (**H4**) means exactly that there exists a unique (proper) line a through X, Y with base point on g, i.e. $a\in\mathcal{L}_g$.

Let $a \in \mathcal{L}_g$, $P \notin a$. Set $b = (P \equiv a)$. If $a \in \Upsilon$ then by (**R2**), b is the only line through P disjoint from a. Either $b \equiv g$, or $b \neq g$ has a base point on g. In both cases $b \in \mathcal{L}_g$. If $a \notin \Upsilon$ then put $b \cap g = \{Q\}$. By (**P1**) and (**P2**), $(Q \equiv b) \equiv b \equiv a$, where $P \in (Q \equiv b)$, and the base points of $(Q \equiv b)$, a are on g. By Lemma 2.2(a), $(Q \equiv b) \cap a = \emptyset$ and by Lemma 2.2(b), only $(Q \equiv b)$ is a line through P, with base point on g and disjoint from a.

$\Rightarrow$: Assume that $\mathbf{A}(g)$ is affine for every $g \in \Upsilon$. In particular, for every $P, Q \notin g$ such that $(P, Q) \in \Gamma$, there exists exactly one line from $\mathcal{L}_g$ passing through P, Q, i.e. a proper line with base point S on g. Thus (**H4**) holds. Suppose that (**V**) does not hold, i.e. for some pairwise distinct points $P, Q, R \in g$ and $S \notin g$ we have $(R \equiv Q \rhd S) \cap P \rhd S = \emptyset$. By Lemma 2.2(a), $Q \rhd S \cap (R \equiv Q \rhd S) = \emptyset$. Thus $P \rhd S$ and $Q \rhd S$ are distinct lines through S, both parallel to $(R \equiv Q \rhd S)$ in the affine plane $\mathbf{A}(g)$, a contradiction.

Proposition 2.3. *For any $\lambda \in \Lambda_1$, $P^1 \in \lambda$, $Q \in \Omega \cup \Omega_2 \setminus \lambda$ there exists a circle μ such that $\lambda \cap \mu = \{P^1\}$ and $Q \in \mu$.*

Proof. Let $\lambda^* = P \rhd S$ for some $S \in \lambda \cap \Omega$. If $Q \in \Omega$ then $(Q \equiv P \rhd S)$ intersects $[P]_1$ in some point R and by (**P2**) we get

$$P \rhd S \equiv (Q \equiv P \rhd S) = Q \rhd R \equiv R \rhd Q = (R \equiv P \rhd S).$$

If $Q = Q^2 \in \Omega_2$ then define $\{R\} = [P^1]_+ \cap [Q^2]_-$. In both cases let $\mu^* = (R \equiv P \rhd S) = b$. Thus $b \equiv P \rhd S$ and the base points are on the common straight line $f_1(P^1)$, so $b \cap P \rhd S = \emptyset$ (cf. Lemma 2.2(a)), whence $\lambda \cap \mu = \{P^1\}$. Of course $Q \in \mu$.

Proposition 2.4. *If* (**V**) *holds in* **NA** *then the following conditions are satisfied*:

(a) *The circle μ from Proposition* 2.3 *is uniquely determined.*
(b) *Let $\lambda^* = P \rhd U$, $\alpha \cap \lambda = \{P^1\}$, $Q^2 \in \alpha$, $\mu \cap \alpha = \{Q^2\}$ and $Q^1 \in \mu$. If $\beta \cap \lambda = \{P^2\}$, $Q^1 \in \beta$, $\nu \cap \beta = \{Q^1\}$ and $Q^2 \in \nu$ then $\mu = \nu$. If $U = Q$ then $P \in \mu$* (see Figure 4)*.*

Proof. From Propositions 2.3 and Lemma 2.2(b) item (a) is immediate. Therefore α, β, μ, ν exist and they are uniquely determined. Set $\{R\} = [P^1]_+ \cap [Q^2]_-$ and $\{S\} = [Q^1]_+ \cap [P^2]_-$. For some points $T \in \alpha$ and $V \in \beta$ we obtain $\alpha^* = R \rhd T$, $\beta^* = S \rhd V$, and the conditions $\alpha \cap \lambda = \{P^1\}$, $\beta \cap \lambda = \{P^2\}$ mean $R \rhd T \equiv P \rhd U \equiv S \rhd V$. Since parallelity is symmetric, we have $S \rhd V \equiv R \rhd T$ and there exists a line $Q \rhd X = \gamma^*$, where $\gamma \cap \beta = \{Q^1\}$ and $\gamma \cap \alpha = \{Q^2\}$. Therefore $\mu = \gamma = \nu$.

Now assume $U = Q$. Then we have $\alpha^* = R \rhd T$, $\lambda^* = P \rhd Q$ and $\mu^* = Q \rhd Y$ for some $Y \in \mu \cap \Omega$. The base points P, R lie on the same straight line and so do R, Q. We obtain $P \rhd Q \cap R \rhd T = \emptyset$ and $R \rhd T \cap Q \rhd Y = \emptyset$. By (**P2**) and Lemma 2.2(b) we get $Q \rhd P \equiv P \rhd Q \equiv R \rhd T \equiv Q \rhd Y$. Now

(**P1**) implies that $Q \rhd P = Q \rhd Y$, i.e. $P \in Q \rhd Y = \mu^*$.

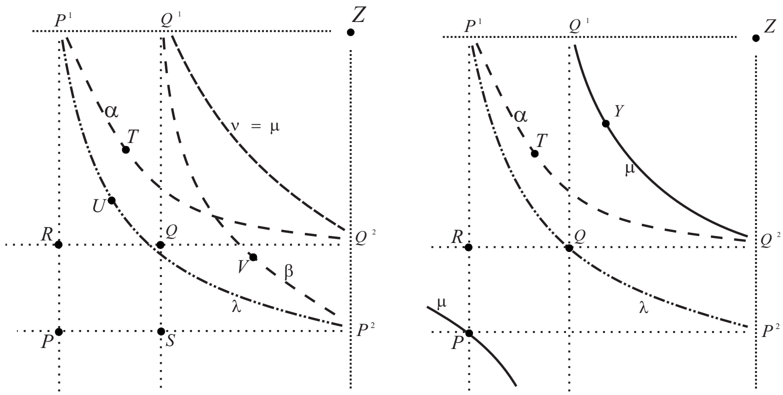

Fig. 4

Propositions 2.3 and 2.4(a) imply that for Veblenian **NA**, **H**(**NA**) satisfies (**T**) with the range of P restricted to $\Omega_1 \cup \Omega_2$. In order to extend this range to $\Omega_1 \cup \Omega_2 \cup \{Z\} = [Z]_+ \cup [Z]_-$ we have to require

(H5) For every $[S,T]$ and $R \notin [S,T]$ there exists a unique $[U,V]$ such that $R \in [U,V]$ and $[S,T] \cap [U,V] = \{Z\}$.

So-called classical models in any geometry are constructed "over a field". Some non-classical models are usually connected with algebraic structures which are weaker than fields. However, even for such weaker structures we often use a field as a tool. In a nearaffine plane over a field every proper line is given by an equation $(x-p)(y-q) = r$. If the field is pseudo–ordered, then this equation may be modified: if $-r$ is positive, then we put $(f(x)-p)(y-q) = r$ (cf. [4, p. 348]).

Example 2.1. Consider a Moulton nearaffine plane, i.e.

$$f(x) = \begin{cases} x & \text{for } x \geq 0, \\ kx & \text{for } x \leq 0 \end{cases}$$

(cf. [4, p. 355]) using the field of rational numbers. Then proper lines are given in the following form:

$$Q \rhd S = \left\{(p,q)\right\} \cup \left\{(x,y);\, r = \begin{cases} (x-p)(y-q) & \text{for } x \geq 0 \\ (kx-p)(y-q) & \text{for } x \leq 0 \end{cases}\right\}$$

for some fixed k, where $0 < k \neq 1$. This nearaffine plane extends to a hyperbola structure [4, Corollary 3.3, p. 359]. But the Veblen condition does not hold in this case. Indeed, let $k=2$, $P=(1,0)$, $Q=(1,6)$, $R=(1,2)$, $S=(0,2)$. We have

$$P \rhd S = \{(1,0)\} \cup \{(x,y)\};\, (x-1)y = -2\},$$

$$Q \rhd S = \{(1,6)\} \cup \left\{(x,y);\ 4 = \begin{cases} (x-1)(y-6) & \text{for } x \geq 0 \\ (2x-1)(y-6) & \text{for } x \leq 0 \end{cases}\right\},$$

$$(R \equiv Q \rhd S) = \left\{(1,2)\} \cup \{(x,y);\ 4 = \begin{cases} (x-1)(y-2) & \text{for } x \geq 0 \\ (2x-1)(y-2) & \text{for } x \leq 0 \end{cases}\right\}.$$

Therefore $P \rhd S \cap (R \equiv Q \rhd S) = \emptyset$. This example shows that nearaffine planes which do not satisfy the Veblen condition may extend to hyperbola structures. But they cannot extend to Minkowski planes (cf. [6, p. 124]). Note that $P \rhd S \not\equiv (R \equiv Q \rhd S)$ (see Lemma 2.2(b)).

We shall show that the Veblen condition is an essential assumption in Proposition 2.4. Let λ, α, μ, β, ν be circles given by the following equations:

$$\lambda:\ (x-1)y = -2, \qquad \alpha:\ (x-1)(y-1) = -2,$$
$$\mu:\ (x+1)(y-1) = -2, \qquad \beta:\ (x+1)y = -2,$$

$$\nu:\ 4 = \begin{cases} (x+1)(y-1) & \text{for } x \geq 0, \\ (2x+1)(y-1) & \text{for } x \leq 0. \end{cases}$$

Set $P = (1,0)$, $Q = (1,1)$. Then $\lambda \cap \alpha = \{P^1\}$, $Q^2 \in \alpha$, $\alpha \cap \mu = \{Q^2\}$, $Q^1 \in \mu$, $\lambda \cap \beta = \{P^2\}$, $Q^1 \in \beta$, $\beta \cap \nu = \{Q^1\}$, $Q^2 \in \nu$ and $\mu \neq \nu$.

For λ^* with base point P and ν^* with base point Q we have $Q \in \lambda$ but $P \notin \nu$.

EXAMPLE 2.2. The field $\mathbb{Q}(\sqrt{2}) = \{p + q\sqrt{2};\ p, q \in \mathbb{Q}\}$ is pseudo-ordered if we declare that $p+q\sqrt{2}$ is positive if $p^2 - 2q^2 \succ 0$. We define $f(x_1 + x_2\sqrt{2}) = x_1 - x_2\sqrt{2}$. Then proper lines are given by

$$(x_1 + x_2\sqrt{2} - (p_1 + p_2\sqrt{2}))(y_1 + y_2\sqrt{2} - (q_1 + q_2\sqrt{2})) = r_1 + r_2\sqrt{2} \quad \text{for } r_1^2 - 2r_2^2 \succ 0$$

and

$$(x_1 - x_2\sqrt{2} - (p_1 + p_2\sqrt{2}))(y_1 + y_2\sqrt{2} - (q_1 + q_2\sqrt{2})) = r_1 + r_2\sqrt{2} \quad \text{for } r_1^2 - 2r_2^2 \prec 0.$$

One can easily prove that this plane satisfies Desargues' postulates (**D1**), (**D2**) (cf. [1, p. 72], [3, p. 339]). Therefore it is a translation plane. By Proposition 2.1, every translation extends to a translation of some hyperbola structure.

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Kinga Cudna-Salmanowicz and Jan Jakóbowski
Faculty of Mathematics and Informatics
University of Warmia and Mazury in Olsztyn
Żołnierska 14
10-561 Olsztyn, Poland
E-mail: salmanowicz@matman.uwm.edu.pl
jjakob@matman.uwm.edu.pl

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