BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 54, No. 3-4, 2006

Almost Weakly Compact Operators

by

Ioana GHENCIU and Paul LEWIS

Presented by Gilles PISIER

Summary. Dunford–Pettis type properties are studied in individual Banach spaces as well as in spaces of operators. Bibasic sequences are used to characterize Banach spaces which fail to have the Dunford–Pettis property. The question of whether a space of operators has a Dunford–Pettis property when the dual of the domain and the codomain have the respective property is studied. The notion of an almost weakly compact operator plays a consistent and important role in this study.

1. Introduction. Throughout this paper X, Y, E, and F will denote real Banach spaces. The unit ball of X will be denoted by B_X unless otherwise specified, and X^* will denote the continuous linear dual of X. An *operator* $T : X \to Y$ will be a continuous and linear function. Of course, it is well known that $T : X \to Y$ is a compact (resp. weakly compact) operator if and only if $T^*(B_{Y^*})$ is relatively compact (resp. relatively weakly compact) in X^*. The following equivalences (which one can easily derive directly from definitions) appear to be less well celebrated:

(a) $T : X \to Y$ is completely continuous iff $T^*(B_{Y^*})$ is an L-subset of X^*.
(b) $T : X \to Y$ is unconditionally converging iff $T^*(B_{Y^*})$ is a V-subset of X^*.
(c) $T : X \to Y$ is limited iff T^* is w^*-norm sequentially continuous.
(d) $T(B_X)$ is a DP subset of Y iff $T^* : Y^* \to X^*$ is completely continuous.
(e) $T(B_X)$ is a V^*-subset of Y iff $T^* : Y^* \to X^*$ is unconditionally converging.

2000 *Mathematics Subject Classification*: 46B20, 46B25, 46B28.
Key words and phrases: Dunford–Pettis property, Dunford–Pettis sets, limited sets, L-sets, w^*-w continuity, almost weakly compact operator.

In this paper we will investigate some structural consequences of (a)–(e) as well as topological properties of distinguished sets naturally associated with the classes of operators mentioned above.

2. Definitions and notation. The set of all continuous linear transformations from X to Y will be denoted by $L(X,Y)$, and the compact operators will be denoted by $K(X,Y)$. The w^*-w continuous (resp. w^*-w continuous compact) maps from X^* to Y will be denoted by $L_{w^*}(X^*,Y)$ (resp. $K_{w^*}(X^*,Y)$).

The bounded subset A of X is called a *Dunford–Pettis subset* (resp. *limited subset*) of X if each weakly null sequence in X^* (resp. w^*-null sequence in X^*) tends to 0 uniformly on A. The bounded subset A of X^* is called an *L-subset* of X^* if each weakly null sequence in X tends to 0 uniformly on A, and the subset S of X is said to be *weakly precompact* provided that every bounded sequence from S has a weakly Cauchy subsequence.

The operator $T : X \to Y$ is *limited* provided that $T(B_X)$ is a limited subset of Y; T is *completely continuous* (or *Dunford–Pettis*) if it maps weakly Cauchy sequences to norm convergent sequences; T is *unconditionally converging* if it maps weakly unconditionally converging series to unconditionally converging series; and T is *almost weakly compact* if $T(B_X)$ is weakly precompact. If every weakly compact operator defined on X is completely continuous, then we say that X has the *Dunford–Pettis property* (= DPP); see [9] and [1] for inventories of classical results related to the DPP. We denote the canonical unit vector basis of c_0 (resp. ℓ_1) by (e_n) (resp. (e_n^*)). The reader should consult Lindenstrauss and Tzafriri [29], Diestel [10], or [3] for undefined terminology and notation.

3. Almost weakly compact operators, limited sets, and Dunford–Pettis sets. Odell, Rosenthal, and Stegall [33, p. 377] showed that an operator $T : X \to Y$ is almost weakly compact if $L \circ T : X \to L_1$ is compact whenever $L : Y \to L_1$ is a completely continuous map. (The converse of this statement is straightforward.) In view of the connection between completely continuous operators and L-sets noted in (a) above, the Odell–Stegall–Rosenthal theorem appears to say that the L-subsets of Y^* are sufficient to identify the almost weakly compact operators $T : X \to Y$. More precisely, we have the following.

THEOREM 3.1. *An operator $T : X \to Y$ is almost weakly compact if and only if $T^*(A)$ is relatively compact in X^* whenever A is an L-subset of Y^*.*

Proof. Suppose that $T : X \to Y$ is almost weakly compact, and let A be an L-subset of Y^*. Let $B(A)$ be the Banach space (sup norm) of all bounded real-valued functions defined on A, and let $J : Y \to B(A)$ be the natural

evaluation map, i.e., $J(y)(a) = a(y)$. Since A is an L-set, J is completely continuous. Then $J \circ T$ is compact, $A \subseteq J^*(B^*)$ (B^* = unit ball of $B(A)^*$), and $T^*J^*(B^*)$ is relatively compact. Thus $T^*(A)$ is relatively compact.

Conversely, suppose that $T^*(A)$ is relatively compact for all L-subsets A of Y^*. Let $L : Y \to L_1$ be a completely continuous operator. Then $L^*(B_{L_\infty})$ is an L-subset of Y^*, and $T^*(L^*(B_{L_\infty}))$ is relatively compact. Therefore $(L \circ T)^*$ and $L \circ T$ are compact. By the Odell–Rosenthal–Stegall theorem, T is almost weakly compact. ∎

COROLLARY 3.2 ([14]). *Every L-subset of X^* is relatively compact iff $\ell_1 \not\hookrightarrow X$.*

Proof. Apply Rosenthal's ℓ_1-theorem and the previous theorem to the identity operator $I : X \to X$. ∎

It is well known that $\ell_1 \hookrightarrow X$ if and only if $L_1 \hookrightarrow X^*$ [10, p. 213], and that L_1 has the DPP [9]. Consequently, any normalized and weakly null sequence in L_1 is a non-relatively compact DP subset (therefore a non-relatively compact L-subset) of L_1. Thus the preceding corollary can be restated.

COROLLARY 3.3 ([2, Cor. 7], [20, Thm. 1]). *The following are equivalent*:

(i) $\ell_1 \not\hookrightarrow X$.
(ii) *Every L-subset of X^* is relatively compact.*
(iii) *Every DP subset of X^* is relatively compact.*
(iv) *Every weakly null DP sequence in X^* is norm null.*
(v) $L_1 \not\hookrightarrow X^*$.
(vi) *The identity map from X to X is almost weakly compact.*

The Odell–Rosenthal–Stegall theorem used completely continuous maps with range in L_1 to identify almost weakly compact operators. Weakly compact maps with range in c_0 can also be employed to identify such operators.

THEOREM 3.4. *If $T : Y \to X$ is an operator and $L \circ T : Y \to c_0$ is compact for all weakly compact operators $L : X \to c_0$, then T is almost weakly compact.*

Proof. Let (y_n) be a sequence in B_Y. Suppose (by way of contradiction) that $(T(y_n))$ has no weakly Cauchy subsequence. By Rosenthal's ℓ_1-theorem, we can assume that $(T(y_n)) \sim (e_n^*)$. Let $S = [T(y_n)]$, an isomorph of ℓ_1. Let $j : S \to c_0$ be the natural inclusion. Since j naturally factors through ℓ_2, j is absolutely 2-summing [32, Satz 2]. Now use the fact that all closed linear subspaces of an L_2-space are complemented and the constructions in Theorem 2 of [32] ($U^0 = (B_{X^*}, w^*)$) and on pp. 60–61 of [10] to obtain a weakly compact operator $J : X \to c_0$ which extends j. Then $JT : Y \to c_0$

is compact. But $JT(y_n) = j(e_n^*) = e_n$, and (e_n) is certainly not relatively compact in c_0. ∎

The weak precompactness of a DP set is well known and has been used extensively by many authors; e.g., see [1], [3], [13] and [20]. The usual proof of this result involves the Odell–Rosenthal–Stegall theorem. Theorem 3.4 yields a particularly quick argument for this fact.

COROLLARY 3.5. *Every Dunford–Pettis subset of X is weakly precompact.*

Proof. If K is a DP subset of X, W is the closed and absolutely convex hull of K, and $T : \ell_1(W) \to X$ is defined by $T(f) = \sum_{w \in W} f(w)w$, then T and T^* are completely continuous, and $K \subseteq T(B_{\ell_1(W)})$. If $L : X \to c_0$ is weakly compact, then a look at the adjoint immediately shows that $L \circ T$ is compact. ∎

It is known that a set A can be isometrically embedded in both E and F ($F \subset E$) and be a limited (Dunford–Pettis) subset of E and fail to be a limited (Dunford–Pettis) subset of F. For example, Phillips' lemma shows that if $j : c_0 \to \ell_\infty$ is the natural inclusion, then $j^* : \ell_\infty^* \to c_0^*$ is w^*-norm sequentially continuous; thus $A := B_{c_0}$ is a limited set in ℓ_∞. Certainly A fails to be a limited subset of c_0.

Also, if B is any bounded subset of a separable reflexive space R and R is viewed as sitting in $C[0,1]$, then the fact that $C[0,1]$ has the DPP guarantees that B is a Dunford–Pettis subset of $C[0,1]$. If R is infinite-dimensional and $B = B_R$, then the identity map $I : R \to R$ clearly shows that B is not a Dunford–Pettis subset of R. In fact, since the continuous linear image of a DP set is a DP set, B is not a DP subset of any space in which R is complemented.

We note that these observations, together with the easily established fact that the continuous linear image of a limited set is a limited set, lead to a particularly simple proof that c_0 is not complemented in ℓ_∞. For if $P : \ell_\infty \to c_0$ were a projection, then $P(B_{c_0}) = B_{c_0}$ would be a limited subset of c_0, and it is not.

The definitive result which determines when a basic sequence (x_n) in X is simultaneously equivalent to (e_n) and limited in X is due to Schlumprecht. See p. 36 of [35] for a proof.

THEOREM 3.6. *If (x_n) is a basic sequence in X which is equivalent to (e_n), then the following are equivalent*:

(a) *(x_n) is a limited sequence in X.*
(b) *If S is an infinite subset of $\mathbb{N}$, then $\overline{\text{span}}\{x_n : n \in S\}$ is not complemented in X.*

The proof of Theorem 3.6 in [35] used Sobczyk's theorem: *If X is separable and $c_0 \hookrightarrow X$, then c_0 is complemented in X.* The proof of 3.6 immediately implies that the following is true.

If B_{X^*} is w^*-sequentially compact, (x_n) is a basic sequence in X which is equivalent to (e_n), and (b_n) is any subsequence of (x_n), then there is a subsequence (c_n) of (b_n) such that $[(c_n)]$ is complemented in X. We do not know if the full sequence $[(x_n)]$ is complemented in X. See Emmanuele [17] for a similar result.

The next result uses bibasic sequences to characterize when (e_n) embeds as a limited sequence in X. In the process, we extend Theorem 4.5 of [3]. We refer the reader to [8], [3], and [36, p. 32] for a discussion of bibasic sequences (x_n, f_n^*) in $X \times X^*$. In particular, note that if (x_n, f_n^*) is a bibasic sequence in X and (x_n^*) is the a.s.c.f. for (x_n), then f_k^* is a continuous linear extension of x_k^* to all of X for each k. The problem of the equivalence of (f_n^*) and (x_n^*) as basic sequences has been studied by several authors; e.g., see [36].

It is known that ℓ_∞ contains limited sets which are not relatively weakly compact (see Example 1.1.8 of [35]). Further, Haydon [25] has given an example of a $C(K)$ space which is Grothendieck and does not contain ℓ_∞. Such a space must contain limited sets S which are not relatively weakly compact [35, pp. 27–28]. We shall see that such limited sets generate bibasic sequences (x_n, f_n^*) for which $(f_n^*) \not\sim (x_n^*)$ in a very strong sense. In the next theorem, $t : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function and $u_{t(n)}^* = x_{t(n)}^*|_{[x_{t(j)} : j \in \mathbb{N}]}$.

THEOREM 3.7. *If S is a limited subset of X which is not relatively weakly compact, then there is a a bibasic sequence (x_n, f_n^*) in $S \times X^*$ so that if $t : \mathbb{N} \to \mathbb{N}$ and $u_{t(n)}^*$ are defined as above, then $(u_{t(n)}^*) \not\sim (f_{t(n)}^*)$. Further, (e_n) embeds as a limited sequence in X iff there is a bounded bibasic sequence (x_n, f_n^*) in $X \times X^*$ so that (x_n) is limited in X, unconditional, and $(f_n^*) \sim (x_n^*)$.*

Proof. Let (x_n) be a sequence in S which has no weakly convergent subsequence. By Pełczyński's version of the Eberlein–Šmulian Theorem [10, p. 41], we may assume that (x_n) is a seminormalized basic sequence. Let (x_n^*) be the sequence of coefficient functionals, and let (f_n^*) be a sequence of Hahn–Banach extensions to all of X. Suppose that (f_n^*) has a w^*-Cauchy subsequence. Without loss of generality, assume that $f_n^* - f_{n+1}^* \xrightarrow{w^*} 0$. Then $\langle f_n^* - f_{n+1}^*, x_n \rangle \to 0$ since (x_n) is limited. However, $\langle f_n^* - f_{n+1}^*, x_n \rangle = 1$ for all n. Therefore (f_n^*) has no w^*-Cauchy subsequence, and thus it has no weakly Cauchy subsequence. By Rosenthal's ℓ_1-theorem, we may assume that $(f_n^*) \sim (e_n^*)$. The bibasic sequence (x_n, f_n^*) satisfies the first conclusion of the theorem. For if $(f_{t(n)}^*)$ were equivalent to $(u_{t(n)}^*)$, then the proof of

Lemma 2.2 in [28] would show that $(x_{t(n)}) \sim (e_n)$ and some subsequence of (x_n) would be weakly null.

If there is an isomorphism $T : c_0 \to X$ so that the sequence $(T(e_n))$ is limited in X and (x_n^*) is the associated sequence of coefficient functionals, then $(x_n^*) \sim (e_n^*)$. If (f_n^*) is any sequence of Hahn–Banach extensions of (x_n^*) to all of X, then the strength of the ℓ_1-norm ensures that $(f_n^*) \sim (x_n^*)$.

Conversely, suppose that the sequence (x_n, f_n^*) satisfies the hypotheses of the concluding statement in the theorem. Let $L : [f_n^*] \to [x_n^*]$ be an isomorphism so that $L(f_n^*) = x_n^*$ for each n. Since (x_n) is a limited sequence in X, some subsequence $(f_{n_i}^*)$ of (f_n^*) is equivalent to (e_i^*). Thus $(x_{n_i}^*) \sim (e_i^*)$. Consequently, we have an unconditional basic sequence (x_n) and a subsequence $(x_{n_i}^*)$ of the coordinate functionals so that $(x_{n_i}^*) \sim (e_i^*)$ in $[(x_n)]^*$. Thus $(x_{n_i}) \sim (e_i)$ (see Lemma 1 in [28]). ∎

Next we use bibasic sequences and the techniques in the proof of 1.3.1 in [35] to produce a strong characterization of spaces which do not have the DPP.

THEOREM 3.8. *The Banach space X fails to have the DPP iff there is a bounded bibasic sequence $(x_n, f_n^*) \in X \times X^*$ so that (x_n), (x_n^*), and (f_n^*) are weakly null.*

Proof. Note first that X has the DPP iff every weakly null sequence in X is a Dunford–Pettis subset of X. Suppose then that X fails to have the DPP, and let (x_n) be a weakly null sequence in X which is not Dunford–Pettis. By passing to a subsequence, we may (and do) assume that (x_n) is a weakly null basic sequence (apply the Bessaga–Pełczyński selection principle [10]) which is not a Dunford–Pettis sequence in X and none of its subsequences is a Dunford–Pettis sequence in X. Thus we may assume that there is a w-null sequence (z_n^*) in X^* and $z_n^*(x_n) = 1$ for each n. Further, by passing to subsequences, we may assume that $(z_n^*(x_m))_{m=1}^{\infty}$ converges for all n. Let $a_n = \lim_m z_n^*(x_m)$ for each n. By passing to another subsequence if necessary, one may assume that one of the following three cases holds:

(1) $a_n = 0$ for each n.
(2) $a_n \neq 0$ for each n, and $a_n \to 0$.
(3) There is an $\varepsilon > 0$ for which $|a_n| > \varepsilon$, $n \in \mathbb{N}$.

In (1) we set $y_n^* = z_n^*$ for each n. Now suppose that (2) holds. By passing to subsequences, we assume that $z_{n-1}^*(x_n) < 1/4$ for each n. Define r_n to be $\langle z_n^* - (a_n/a_{n-1})z_{n-1}^*, x_n\rangle$ for $n \in \mathbb{N}$. Since $a_n/a_{n-1} \to 1$, we may assume that $r_n \geq 1/2$ for each n. Set $y_n^* = (1/r_n)(z_n^* - (a_n/a_{n-1})z_{n-1}^*)$, and note that $y_n^* \xrightarrow{w} 0$, $\langle y_n^*, x_n\rangle = 1$, and $\lim_m y_n^*(x_m) = 0$ for each n.

Suppose next that (3) holds. In this case, the sequence (a_n/a_{n+1}) is bounded. Set $r_n = \langle z_n^* - (a_n/a_{n+1})z_{n+1}^*, x_n\rangle$. Since (z_n^*) is weakly null, we

may assume (as above) that $r_n \geq 1/2$ for each n. Set $y_n^* = (1/r_n)(z_n^* - (a_n/a_{n+1})z_{n+1}^*)$. Then, as in the preceding paragraph, $y_n^*(x_n) = 1$, $y_n^* \overset{w}{\to} 0$, and $y_n^*(x_m) \overset{m}{\to} 0$ for each n.

The reader should note that in all three cases one has $\lim_m y_m^*(x_n) = \lim_m y_n^*(x_m) = 0$ for each n. By passing to further subsequences if necessary, we assume that $\sum_{m,m\neq n} |y_n^*(x_m)| < 1/2$, $n \in \mathbb{N}$.

Define $T : X \to c_0$ by $T(x) = (y_n^*(x))$. It is not difficult to check that $(T(x_n)) \sim (e_n)$. Further, $T^*(e_n^*) = y_n^*$ for each n. Thus T^* and T are weakly compact operators. Let $Y_0 = \overline{\text{span}}\{T(x_n) : n \in \mathbb{N}\} = [(T(x_n))]$. Define $S_0 : Y_0 \to c_0$ by $S_0(T(x_n)) = e_n$ for each n. Now Y_0 is (isomorphic to) a separable subspace of c_0, and c_0 is separably injective. Let $S : c_0 \to c_0$ be a continuous linear extension of S_0, and consider the operator $ST : X \to c_0$. Let $f_n^* = (ST)^*(e_n^*)$ for each n. Since ST is weakly compact and $e_n^* \overset{w^*}{\to} 0$, it follows that $f_n^* \overset{w}{\to} 0$; i.e., the adjoint of a weakly compact operator is w^*-w continuous. Further, $f_n^*(x_m) = \langle e_n^*, e_m \rangle = \delta_{nm}$. Another application of the Bessaga–Pełczyński selection principle allows us to assume that (f_n^*) is a weakly null basic sequence in X^*. Let $x_n^* = f_n^*|_{[(x_n)]}$. To see that (x_n^*) is weakly null, notice that the restriction operator $R : X^* \to [(x_n)]^*$ maps weakly null sequences to weakly null sequences. This completes the proof of one implication in the theorem, and the converse implication follows from the definition. ∎

Remark. By localizing the preceding argument, one sees that if K is a bounded subset which is not a Dunford–Pettis subset of X, then one can find a sequence (x_n) in K and a weakly null sequence (x_n^*) in X^* so that $x_n^*(x_m) = \delta_{nm}$. Lemma 1.3.1 of [35] showed that if a bounded subset K of X is not limited in X, then there exists an operator $T : X \to c_0$ so that $\{e_n : n \in \mathbb{N}\} \subseteq T(K)$. The proof of 3.8 shows that K is not a DP subset of X iff there is a weakly compact operator $T : X \to c_0$ so that $\{e_n : n \in \mathbb{N}\} \subseteq T(K)$.

If (x_n) is a basic sequence in X and $(x_n) \sim (e_n)$, then (x_n) is always a DP sequence in X, and Theorems 3.6 and 3.7 investigate when (x_n) is a limited sequence in X. If $(x_n) \sim (e_n^*)$, then (x_n) is never a limited or a DP sequence in X. However, (e_n^*) may embed as a V^*-subset of X. Also, (e_n^*) may embed as a V-subset of X^*.

A bounded subset A of X (resp. A of X^*) is called a *V^*-subset* of X (resp. a *V-subset* of X^*) provided that

$$\lim_n(\sup\{|x_n^*(x)| : x \in A\}) = 0$$

$$(\text{resp. } \lim_n(\sup\{|x^*(x_n)| : x^* \in A\}) = 0)$$

for each *wuc* series $\sum x_n^*$ in X^* (resp. *wuc* series $\sum x_n$ in X). A Banach space X has *property V* if every V-subset of X^* is relatively weakly com-

pact, and X has *property* V^* if every V^*-subset of X is relatively weakly compact [31].

Emmanuele [15] and Bombal [4] determined when (e_n^*) embeds as a V^*-sequence in X.

THEOREM 3.9. *If (x_n) is a sequence in X and $(x_n) \sim (e_n^*)$, then the following are equivalent*:

(1) *$\{x_n : n \geq 1\}$ is a V^*-set in X.*
(2) *If N is an infinite subset of $\mathbb{N}$, then $[x_n : n \in N]$ is not complemented in X.*

Since every limited subset of X is a DP subset of X, it is clear that every limited subset of X is weakly precompact. Theorem 4.2 of [3] points out that the class of bounded and weakly precompact sets is properly contained in the class of V^*-sets. Since every bounded subset of ℓ_∞ is a V-subset of ℓ_∞, it is immediate that any isomorphic copy of (e_n^*) contained in ℓ_∞ is a V-subset of ℓ_∞ which is not weakly precompact. Further, a direct interpretation of Theorem 3.10 in terms of V-sets fails: one can have $(x_n^*) \sim (e_n^*)$, (x_n^*) is a V-sequence in X^*, and $[x_n^* : n \in A]$ is complemented in X^* for all non-empty $A \subseteq \mathbb{N}$.

Pełczyński gave an operator-theoretic characterization of spaces X which have property V: *Every unconditionally converging operator with domain X is weakly compact iff every V-subset of X^* is relatively weakly compact.* In the next theorem, we give elementary operator-theoretic characterizations of weak precompactness, relative weak compactness, and relative norm compactness for both V-sets and V^*-sets. The argument contains the theorem in [31] just cited.

THEOREM 3.10.

(1) *The following are equivalent*:
  (1)(i) *If $T : Y \to X^*$ is an operator and $T^*|_X$ is unconditionally converging, then T is almost weakly compact (weakly compact, resp. compact).*
  (1)(ii) *Same as (1)(i) with $Y = \ell_1$.*
  (1)(iii) *Every V-subset of X^* is weakly precompact (relatively weakly compact, resp. relatively compact).*
(2) *The following are equivalent*:
  (2)(i) *If $T : Y \to X$ is an operator so that $T^* : X^* \to Y^*$ is unconditionally converging, then T is almost weakly compact (weakly compact, resp. compact).*
  (2)(ii) *Same as (2)(i) with $Y = \ell_1$.*

(2)(iii) *Every V^*-subset of X is weakly precompact (relatively weakly compact, resp. relatively compact).*

Proof. We will show that (1)(i)⇒(1)(ii)⇒(1)(iii)⇒(1)(i) in the weakly precompact case, as well as (2)(i)⇒(2)(ii)⇒(2)(iii)⇒(2)(i) in the compact case. These two arguments are similar, and the arguments for all the remaining implications in the theorem follow the same pattern.

(1) (*weakly precompact*) Certainly (i) implies (ii). Now suppose that A is a V-subset of X^*, and let (x_n^*) be a sequence in A. Define $T : \ell_1 \to X^*$ by $T(b) = \sum b_i x_i^*$. Since the closed absolutely convex hull of (x_i^*) is a V-subset of X^*, $T^*|_X$ is unconditionally converging. Thus T is almost weakly compact, and $(T(e_n^*)) = (x_n^*)$ has a weakly Cauchy subsequence. To see that (iii)⇒(i), note that $T(B_Y)$ is a V-subset of X^* iff $T^*|_X$ is unconditionally converging.

(2) (*compact*) Certainly (i) implies (ii). Suppose that A is a V^*-subset of X, and let (x_n) be a sequence in A. Define $T : \ell_1 \to X$ as above, and note that T^* is unconditionally converging. Thus T is compact, and (x_n) has a norm converging subsequence. To see that (iii)⇒(i), use (e) in the introduction. ∎

Theorem 3.10 facilitates a particularly quick proof of Theorems 1 and 1′ of [31].

COROLLARY 3.11. *If H is a compact Hausdorff space and X is reflexive, then $C(H, X)$ has property V.*

Proof. If $T : Y \to C(H, X)^*$ is an operator and $L = T^*|_{C(H,X)}$ is unconditionally converging, then L is strongly bounded [7], [37], [12]. Since X is reflexive, L is weakly compact ([7, Thm. 4.1]). Therefore T is weakly compact, and the desired conclusion follows from part (1) of the preceding theorem. ∎

Observations (c) and (d) in the introduction, the fact that the closed absolutely convex hull of a DP (resp. limited) subset of X is a DP (resp. limited) subset of X, and the arguments in 3.10 yield the next result.

THEOREM 3.12.

(1) *The following are equivalent*:

- (1)(i) *If $T : Y \to X$ is an operator and $T^* : X^* \to Y^*$ is completely continuous, then T is weakly compact (resp. compact).*
- (1)(ii) *Same as* (1)(i) *with $Y = \ell_1$.*
- (1)(iii) *Every DP subset of X is relatively weakly compact (resp. relatively compact).*

(2) *The following are equivalent*:

(2)(i) *If $T : Y \to X$ is an operator and $T^* : X^* \to Y^*$ is w^*-norm sequentially continuous, then T is weakly compact (resp. compact).*

(2)(ii) *Same as* (2)(i) *with $Y = \ell_1$.*

(2)(iii) *Every limited subset of X is relatively weakly compact (resp. relatively compact).*

COROLLARY 3.13 ([6, parts 5 and 6]). *If X is reflexive or separable, then every limited subset of X is relatively compact.*

Proof. In either case, a w^*-norm sequentially continuous adjoint $T^* : X^* \to Y^*$ must be compact. ∎

COROLLARY 3.14. *If X^* is isomorphic to a $C(K)$-space, then every DP subset of X is relatively weakly compact.*

Proof. If X^* is a $C(K)$-space and $T^* : X^* \to Y^*$ is completely continuous, then T^* is unconditionally converging and thus weakly compact [31]. ∎

4. Spaces of operators. This section will be primarily concerned with studying the following two (related) questions:

(A) Does a space of operators have a certain Dunford–Pettis type property when the codomain and the dual of the domain have this property?

(B) Does c_0 embed isomorphically into a space of operators?

If X is infinite-dimensional and $c_0 \hookrightarrow L(X,Y)$, then $\ell_\infty \hookrightarrow L(X,Y)$ [27], and thus $L(X,Y)$ must contain a limited copy of (e_n) as well as limited sets which are not relatively weakly compact. In 1.3.6 of [35], Schlumprecht showed that if X contains a copy of c_0—whether the corresponding copy of (e_n) is limited or not—then (e_n) does embed as a non-limited sequence in the least crossnorm tensor product completion $X \otimes_\lambda Y$ of X and Y. We say that X has (DPrcP) if every DP subset of X is relatively compact [20]; X has RDP* (or $X \in$ (RDP*)) if every DP subset of X is relatively weakly compact [3]; X is a *Gelfand–Phillips space* if every limited subset of X is relatively compact, and X has the (BD) *property* if every limited subset of X is relatively weakly compact. If $\ell_1 \not\hookrightarrow X$, then X has the (BD) property [6]. X has RDP* whenever X has V^* or X is weakly sequentially complete [3]. Also, X^* has RDP* whenever X has V. If $c_0 \hookrightarrow L(X,Y)$, then $L(X,Y)$ fails to belong to any of the classes of spaces just defined.

One can find numerous references to papers which study the embedability of c_0 in spaces of operators in [19], [18], and [27]. Specifically we note that [19], [18], and [27] point out that if the Banach space X has an unconditional finite-dimensional decomposition, then $c_0 \hookrightarrow K(X,X)$. However, is c_0 com-

plemented in these spaces? Emmanuele gave a positive answer in Theorem 18 of [21].

THEOREM 4.1. *Let c_0 embed in $K_{w^*}(X^*, Y)$. If Y is a Gelfand–Phillips space, then c_0 embeds complementably into $K_{w^*}(X^*, Y)$.*

The techniques of Schlumprecht [35, Corollary 1.3.6] which were used to prove Theorem 4.1 can also be employed with different hypotheses to produce complemented copies of c_0 in $K_{w^*}(X^*, Y)$.

THEOREM 4.2. *Suppose that X and Y are infinite-dimensional Banach spaces.*

(i) *If $c_0 \hookrightarrow X$ or $c_0 \hookrightarrow Y$, then $K_{w^*}(X^*, Y)$ contains a copy of c_0 which is complemented in $K_{w^*}(X^*, Y)$.*
(ii) *If $c_0 \not\hookrightarrow X$, B_{Y^*} is w^*-sequentially compact, and $c_0 \hookrightarrow K_{w^*}(X^*, Y)$, then c_0 is complemented in $K_{w^*}(X^*, Y)$.*

For completeness, we give a proof of Theorem 4.2.

Proof. (i) Let (x_n) be a sequence of normalized vectors in X so that $(x_n) \sim (e_n)$. Let (x_n^*) be a sequence of norm one functionals in X^* so that $x_n^*(x_n) = \|x_n\|$ for each n, and suppose that (y_n^*) is a w^*-null sequence of norm one in Y^*. For each $n \in \mathbb{N}$, choose $y_n \in Y$ so that $1 \leq \|y_n\| \leq 2$ and $y_n^*(y_n) = 1$. Let $T \in K_{w^*}(X^*, Y)$, and observe that T^* is also w^*-w continuous and compact. Therefore $\|T^*(y_n^*)\| \to 0$. Consequently, if one defines $x_n^* \otimes y_n^*(T)$ to be $y_n^*(T(x_n^*)) = \langle T^*(y_n^*), x_n^* \rangle$, then $(x_n^* \otimes y_n^*)$ is a w^*-null sequence in the dual of $K_{w^*}(X^*, Y)$. Moreover, $(x_n \otimes y_n)$ has a natural interpretation as a sequence in $K_{w^*}(X^*, Y)$ [11], and then the argument in 1.3.6 of [35] shows that $(x_n \otimes y_n) \sim (e_n)$. Now $\langle x_n^* \otimes y_n^*, x_n \otimes y_n \rangle = x_n^*(x_n) y_n^*(y_n) = \|x_n\| \not\to 0$, and $(x_n \otimes y_n)$ is not limited in $K_{w^*}(X^*, Y)$. Note that we obtain the same construction if we assume that $c_0 \hookrightarrow Y$. An application of Theorem 3.6 gives that $[(x_n \otimes y_n)]$ is complemented in $K_{w^*}(X^*, Y)$.

(ii) Suppose that $(T_n) \sim (e_n)$ in $K_{w^*}(X^*, Y)$. Then $\sum T_n$ is weakly unconditionally convergent. Without loss of generality, suppose that $\|T_n\| = 1$ for each n, and let (x_n^*) be a sequence in B_{X^*} so that $||T_n(x_n^*)|| > 1/2$. Now let (y_n^*) be a sequence in B_{Y^*} so that

$$\langle y_n^*, T_n(x_n^*) \rangle = \|T_n(x_n^*)\| > 1/2.$$

Further, since B_{Y^*} is w^*-sequentially compact, we may assume that $y_n^* \overset{w^*}{\to} y^*$. Now use the fact that $\sum T_n$ is weakly unconditionally convergent to see that $\sum T_n^*(y^*)$ is weakly unconditionally convergent—and thus unconditionally convergent—in X. Therefore $\langle y^*, T_n(x_n^*) \rangle \leq \|T_n^*(y^*)\| \to 0$. Consequently,

$$\langle y_n^* - y^*, T_n(x_n^*) \rangle = \langle y_n^*, T_n(x_n^*) \rangle - \langle y^*, T_n(x_n^*) \rangle \geq 1/2 - 1/4$$

for large n. Specifically,

$$\langle T_n, x_n^* \otimes (y_n^* - y^*)\rangle \not\to 0.$$

Since $(x_n^* \otimes (y_n^* - y^*))$ is w^*-null in the dual of $K_{w^*}(X^*, Y)$, (T_n) is not limited in this space of operators. Apply Theorem 3.6. ∎

REMARK. Theorems 3.7, 4.2, and 3.6 (Schlumprecht's result) make it clear that a Banach space X may contain a copy of (e_n) which is limited in X as well as a copy which fails to be limited in X. In particular, if there are copies of (e_n) in X (or Y) which are limited in X (or Y), then the natural isometric copies of these spaces in $K_{w^*}(X^*, Y)$ will produce copies of (e_n) which are limited in this space of operators. The complemented copy of (e_n) is not limited in $K_{w^*}(X^*, Y)$. Since $K(X, Y) \cong K_{w^*}(X^{**}, Y)$, notice also that if $c_0 \not\hookrightarrow X^*$, B_{Y^*} is w^*-sequentially compact, and $c_0 \hookrightarrow K(X, Y)$, then $c_0 \overset{c}{\hookrightarrow} K(X, Y)$. Consequently, $c_0 \overset{c}{\hookrightarrow} K(\ell_2, \ell_2)$, and thus (e_n) embeds as a non-limited sequence in $K(\ell_2, \ell_2)$.

Recall that X has the Schur property precisely when weak and norm convergence of sequences in X coincide. Lust [30] and Ryan [34] showed that $L_{w^*}(X^*, Y)$ (resp. $L(X, Y)$) has the Schur property if and only if X and Y (resp. X^* and Y) have the Schur property. Thus $L(c_0, \ell_1)$ is a Schur space, and $c_0 \not\hookrightarrow L(c_0, \ell_1)$. Note also that $L(c_0, \ell_1) = K(c_0, \ell_1)$. This equality is common to all spaces $L(X, Y)$ which have the Schur property. We give a brief argument for completeness.

THEOREM 4.3. *If $L(X, Y)$ is a Schur space, then $L(X, Y) = K(X, Y)$.*

Proof. Suppose that $T \in L(X, Y)$ and T is not compact. Since Y has the Schur property, T cannot be almost weakly compact. Thus there must be a sequence in B_X without a weakly Cauchy subsequence, and Rosenthal's ℓ_1-theorem guarantees that $\ell_1 \hookrightarrow X$. Therefore $L_1 \hookrightarrow X^*$ [10, p. 212], and X^* cannot be a Schur space. ∎

We remark that the proof of Theorem 4.3 could have been given using the fact that X^* has the Schur property iff X has the DPP and $\ell_1 \not\hookrightarrow X$.

In [23] we proved the following theorem.

THEOREM 4.4. *If $L_{w^*}(E^*, F) = K_{w^*}(E^*, F)$ and E and F are weakly sequentially complete, then $K_{w^*}(E^*, F)$ is weakly sequentially complete.*

As a consequence, we deduce that $K_{w^*}(E^*, F)$ has RDP* if $L_{w^*}(E^*, F) = K_{w^*}(E^*, F)$ and both E and F are weakly sequentially complete.

Next we investigate sufficient conditions for a subset of $K_{w^*}(X^*, Y)$ to be relatively weakly compact. If $H \subseteq L_{w^*}(X^*, Y)$, $x^* \in X^*$ and $y^* \in Y^*$, let $H(x^*) = \{T(x^*) : T \in H\}$ and $H^*(y^*) = \{T^*(y^*) : T \in H\}$. We begin with three lemmas which are similar to results in [26]. In Corollary 2 of [26], Kalton used these results to show that if X and Y are reflexive

and $L(X,Y) = K(X,Y)$, then $K(X,Y)$ is reflexive. Kalton's lemmas are extended to the w^*-w operator case. Theorem 4.8 below extends Corollary 2 of [26].

Let U denote the unit ball of X^* with the w^* topology and V denote the unit ball of Y^* with the w^* topology. For T in $L_{w^*}(X^*,Y)$, define $\chi_T : U \times V \to \mathbb{R}$ by $\chi_T(x^*,y^*) = y^*(Tx^*)$.

LEMMA 4.5. *The mapping $T \mapsto \chi_T$ defines a linear isometry of the space $K_{w^*}(X^*,Y)$ onto a closed subspace of $C(U \times V)$.*

Proof. Suppose $x^*_\alpha \overset{w^*}{\to} x^* \in U$ and $y^*_\alpha \overset{w^*}{\to} y^* \in V$. We have

$$\begin{aligned}|\chi_T(x^*_\alpha,y^*_\alpha) - \chi_T(x^*,y^*)| &= |y^*_\alpha(Tx^*_\alpha) - y^*(Tx^*)| \\ &\le |y^*_\alpha(Tx^*_\alpha - Tx^*)| + |(y^*_\alpha - y^*)(Tx^*)| \\ &\le \|Tx^*_\alpha - Tx^*\| + |(y^*_\alpha - y^*)(Tx^*)|.\end{aligned}$$

Since T is w^*-w continuous and compact, T is w^*-norm continuous, and thus $\|Tx^*_\alpha - Tx^*\| \to 0$. Also, $|(y^*_\alpha - y^*)(Tx^*)| \to 0$ because $y^*_\alpha \overset{w^*}{\to} y^*$. Thus $\chi_T(x^*_\alpha,y^*_\alpha) \to \chi_T(x^*,y^*)$ and $\chi_T \in C(U \times V)$. Since $\|\chi_T\| = \|T\|$ and $T \mapsto \chi_T$ is linear, the conclusion follows. ∎

Let *wot* denote the weak operator topology on $L(X,Y)$: $T_n \to T$ (*wot*) provided that $\langle T_n(x), y^*\rangle \to \langle T(x), y^*\rangle$ for all $x \in X$ and $y^* \in Y^*$. Following [26], we let w' denote the dual weak operator topology defined by the functionals $T \mapsto x^{**}T^*(y^*)$, $x^{**} \in X^{**}$, $y^* \in Y^*$.

LEMMA 4.6. *Let A be a subset of $K_{w^*}(X^*,Y)$. Then A is weakly compact if and only if A is wot-compact.*

Proof. Suppose A is *wot*-compact and let $\chi(A)$ be $\{\chi_T : T \in A\}$. Let (T_α) be a net in A and (T_β) be a *wot*-convergent subnet. If $T_\beta \to T$ (*wot*), then $\chi_{T_\beta}(x^*,y^*) \to \chi_T(x^*,y^*)$ for all $x^* \in X^*$ and $y^* \in Y^*$, and $\chi(A)$ is compact in the topology of pointwise convergence on $C(U \times V)$. By Grothendieck's result [24], $\chi(A)$ is weakly compact in $C(U \times V)$, and A is weakly compact by the preceding result.

The other implication is clear since *wot* is weaker than the weak topology on $K_{w^*}(X^*,Y)$. ∎

LEMMA 4.7. *Let (T_n) be a sequence of w^*-w continuous compact operators such that $T_n \to T$ (wot), where T is w^*-w continuous and compact. Then $T_n \to T$ weakly.*

Proof. Let $A = \{(T_n), T\}$ and apply the previous lemma to this *wot*-compact set. ∎

Next we apply 4.5–4.7 to obtain an extension of Corollary 2 of [26].

THEOREM 4.8. *If $H \subseteq L_{w^*}(E^*, F)$, $H(x^*)$ is relatively weakly compact for each $x^* \in E^*$, $H^*(y^*)$ is relatively weakly compact for each $y^* \in F^*$, and $L_{w^*}(E^*, F) = K_{w^*}(E^*, F)$, then H is relatively weakly compact.*

Proof. Let (h_n) be a sequence in H, and let $S = [h_n(E^*) : n \geq 1]$. The compactness of each h_n implies that S is separable. Therefore (B_{S^*}, w^*) is a compact metric space. Let $Y = (y_n^*)$ be a w^*-dense sequence in S^*. By hypotheses, $\{h_n^*(y_i^*) : n \geq 1\}$ is relatively weakly compact for each i. By diagonalization, we may (and do) assume that (h_{n_i}) is a subsequence of (h_n) so that $(h_{n_i}^*(y_k^*))_{i=1}^{\infty}$ is weakly convergent for each k. In fact, without loss of generality, we assume that $(h_n^*(y^*))$ is weakly convergent for each $y^* \in Y$.

Now consider $(h_n(x^*))$, where $x^* \in E^*$. This sequence must have a weakly convergent subsequence. Suppose that z_1 and z_2 are weak sequential cluster points of this sequence. Certainly $z_1, z_2 \in S$. Suppose that $h_{k(n)}(x^*) \xrightarrow{w} z_1$, $h_{p(n)}(x^*) \xrightarrow{w} z_2$, and $y^* \in Y$. Now

$$\begin{aligned}\langle z_1, y^* \rangle &= \lim_n \langle h_{k(n)}(x^*), y^* \rangle = \lim_n \langle x^*, h_{k(n)}^*(y^*) \rangle \\ &= \lim_n \langle x^*, h_{p(n)}^*(y^*) \rangle = \lim_n \langle h_{p(n)}(x^*), y^* \rangle = \langle z_2, y^* \rangle,\end{aligned}$$

and $z_1 = z_2$ since Y is w^*-dense in S^*. Therefore $(h_n(x^*))$ is w-convergent for all $x^* \in E^*$. Set

$$h(x^*) = w\text{-}\lim h_n(x^*) \quad \text{for } x^* \in E^*.$$

Next suppose that $y^* \in F^*$ and consider $h^*(y^*)$. Without loss of generality, suppose that $h_n^*(y^*) \xrightarrow{w} x \in E$, and let $x^* \in E^*$. Then $\langle h(x^*), y^* \rangle = \lim_n \langle h_n(x^*), y^* \rangle = \lim_n \langle x^*, h_n^*(y^*) \rangle = x^*(x)$. Thus $h^*(y^*) = x$, and h is w^*-w continuous. Therefore h is compact and w^*-w continuous. Lemma 4.7 then shows that $h_n \xrightarrow{w} h$. ∎

COROLLARY 4.9. *If E and F are reflexive and $L(E, F) = K(E, F)$, then $K(E, F)$ is reflexive.*

The assumption that H is weakly precompact allows an interpretation of Theorem 4.8 for the space $L(X, Y)$.

THEOREM 4.10. *Suppose that $K(X, Y) = L(X, Y)$ and H is a weakly precompact subset of $K(X, Y)$ so that*

(i) *$H(x)$ is relatively weakly compact for all $x \in X$,*
(ii) *$H^*(y^*)$ is relatively weakly compact for all $y^* \in Y^*$.*

Then H is relatively weakly compact.

Proof. Let (T_n) be a sequence in H. Without loss of generality, suppose that (T_n) is weakly Cauchy, and let $x \in X$. Since $(T_n(x))$ has a weakly convergent subsequence and it is weakly Cauchy, it follows that it is weakly

convergent. Similarly, $(T_n^*(y^*))$ is weakly convergent for all $y^* \in Y^*$. Let $T \in L(X,Y)$ be such that $T_n(x) \xrightarrow{w} T(x)$ for all $x \in X$.

Since $T_n^*(y^*) \xrightarrow{w^*} T^*(y^*)$ and $\{T_n^*(y^*) : n \geq 1\}$ is relatively weakly compact, one can show that $T_n^*(y^*) \xrightarrow{w} T^*(y^*)$, i.e. $T_n \xrightarrow{w'} T$ in $K(X,Y)$. By Corollary 3 of Kalton [26], we obtain $T_n \to T$ weakly. Thus H is relatively weakly compact. ∎

COROLLARY 4.11. *Suppose that $K_{w^*}(X^*,Y) = L_{w^*}(X^*,Y)$. If both X and Y have property RDP* (resp. the (BD) property), then $K_{w^*}(X^*,Y)$ has property RDP* (resp. the (BD) property).*

Proof. We present the proof of the result relative to the RDP* property; the other case is similar. Let H be a DP set in $K_{w^*}(X^*,Y)$. For each $x^* \in X^*$, $H(x^*)$ is a DP set in Y, thus relatively weakly compact. Similarly $H^*(y^*)$ is relatively weakly compact for each $y^* \in Y^*$. Apply Theorem 4.8. ∎

COROLLARY 4.12. *Suppose that $L(X,Y) = K(X,Y)$ and both X^* and Y have RDP* (resp. the (BD) property). Then $K(X,Y)$ has RDP* (resp. the (BD) property).*

COROLLARY 4.13. *Suppose that $L(X,Y^*) = K(X,Y^*)$ and both X^* and Y^* have RDP*. Then $K(X,Y^*)$ has RDP* and $\ell_1 \overset{c}{\not\hookrightarrow} X \otimes_\gamma Y$.*

Proof. $K(X,Y^*)$ has RDP* by 4.12. Note that $c_0 \not\hookrightarrow (X \otimes_\gamma Y)^*$ since $L(X,Y^*) \cong (X \otimes_\gamma Y)^*$ has RDP* (c_0 does not have RDP* since $(z_n) = (\sum_{i=1}^n e_i)$ is a DP set which is not relatively weakly compact). By a result of Bessaga and Pełczyński, $\ell_1 \overset{c}{\not\hookrightarrow} X \otimes_\gamma Y$. ∎

The next three theorems continue to concentrate on conditions which ensure that spaces of operators have the RDP* or the (BD) property.

THEOREM 4.14. *Suppose that $L(X,Y^*) = K(X,Y^*)$ and Y^* has RDP*. The following statements are equivalent*:

(i) *X^* has RDP* and $\ell_1 \not\hookrightarrow X$ or $\ell_1 \not\hookrightarrow Y$.*
(ii) *$K(X,Y^*)$ has RDP*.*

Proof. Suppose first that X^* has RDP* and $\ell_1 \not\hookrightarrow X$ or $\ell_1 \not\hookrightarrow Y$. By 4.13, $K(X,Y^*)$ has RDP*.

Now suppose that $K(X,Y^*)$ has RDP*. Then $K(Y,X^*)$ has RDP* since $K(Y,X^*) \cong K(X,Y^*)$. Thus X^* has RDP* since the property RDP* is inherited by subspaces. We will show that $\ell_1 \not\hookrightarrow X$ or $\ell_1 \not\hookrightarrow Y$. Suppose that $\ell_1 \hookrightarrow X$ and $\ell_1 \hookrightarrow Y$. Hence L^1 and thus ℓ_2 embeds in X^* and similarly ℓ_2 embeds in Y^*. Hence $\ell_2 \otimes_\lambda \ell_2$ is isomorphic to a subspace of $L(X,Y^*) = K(X,Y^*)$ and thus $\ell_2 \otimes_\lambda \ell_2$ has RDP*. By Lemma 3.2 of [23] (as well as a result of Emmanuele [18]), $c_0 \hookrightarrow \ell_2 \otimes_\lambda \ell_2$, and we have a contradiction since c_0 fails RDP*. ∎

THEOREM 4.15.

(i) *Suppose that X has the Schur property and Y has RDP*. Then $L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$ has RDP* and $c_0 \not\hookrightarrow K_{w^*}(X^*, Y)$.*

(ii) *Suppose that X has RDP* and Y has the Schur property. Then $L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$ has RDP* and $c_0 \not\hookrightarrow K_{w^*}(X^*, Y)$.*

(iii) *Suppose that X has the Schur property and Y has the (BD) property. Then $L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$ has the (BD) property.*

(iv) *Suppose that X has the (BD) property and Y has the Schur property. Then $L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$ has the (BD) property.*

Proof. We present the proof of (i); the other cases are similar. Let $T \in L_{w^*}(X^*, Y)$. Then T is weakly compact since T^* is w^*-w continuous. Hence T^* is compact since X is a Schur space. An application of 4.11 shows that $K_{w^*}(X^*, Y)$ has RDP*. ∎

THEOREM 4.16.

(i) *Suppose that X^* is a Schur space and Y has RDP*. Then $K(X, Y) = L(X, Y)$ has RDP* and $c_0 \not\hookrightarrow K(X, Y)$. Moreover, if $Y = Z^*$, then $\ell_1 \overset{\mathcal{C}}{\not\hookrightarrow} X \otimes_\gamma Z$.*

(ii) *Suppose that X^* has RDP* and Y is a Schur space. Then $K(X, Y) = L(X, Y)$ has RDP* and $c_0 \not\hookrightarrow K(X, Y)$. Moreover, if $Y = Z^*$, then $\ell_1 \overset{\mathcal{C}}{\not\hookrightarrow} X \otimes_\gamma Z$.*

(iii) *Suppose that X^* is a Schur space and Y has the (BD) property. Then $K(X, Y) = L(X, Y)$ has the (BD) property.*

(iv) *Suppose that X^* has the (BD) property and Y is a Schur space. Then $K(X, Y) = L(X, Y)$ has the (BD) property.*

Proof. We present the proofs for (i) and (ii) only.

Let $T \in L(X, Y)$. Since X^* is a Schur space, T^* is completely continuous, and $T(B_X)$ is a DP set in Y, thus relatively weakly compact. Then T and hence T^* is weakly compact, and therefore compact because X^* has the Schur property. By 4.12, $K(X, Y) = L(X, Y)$ has RDP* and $c_0 \not\hookrightarrow K(X, Y)$.

Suppose now that $Y = Z^*$. Then $c_0 \not\hookrightarrow K(X, Z^*) \cong (X \otimes_\gamma Z)^*$ and thus $\ell_1 \overset{\mathcal{C}}{\not\hookrightarrow} X \otimes_\gamma Z$. Another application of 4.12 gives (ii). ∎

REMARK. Theorems 4.15 and 4.16 provide examples of spaces of operators which contain ℓ_1 and have the (BD) property.

COROLLARY 4.17. *Suppose that X has RDP* (resp. the (BD) property). Then the space $l^1[X]$ of all unconditionally convergent series in X with the norm*

$$||(x_n)|| = \sup\Big\{\sum |x^*(x_n)| : x^* \in B_{X^*}\Big\}$$

has RDP (resp. the (BD) property).*

Proof. It is known that $\ell_1[X]$ is isometrically isomorphic to $K(c_0, X)$ [20]. Since X has RDP* and $c_0^* = \ell_1$ has the Schur property, Theorem 4.16 gives the conclusion. ∎

The fact that the continuous linear image of a Dunford–Pettis (resp. limited) set is Dunford–Pettis (resp. limited) produces the following result for quotient spaces.

THEOREM 4.18. *If X^* has RDP* (resp. the (BD) property) and Z is a quotient of X, then Z^* has RDP* (resp. the (BD) property).*

Proof. Let $Q : X \to Z$ be a quotient map. Then $Q^* : Z^* \to X^*$ is an isomorphism. If K is a DP subset of Z^*, then $Q^*(K)$ is a DP subset of X^*. Thus $Q^*(K)$ and K must be relatively weakly compact. ∎

COROLLARY 4.19. *Suppose that $L(X^*, Y^*) = K(X^*, Y^*)$ and both X^{**} and Y^* have RDP*. Then the dual of the space $N_1(X, Y)$ of all nuclear operators has RDP* and hence $\ell_1 \overset{c}{\not\hookrightarrow} N_1(X, Y)$.*

Proof. It is known that $N_1(X, Y)$ is a quotient of $X^* \otimes_\gamma Y$ ([16]). By Corollary 4.13, $(X^* \otimes_\gamma Y)^* \cong L(X^*, Y^*)$ has RDP*. An application of 4.18 gives that $(N_1(X, Y))^*$ has RDP*. Thus $c_0 \not\hookrightarrow (N_1(X, Y))^*$ and $\ell_1 \overset{c}{\not\hookrightarrow} N_1(X, Y)$. ∎

Feder [22] and Emmanuele [19] asked if there exist spaces X and Y so that $c_0 \not\hookrightarrow K(X, Y)$ yet $L(X, Y) \neq K(X, Y)$. Emmanuele gave a positive solution to this question in the concluding "added in proof" remark in [19] using a fundamental space created by Bourgain and Delbaen [5]. Results of this note highlight additional properties of the Bourgain–Delbaen spaces as well as operators defined on these spaces. We denote the two spaces by $\mathcal{X}$ and $\mathcal{Y}$. Both $\mathcal{X}$ and $\mathcal{Y}$ are infinite-dimensional spaces with the Dunford–Pettis property. The space $\mathcal{X}$ is a separable $\mathcal{L}_\infty$-space which is a Schur space, and both $\mathcal{X}$ and $\mathcal{X}^*$ are weakly sequentially complete. Since $\mathcal{X}$ is a Schur space and $\mathcal{X}^*$ is weakly sequentially complete, $\mathcal{X}$ contains a copy of (e_n^*) no subsequence of which can be complemented in $\mathcal{X}$. Thus by Theorem 3.9, (e_n^*) embeds as a V^*-sequence in $\mathcal{X}$.

The space $\mathcal{Y}$ is a separable $\mathcal{L}_\infty$-space which is somewhat reflexive. Further, $\mathcal{Y}^*$ is isomorphic to ℓ_1. Consequently, there is a sequence (y_n^*) in $\mathcal{Y}^*$ so that $(y_n^*) \sim (e_n^*)$, (y_n^*) is a V-sequence in $\mathcal{Y}^*$, and $[y_n^* : n \in A]$ is complemented in $\mathcal{Y}^*$ for each non-empty $A \subseteq \mathbb{N}$.

If the Banach spaces Z and Z^* are weakly sequentially complete and

$$L_{w^*}(Z^{**}, Z) = K_{w^*}(Z^{**}, Z),$$

then $K_{w^*}(Z^{**}, Z)$ is weakly sequentially complete. Since

$$K(Z, Z) \cong K_{w^*}(Z^{**}, Z),$$

$K(Z,Z)$ is weakly sequentially complete. If Z is infinite-dimensional, then certainly $L(Z,Z) \neq K(Z,Z)$ and $c_0 \not\hookrightarrow K(Z,Z)$. Setting $Z = \mathcal{X}$, one obtains Emmanuele's solution [19].

Alternatively, suppose that X^* is weakly sequentially complete, Y is a Schur space, and $L(X,Y) \neq K(X,Y)$. Then $K(X,Y) \cong K_{w^*}(X^{**},Y) = L_{w^*}(X^{**},Y)$, and 4.4 implies that $L_{w^*}(X^{**},Y)$ is weakly sequentially complete. Thus $c_0 \not\hookrightarrow K(X,Y)$.

If $\ell_1 \not\hookrightarrow E$ and F is arbitrary, then every operator $T : E \to F$ is almost weakly compact. If F is weakly sequentially complete, then the operator T is weakly compact. If F is Schur, then T is compact. Consequently, if F is Schur, then $L(\mathcal{Y},F) = K(\mathcal{Y},F)$. Specifically, note that $L(\mathcal{Y},\mathcal{X}) = K(\mathcal{Y},\mathcal{X})$. Further, an immediate application of the Lust–Ryan theorem stated earlier shows that $L(\mathcal{Y},\mathcal{X})$ is a Schur space and thus is weakly sequentially complete. Consequently,

$$c_0 \not\hookrightarrow L(\mathcal{Y},\mathcal{X}).$$

Moreover, another application of Theorem 4.4 shows that

$$K_{w^*}(\mathcal{X}^{**}, L(\mathcal{Y},\mathcal{X})) = L_{w^*}(\mathcal{X}^{**}, L(\mathcal{Y},\mathcal{X}))$$

is weakly sequentially complete. Therefore

$$c_0 \not\hookrightarrow K(\mathcal{X}, L(\mathcal{Y},\mathcal{X})) \cong K_{w^*}(\mathcal{X}^{**}, L(\mathcal{Y},\mathcal{X})).$$

Since $\mathcal{X}$ embeds isometrically in $L(\mathcal{Y},\mathcal{X})$,

$$L(\mathcal{X}, L(\mathcal{Y},\mathcal{X})) \neq K(\mathcal{X}, L(\mathcal{Y},\mathcal{X})),$$

and we have another example of a solution to the Feder–Emmanuele problem discussed previously.

The proof of Theorem 4.14 made use of the fact that $c_0 \hookrightarrow \ell_2 \otimes_\lambda \ell_2$ and Corollary 1.3.6 of [35] showed that if $c_0 \hookrightarrow X$, then c_0 is complemented in $X \otimes_\lambda X$. Certainly $c_0 \not\hookrightarrow \mathcal{X}$, and the Lust–Ryan theorem (or Theorem 4.4) easily shows that the least crossnorm tensor product completion of any Schur space with itself does not contain c_0. Specifically, $\mathcal{X} \otimes_\lambda \mathcal{X} \hookrightarrow L_{w^*}(\mathcal{X}^*,\mathcal{X})$, and Theorem 4.4 shows that this space of operators is weakly sequentially complete, while the Lust–Ryan result shows even more. This space of operators is a Schur space; thus $\mathcal{X} \otimes_\lambda \mathcal{X}$ is a Schur space.

References

[1] K. Andrews, *Dunford–Pettis sets in the space of Bochner integrable functions*, Math. Ann. 241 (1979), 35–41.

[2] E. M. Bator, *Remarks on completely continuous operators*, Bull. Polish Acad. Sci. Math. 37 (1989), 409–413.

[3] E. Bator, P. Lewis, and J. Ochoa, *Evaluation maps, restriction maps, and compactness*, Colloq. Math. 78 (1998), 1–17.

[4] F. Bombal, *On V^* sets and Pełczyński's property V^**, Glasgow Math. J. 32 (1990), 109–120.

[5] J. Bourgain and F. Delbaen, *A class of special $\mathcal{L}_\infty$ spaces*, Acta Math. 145 (1981), 155–176.

[6] J. Bourgain and J. Diestel, *Limited operators and strict cosingularity*, Math. Nachr. 119 (1984), 55–58

[7] J. Brooks and P. Lewis, *Linear operators and vector measures*, Trans. Amer. Math. Soc. 192 (1974), 139–162.

[8] W. J. Davis, D. W. Dean, and B.-L. Lin, *Bibasic sequences and norming basic sequences*, Trans. Amer. Math. Soc. 176 (1974), 89–102.

[9] J. Diestel, *A survey of results related to the Dunford–Pettis property*, in: Contemp. Math. 2, Amer. Math. Soc., 1980, 15–60.

[10] —, *Sequences and Series in Banach Spaces*, Grad. Texts in Math. 92, Springer, Berlin, 1984.

[11] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., 1977.

[12] I. Dobrakov, *On representation of linear operators on $C_0(T, X)$*, Czechoslovak Math. J. 21 (1971), 13–30.

[13] L. Drewnowski and G. Emmanuele, *On Banach spaces with the Gelfand–Phillips property, II*, Rend. Circ. Mat. Palermo 38 (1989), 377–391.

[14] G. Emmanuele, *A dual characterization of Banach spaces not containing ℓ^1*, Bull. Polish Acad. Sci. Math. 34 (1986), 155–160.

[15] —, *On Banach spaces with the property (V^*) of Pełczyński*, Ann. Mat. Pura Appl. 152 (1988), 171–181.

[16] —, *The reciprocal Dunford–Pettis property and projective tensor products*, Math. Proc. Cambridge Philos. Soc. 109 (1991), 161–166.

[17] —, *On Banach spaces containing complemented copies of c_0*, Extracta Math. 3 (1988), 98–100.

[18] —, *On the containment of c_0 in spaces of compact operators*, Bull. Sci. Math. 115 (1991), 177–184.

[19] —, *A remark on the containment of c_0 in spaces of compact operators*, Math. Proc. Cambridge Philos. Soc. 111 (1992), 331–335.

[20] —, *Banach spaces in which Dunford–Pettis sets are relatively compact*, Arch. Math. (Basel) 58 (1992), 477–485.

[21] —, *About the position of $K_{w^*}(E^*, F)$ inside $L_{w^*}(E^*, F)$*, Atti Sem. Mat. Fis. Univ. Modena 42 (1994), 123–133.

[22] M. Feder, *On the nonexistence of a projection onto the space of compact operators*, Canad. Math. Bull. 25 (1982), 78–81.

[23] I. Ghenciu and P. Lewis, *Strong Dunford–Pettis sets and spaces of operators*, Monatsh. Math. 144 (2005), 275–284.

[24] A. Grothendieck, *Critères de compacité dans les espaces fonctionnels généraux*, Amer. J. Math. 74 (1952), 168–186.

[25] R. Haydon, *A non-reflexive Grothendieck space that does not contain ℓ_∞*, Israel J. Math. 40 (1981), 65–73.

[26] N. Kalton, *Spaces of compact operators* Math. Ann. 208 (1974), 267–278.

[27] P. Lewis, *Spaces of operators and c_0*, Studia Math. 145 (2001), 213–218.

[28] —, *Dunford–Pettis sets*, Proc. Amer. Math. Soc. 129 (2001), 3297–3302.

[29] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, I*, Ergeb. Math. Grenzgeb. 92, Springer, Berlin, 1977.

[30] F. Lust, *Produits tensoriels injectifs d'espaces de Sidon*, Colloq. Math. 32 (1975), 286–289.

[31] A. Pełczyński, *Banach spaces on which every unconditionally converging operator is weakly compact*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 10 (1962), 641–648.

[32] A. Pietsch, *Absolut p-summierende Abbildungen in normierten Räumen*, Studia Math. 28 (1966/1967), 333–353.

[33] H. Rosenthal, *Pointwise compact subsets of the first Baire class*, Amer. J. Math. 99 (1977), 362–377.

[34] R. Ryan, *The Dunford–Pettis property and projective tensor products*, Bull. Polish Acad. Sci. Math. 35 (1987), 785–792.

[35] T. Schlumprecht, *Limited sets in Banach spaces*, Dissertation, Munich, 1987.

[36] I. Singer, *Bases in Banach Spaces, II*, Springer, 1981.

[37] C. Swartz, *Unconditionally converging operators on the space of continuous functions*, Rev. Roumaine Math. Pures Appl. 17 (1973), 123–131.

Ioana Ghenciu
Department of Mathematics
University of Wisconsin at River Falls
River Falls, WI 54022, U.S.A.
E-mail: ioana.ghenciu@uwrf.edu

Paul Lewis
Department of Mathematics
University of North Texas
Box 311430
Denton, TX 76203-1430, U.S.A.
E-mail: lewis@unt.edu

Received June 9, 2006;
received in final form November 22, 2006 (7532)