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FUNCTIONAL ANALYSIS

Intertwining Multiplication Operators on Function Spaces

by

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Summary. Suppose that X is a Banach space of analytic functions on a plane domain Ω. We characterize the operators T that intertwine with the multiplication operators acting on X.

Introduction. Suppose that the set of analytic polynomials is dense in a Banach space X of functions analytic on a plane domain Ω, and suppose that for each $\lambda \in \Omega$ the linear functional of evaluation at λ, $e(\lambda)$, is bounded on X. We further assume that X contains the constant functions and that multiplication by the independent variable z defines a bounded linear operator M_z on X. Also, we assume that for any fixed $n \in \mathbb{N}$, every f in X has a unique decomposition $f = \bigoplus_{i=0}^{n-1} f_i$ where $f_i \in X_i$ and X_i is the closed linear span of the set $\{z^{nk+i} : k \geq 0\}$ in X for $i = 0, 1, \ldots, n-1$.

The weighted Hardy spaces, $H^p(\beta)$, are examples of such spaces. For more information on such spaces see [5, 7, 8].

Throughout this paper, by a Banach space of analytic functions on a plane domain Ω we mean one satisfying the above conditions. For some results on such spaces see [1, 2, 3, 6, 10].

A complex-valued function φ on Ω for which $\varphi f \in X$ for every $f \in X$ is called a *multiplier* of X. Every multiplier φ of X determines a multiplication operator M_φ on X by $M_\varphi f = \varphi f$, $f \in X$. The set of all multipliers of X is denoted by $M(X)$. Clearly $M(X) \subset H^\infty(\Omega)$, where $H^\infty(\Omega)$ is the space of all bounded analytic functions on Ω. In fact $\|\varphi\|_\infty \leq \|M_\varphi\|$. A good source on this topic is [4].

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Let $B(X)$ be the set of all bounded operators on X. If $T \in B(X)$ and $TM_\varphi = -M_\varphi T$ or $TM_{\varphi^2} = M_{\varphi^2} T$ where $\varphi \in H^\infty(\Omega)$, then under suitable conditions the structure of T was determined in [9]. In this paper we want to characterize the operators T satisfying $TM_\varphi = M_\psi T$ where φ and ψ are multipliers of X. Specially, the case $\psi = a\varphi$ is considered.

Main results. In this section we will characterize the structure of operators that intertwine with the multiplication operators acting on function spaces.

Note that by our assumptions each $f \in X$ has a unique decomposition $f = \bigoplus_{i=0}^{n-1} f_i$ where $f_i \in X_i$ and X_i is the closed linear span of the set $\{z^{nk+i} : k \geq 0\}$ in X for $i = 0, \ldots, n-1$. From now on suppose that X is a Banach space of analytic functions on the open unit disc U and $0 < |a| \leq 1$. Assume further that the composition operators C_φ and $C_{a\varphi}$ are bounded on X where φ is a multiplier of X.

Theorem 1. *Suppose that φ is a multiplier of X and let $T \in B(X)$ be such that $TM_{\varphi^n} = a^n M_{\varphi^n} T$ for some positive integer n. Also consider f in X with decomposition $\bigoplus_{i=0}^{n-1} f_i$ where $f_i \in X_i$ for $i = 0, \ldots, n-1$. Then there exist functions $u_0, u_1, \ldots, u_{n-1}$ such that*

$$TC_\varphi f = u_0 C_{a\varphi} f_0 + u_1 C_{a\varphi} f_1 + \cdots + u_{n-1} C_{a\varphi} f_{n-1}.$$

Proof. Let $u_0 = T(1)$ and put

$$(*) \qquad \psi_i = (TM_{\varphi^i} - a^i M_{\varphi^i} T)(1)$$

for $i = 1, \ldots, n-1$. For all integers $k \geq 0$ we have

$$\begin{aligned} TC_\varphi z^{nk} &= T(\varphi^{nk}) = (TM_\varphi^{nk})(1) \\ &= (a^{nk} M_\varphi^{nk} T)(1) = u_0 a^{nk} C_\varphi z^{nk} = u_0 C_{a\varphi} z^{nk}. \end{aligned}$$

Also, for all $i = 1, \ldots, n-1$, by using $(*)$ we get

$$\begin{aligned} TC_\varphi z^i = T(\varphi^i) = (TM_{\varphi^i})(1) &= \psi_i + (a^i M_{\varphi^i} T)(1) \\ &= \psi_i + u_0 a^i C_\varphi z^i \\ &= \psi_i + u_0 C_{a\varphi} z^i. \end{aligned}$$

Therefore

$$\begin{aligned} TC_\varphi z^{nk+i} &= (TM_\varphi^{nk+i})(1) \\ &= a^{nk} M_\varphi^{nk} (\psi_i + u_0 C_\varphi C_{az} z^i) \\ &= \psi_i C_\varphi C_{az} z^{nk} + u_0 C_\varphi C_{az} z^{nk+i} \\ &= \left(u_0 + \frac{\psi_i}{(a\varphi)^i}\right) C_{a\varphi} z^{nk+i} \end{aligned}$$

for all integers $k \geq 0$ and $i = 1, \ldots, n-1$. Now consider a polynomial p with decomposition $p = \bigoplus_{i=0}^{n-1} p_i$ where $p_i \in X_i$ for $i = 0, \ldots, n-1$. So we have

$$\begin{aligned} TC_\varphi p &= TC_\varphi p_0 + TC_\varphi p_1 + \cdots + TC_\varphi p_{n-1} \\ &= u_0 C_{a\varphi} p_0 + u_1 C_{a\varphi} p_1 + \cdots + u_{n-1} C_{a\varphi} p_{n-1} \end{aligned}$$

where $u_i = u_0 + \psi_i/(a\varphi)^i$ for $i = 1, \ldots, n-1$. Since the set of analytic polynomials is dense in X, we get

$$TC_\varphi f = u_0 C_{a\varphi} f_0 + u_1 C_{a\varphi} f_1 + \cdots + u_{n-1} C_{a\varphi} f_{n-1}.$$

This completes the proof. ∎

The following corollary is an immediate consequence of the proof of Theorem 1, which gives the structure of the functions u_i in Theorem 1.

Corollary 2. *Under the conditions of Theorem* 1, *for* $f = \bigoplus_{i=0}^{n-1} f_i$ *in* X, *we have*

$$TC_\varphi f = T(1) C_{a\varphi} f_0 + \sum_{i=1}^{n-1} \left(T(1) + \frac{(TM_{\varphi^i} - a^i M_{\varphi^i} T)(1)}{(a\varphi)^i} \right) C_{a\varphi} f_i.$$

Theorem 3. *Let* φ *be a multiplier of* X *and* $T \in B(X)$ *be such that* $TM_{\varphi^n} = a^n M_{\varphi^n} T$. *If* $C_{a\varphi}$ *is invertible and* $TM_\varphi - aM_\varphi T$ *is compact*, *then* $TC_\varphi = M_{u_0} C_{a\varphi}$ *where* $u_0 = T(1)$.

Proof. Let $f \in X$ and $f = \bigoplus_{i=0}^{n-1} f_i$ where $f_i \in X_i$ for $i = 0, 1, \ldots, n-1$. By Theorem 1, we have

$$(**) \qquad TC_\varphi f = u_0 C_{a\varphi} f_0 + u_1 C_{a\varphi} f_1 + \cdots + u_{n-1} C_{a\varphi} f_{n-1}$$

where

$$\begin{aligned} u_0 &= T(1), \\ u_i &= u_0 + \psi_i/(a\varphi)^i, \\ \psi_i &= (TM_{\varphi^i} - a^i M_{\varphi^i} T)(1) \end{aligned}$$

for $i = 1, \ldots, n-1$. Put

$$S = (TM_\varphi - aM_\varphi T) C_\varphi.$$

Then S is compact and we have

$$\begin{aligned} Sf &= TC_\varphi(zf) - aM_\varphi TC_\varphi f \\ &= u_0 C_{a\varphi}(zf_{n-1}) + u_1 C_{a\varphi}(zf_0) + \cdots + u_{n-1} C_{a\varphi}(zf_{n-2}) \\ &\quad - a\varphi(u_0 C_{a\varphi} f_0 + \cdots + u_{n-1} C_{a\varphi} f_{n-1}). \end{aligned}$$

Now by substituting $u_0 + \psi_i/(a\varphi)^i$ for u_i in the above relation, we get

$$\begin{aligned} Sf = \psi_1 C_{a\varphi} f_0 &+ \left(\frac{\psi_2}{a\varphi} - \psi_1 \right) C_{a\varphi} f_1 + \cdots \\ &+ \left(\frac{\psi_{n-1}}{(a\varphi)^{n-2}} - \frac{\psi_{n-2}}{(a\varphi)^{n-3}} \right) C_{a\varphi} f_{n-2} - \frac{\psi_{n-1}}{(a\varphi)^{n-2}} C_{a\varphi} f_{n-1}. \end{aligned}$$

Since S is compact, so also are

$$M_{\psi_1} C_{a\varphi}|_{X_0} = S|_{X_0} \quad \text{and} \quad M_{\psi_1} M_{(a\varphi)^n} C_{a\varphi}|_{X_0}.$$

But

$$M_{\psi_1} M_{(a\varphi)^n} C_{a\varphi}|_{X_i} = M_{(a\varphi)^i} M_{\psi_1} C_{a\varphi}|_{X_0} M_{z^{n-i}}|_{X_i},$$

and thus indeed $M_{\psi_1 (a\varphi)^n} C_{a\varphi}$ is compact on X. This implies that $M_{\psi_1 (a\varphi)^n}$ is compact on X, since $C_{a\varphi}$ is invertible. Now by the Fredholm alternative we get $(a\varphi)^n \psi_1 = 0$, which implies that $\psi_1 = 0$, because $a\varphi$ is univalent. By the same method we can see that $\psi_1 = \psi_2 = \cdots = \psi_{n-1} = 0$, so $u_i = u_0$ for all $i = 1, \ldots, n-1$. Now $(**)$ implies that $TC_\varphi = M_{u_0} C_{a\varphi}$ and this completes the proof. ■

Corollary 4. *Let $T \in B(X)$ be such that $TM_{z^n} = a^n M_{z^n} T$ where $0 < |a| \leq 1$. If $TM_z - aM_zT$ is compact, then $T = M_{u_0} C_{az}$ where $u_0 = T(1)$.*

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