BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 61, No. 3-4, 2013

NUMBER THEORY

Extensions of Three Theorems of Nagell

by

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Summary. Three theorems of Nagell of 1923 concerning integer values of certain sums of fractions are extended.

Nagell [3] has proved the following theorems.

1. *If m, n and x are integers, $m > 0$, $n > 0$, $x \geq 0$, then except for $m = 1$, $x = 0$, the sum $\sum_{k=0}^{x} \frac{1}{m+kn}$ is never an integer.*

2. *Let a, b, c be integers. Then the sum $\sum_{k=0}^{x} \frac{c}{b+ka}$ is an integer only for finitely many integers x.*

3. *Let a, b, c and d be integers, $a > 0$, $c^2 + d^2 > 0$ and $-ab$ be not a perfect square. Then the sum*

$$\sum_{k=1}^{x} \frac{ck + d}{ak^2 + b}$$

is an integer for only finitely many integers x.

In statement 2 it was probably meant that a, b, c, x are positive integers. Otherwise, the statement is not true, e.g. for $c = 0$ or $b = -xa/2$ (x odd).

The aim of this paper is to extend the above theorems as follows.

THEOREM 1. *If m, n and x are integers, $m > 0$, $n > 0$, $x \geq 0$, $\varepsilon_k \in \{-1, 1\}$ $(0 \leq k \leq x)$, then except for $m = 1$, $x = 0$ the sum*

$$S_1 = \sum_{k=0}^{x} \frac{\varepsilon_k}{m + kn}$$

is never an integer.

2010 *Mathematics Subject Classification*: Primary 11D68.
Key words and phrases: algebraic sum, arithmetic progression.

DOI: 10.4064/ba61-3-1

THEOREM 2. *Let c be a positive integer. Then the sum*

$$S_2 = \sum_{k=0}^{x} \frac{c_k}{b+ka},$$

where a, b are positive integers, c_k are integers satisfying $0 < |c_k| \le c$ $(k = 0, 1, \ldots, x)$, is an integer only for finitely many positive integers x and possibly infinitely many pairs (a, b).

The following example shows that for $c = 2$ the sum S_2 can be an integer for $x = 1$ and for infinitely many pairs (a, b): $c_0 = 1$, $c_1 = -2$, $a = b$.

THEOREM 3. *Let a, b, c, d be integers, $a > 0$, $c^2 + d^2 > 0$, and $-ab$ be not a perfect square. Then the sum*

$$S_3 = \sum_{k=1}^{x} \frac{c_k x + d_k}{ak^2 + b}$$

is an integer for only finitely many positive integers x, where c_k and d_k are integers satisfying $|c_k| \le c$, $|d_k| \le d$, $c_k^2 + d_k^2 > 0$ $(1 \le k \le x)$.

The proofs follow Nagell's arguments supplemented by the following lemmas, in which $P(N)$ denotes the greatest prime factor of N, and $\pi(x)$ is the number of primes $\le x$.

LEMMA 1. *If $x > 0$, $(m, n) = 1$, and*

$$(m+n)(m+2n)\ldots(m+(x-\nu_1)n) > x!, \tag{1}$$

where ν_1 is the number of primes not exceeding x and not dividing n, then

$$P\big((m+n)(m+2n)\ldots(m+xn)\big) > x. \tag{2}$$

Proof. See Sylvester [4, p. 688]; we have changed Sylvester's i to n and n to x to be in agreement with Nagell's notation. ∎

LEMMA 2. *If $(m, n) = 1$, $m \ge x > 0$, then (2) holds.*

Proof. This is Sylvester's theorem [4, p. 703] quoted also by Dickson [1, p. 437]. ∎

LEMMA 3. *For $x \ge 14$ we have $\pi(x) < \frac{3}{8}x + 1$.*

Proof. The primes are $2, 3$ or $6k \pm 1$ $(k > 0)$. The number of such numbers up to x does not exceed $\frac{x-1}{3} + 2$. Now

$$\frac{x-1}{3} + 2 < \frac{3}{8}x + 1 \quad \text{for } x > 16.$$

For $x = 14, 15, 16$ the lemma is verified directly. ∎

LEMMA 4. *For $x \ge 14$ the function $\left(\frac{x+1}{3}\left(x - \frac{t}{2} + 1\right)\right)^{t-1}$ is a strictly increasing function of $t \le \frac{3}{8}x + 1$.*

Proof. By differentiation. ∎

Lemma 5. *If $3n \geq x+2$, $2 \mid n$ and $(m,n)=1$, then (2) holds.*

Proof. By Descartes's rule of signs the polynomial

$$\left(\frac{x+1}{3}\right)^5 - \left(\frac{13}{16}x + \frac{1}{2}\right)^3$$

has only one positive zero. Hence the inequality

$$\left(\frac{14+1}{3}\right)^5 > \left(\frac{13}{16}\cdot 14 + \frac{1}{2}\right)^3$$

implies

$$\left(\frac{x+1}{3}\right)^5 > \left(\frac{13}{16}x + \frac{1}{2}\right)^3$$

for all $x \geq 14$. Hence

$$\left(\frac{x+1}{3}\right)^x > \left(\frac{x+1}{3}\left(\frac{13}{16}x + \frac{1}{2}\right)\right)^{\frac{3}{8}x}.$$

By Lemmas 3 and 4 the right-hand side is greater than

$$\left(\frac{x+1}{3}\left(x - \frac{\pi(x)}{2} + 1\right)\right)^{\pi(x)-1},$$

thus we obtain

$$\left(\frac{x+1}{3}\right)^{x-\pi(x)+1} > \left(x - \frac{\pi(x)}{2} + 1\right)^{\pi(x)-1}.$$

By the assumption the left hand side is less than $n^{x-\pi(x)+1}$, on the other hand by the inequality of the arithmetic and geometric mean the right hand side is no smaller than $\frac{x!}{(x-\pi(x)+1)!} = \prod_{i=0}^{\pi(x)-2}(x-i)$. Thus we obtain

$$n^{x-\pi(x)+1}(x-\pi(x)+1)! > x!.$$

However, by the assumption $2 \mid n$ we have $\nu_1 \leq \pi(x)-1$, hence the left hand side is less than or equal to

$$n \cdot 2n \cdot \ldots \cdot (x-\nu_1)n < (m+n)(m+2n)\ldots(m+(x-\nu_1)n)$$

and by Lemma 1 we obtain (2) for all $x \geq 14$. For $x < 14$ it is enough to prove (2) for x prime, i.e., for $x = 2, 3, 5, 7, 11, 13$. In each case by Lemma 1 it is enough to check even n in the interval

$$\frac{x+2}{3} \leq n < \left(\frac{x!}{(x-\pi(x)+1)!}\right)^{1/(\pi(x)-1)},$$

and by Lemma 2 it is enough to check $m < x$. A finite computation completes the proof. ■

Proof of Theorem 1. It is enough to assume that $(m,n)=1$, $m>1$, $x>0$. Consider first n odd. Then there is at least one even number in the sequence

$$m, m+n, \ldots, m+xn. \tag{3}$$

Let 2^μ be the highest power of 2 which divides any number of the sequence (3), and let further $m+kn$ be the first number of the sequence (3) which is divisible by 2^μ. Then

$$m+kn = 2^\mu(2h+1).$$

The next number of the form $m+tn$ that is divisible by 2^μ is

$$m+(k+2^\mu)n = 2^\mu(2h+n+1).$$

Since n is odd, this number is divisible by $2^{\mu+1}$, hence it does not belong to the sequence (3). Therefore in the sum S_1 there exists only one term with denominator divisible by 2^μ, namely $\frac{\varepsilon_k}{m+kn}$. We obtain

$$\frac{1}{2}(m+kn)S_1 = \frac{a}{b} \pm \frac{1}{2},$$

where b is odd. It follows that S_1 is not an integer, thus Theorem 1 is proved for n odd.

Now consider n even, thus m is odd ≥ 3.

Let q be a prime factor of $m+kn$, where $0 \leq k \leq x$. If no other term of the sequence (3) is divisible by q, then we obtain

$$\frac{1}{q}(m+kn)S_1 = \frac{a}{c} \pm \frac{1}{q},$$

where $q \nmid c$. Hence S_1 is not an integer. In order that S_1 be an integer at least two terms of the sequence (3) should be divisible by q, thus $q \leq x$. Taking $q = P\big((m+n)(m+2n)\ldots(m+xn)\big)$, by Lemma 2 we obtain $x > m$ and, by Lemma 5, $x \geq 3n-1$.

By Chebyshev's theorem there exists a prime q such that

$$\frac{1}{2}(x+3) < q \leq x+1. \tag{4}$$

Then there is a term of the sequence (3) divisible by q, since we have

$$q > \frac{1}{2}(x+3) \geq 3\frac{n}{2} + 1 > n, \tag{5}$$

and the numbers of the sequence (3) represent all residues modulo q.

Let $m+kn$ be the least term of the sequence (3) divisible by q. Then

$$m+kn = qT, \tag{6}$$

where $k < q$.

According to a previous remark, also the number $m+(k+q)n =: m+ln$ occurs in the sequence (3), thus

$$m+ln = q(T+n). \tag{7}$$

The number $m + (k + 2q)n$ does not occur in (3), since by (4) we have $k + 2q \geq 2q > x$. Therefore, the numbers (6) and (7) are the only terms of the sequence (3) divisible by q. We have

$$\frac{\varepsilon_k}{m+kn} + \frac{\varepsilon_l}{m+ln} = \frac{\varepsilon_k}{qT(T+n)} \cdot \begin{cases} 2T+n & \text{if } \varepsilon_l = \varepsilon_k, \\ n & \text{if } \varepsilon_l = -\varepsilon_k, \end{cases}$$

$$T(T+n)S_1 = \frac{a}{b} \pm \begin{cases} \frac{2T+n}{q} & \text{if } \varepsilon_l = \varepsilon_k, \\ \frac{n}{q} & \text{if } \varepsilon_l = -\varepsilon_k, \end{cases}$$

where $q \nmid b$. If S_1 is an integer, we have $q \mid 2T + n$ or $q \mid n$. The latter is impossible by (5), and the former, since n is even, gives $q \mid T + n/2$. However, since $x > m$, $q > k$ and $q > \frac{1}{2}(x + 3)$ we obtain

$$T = \frac{m+kn}{q} < \frac{2x}{x+3} + n < 2 + n, \quad \text{i.e.} \quad T \leq n + 1,$$

and by (5),

$$T + \frac{n}{2} \leq 3\frac{n}{2} + 1 < q.$$

The contradiction obtained proves Theorem 1. ■

Proof of Theorem 2. The proof follows in general the proof of Theorem 1. However, the first part of that proof now fails, thus it is not possible to assume a even. Hence instead of $T + a/2$ we have to deal with $2T + a$ and instead of the inequality $x \geq 3a - 1$ we have to assume $x \geq 6a + 1$. Moreover, $\nu_1 \leq \pi(x)$ instead of $\nu_1 \leq \pi(x) - 1$. Therefore, instead of Lemma 3 we use the inequality $\pi(x) \leq \frac{3}{8}x$ for $x \geq 24$ and in order to apply the assertion of Lemma 5 we have to use, instead of the inequality

$$\left(\frac{x+1}{3}\right)^5 > \left(\frac{13}{16}x + \frac{1}{2}\right)^3$$

valid for $x \geq 14$, the inequality

$$\left(\frac{x}{6}\right)^5 > \left(\frac{13}{16}x\right)^3$$

valid for $x \geq 65$. Thus the proof of Theorem 1 works for

$$x \geq \max\{65, 2c - 3\}.$$

The desired finitely many x consist of

$$x < \max\{65, 2c - 3\}. \ ■$$

Proof of Theorem 3. Let x_0 be the least positive solution of the congruence

(8) $$ax^2 + b \equiv 0 \pmod{p},$$

where p is an odd prime, not a divisor of ab, thus $0 < x_0 < \frac{1}{2}p$. Then the next positive solution of (8) is $p - x_0$, hence $> \frac{1}{2}p$. Now, Nagell's theorem [2, §1] implies that for all sufficiently large x,

$$P_x = P\Big(\prod_{k=1}^{x}(ak^2+b)\Big) > 2x.$$

Therefore, if x is large enough only one of the numbers ak^2+b $(1 \le k \le x)$ is divisible by P_x. Let it be ax_0^2+b. Then $P_x \mid c_{x_0}x_0+d_{x_0}$ implies $P_x \mid ad_{x_0}^2+bc_{x_0}^2$. By the assumptions $ad_{x_0}^2+bc_{x_0}^2 \ne 0$, hence $2x \le |ad_{x_0}^2+bc_{x_0}^2| \le ad^2+|b|c^2$. If $2x > ad^2+|b|c^2$, then we obtain

$$\frac{1}{P_x}(ax_0^2+b)S_3 = \frac{c_{x_0}x_0+d_{x_0}}{P_x} + \frac{T}{N}$$

where $P_x \nmid (c_{x_0}x_0+d_{x_0})N$. Thus S_3 cannot be an integer. ∎

References

[1] L. E. Dickson, *History of the Theory of Numbers*, Vol. 1, reprint, Chelsea, 1952.

[2] T. Nagell, *Zur Arithmetik der Polynome*, Abh. Math. Sem. Hamburg 1 (1922), 179–194; also: Collected Papers of Trygve Nagell, Queen's Univ., Kingston, 2002, 211–228.

[3] T. Nagell, *Eine Eigenschaft gewisser Summen*, in: Zahlentheoretische Notizen III, Vid.-selsk. Kristiania Skrifter, Matem.-Naturv. Kl. (1923), no. 13, 10–15; also: Collected Papers of Trygve Nagell, Queen's Univ., Kingston, 2002, 358–363.

[4] J. J. Sylvester, *On arithmetical series*, Messenger Math. 21 (1891–2), 1–19; also: The Collected Papers of James Joseph Sylvester, Vol. 4 (1882–1897), Cambridge, at the Univ. Press, 1912, 687–703.

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Received July 19, 2013;
received in final form October 17, 2013 (7945)