BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 61, No. 3-4, 2013

COMMUTATIVE ALGEBRA

Rational Constants of Generic LV Derivations and of Monomial Derivations

by

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Summary. We describe the fields of rational constants of generic four-variable Lotka–Volterra derivations. Thus, we determine all rational first integrals of the corresponding systems of differential equations. Such systems play a role in population biology, laser physics and plasma physics. They are also an important part of derivation theory, since they are factorizable derivations. Moreover, we determine the fields of rational constants of a class of monomial derivations.

1. Introduction. The main result of the paper is Theorem 2, which gives the description of the field of rational constants of a generic four-variable Lotka–Volterra derivation. Moreover, in Section 4 we describe the fields of rational constants of some class of four-variable monomial derivations (Theorem 5). All our considerations are over an arbitrary field k of characteristic zero.

Let us fix some notation:

- $\mathbb{Q}_+$, the set of positive rationals,
- $\mathbb{Q}_-$, the set of negative rationals,
- $\mathbb{N}$, the set of nonnegative integers,
- $\mathbb{N}_+$, the set of positive integers,
- n, an integer ≥ 3,
- $k[X] := k[x_1, \ldots, x_n]$, the ring of polynomials in n variables,
- $k(X) := k(x_1, \ldots, x_n)$, the field of rational functions in n variables.

2010 *Mathematics Subject Classification*: Primary 13N15; Secondary 12H05, 92D25, 34A34.
Key words and phrases: Lotka–Volterra derivation, factorizable derivation, monomial derivation, rational constant, rational first integral.

DOI: 10.4064/ba61-3-2

Recall that if R is a commutative k-algebra, then a k-linear mapping $d : R \to R$ is called a *derivation* of R if for all $a, b \in R$,

$$d(ab) = ad(b) + d(a)b.$$

We call $R^d = \ker d$ the *ring of constants* of the derivation d. Then $k \subseteq R^d$, and a *nontrivial* constant of d is an element of the set $R^d \setminus k$. If $f_1, \ldots, f_n \in k[X]$, then there exists exactly one derivation $d : k[X] \to k[X]$ such that $d(x_1) = f_1, \ldots, d(x_n) = f_n$. A derivation $d : k[X] \to k[X]$ is said to be *factorizable* if $d(x_i) = x_i f_i$, where the polynomials f_i are of degree 1 for $i = 1, \ldots, n$. We may associate a factorizable derivation with any given derivation of $k[X]$; that construction helps to obtain new facts on constants, especially rational constants, of the initial derivation (see, for instance, [5], [7]).

There is no general procedure for determining all constants of a derivation. Even for a given derivation the problem may be difficult: see for instance counterexamples to Hilbert's fourteenth problem (all of them are of the form $k[X]^d$, but it took more than a half century to find at least one of them; for more details we refer the reader to [6], [4]) or Jouanolou derivations (where the rings and fields of constants are trivial, see [5], [6]).

The main motivations of our study are the following:

- applications of Lotka–Volterra systems in population biology, laser physics and plasma physics (see, for instance, [1], [2], [3]);
- Lagutinskii's procedure of associating a factorizable derivation (examples of such derivations are Lotka–Volterra derivations) with any given derivation (see also Section 4);
- relations to invariant theory, mainly to connected algebraic groups (see [6]).

2. Lotka–Volterra derivations and Darboux polynomials. Let $C_1, \ldots, C_n \in k$. From now on, $d : k[X] \to k[X]$ is a derivation of the form

$$d(x_i) = x_i(x_{i-1} - C_i x_{i+1})$$

for $i = 1, \ldots, n$ (we adopt the convention that $x_{n+1} = x_1$ and $x_0 = x_n$). We call d a *Lotka–Volterra derivation* with parameters $C_1, \ldots, C_n$.

A polynomial $g \in k[X]$ is said to be *strict* if it is homogeneous and not divisible by the variables $x_1, \ldots, x_n$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote by X^α the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n} \in k[X]$. Every nonzero homogeneous polynomial $f \in k[X]$ has a unique representation $f = X^\alpha g$, where X^α is a monomial and g is strict.

We call a nonzero polynomial $f \in k[X]$ a *Darboux polynomial* of a derivation $\delta : k[X] \to k[X]$ if $\delta(f) = \Lambda f$ for some $\Lambda \in k[X]$. We will call Λ a *cofactor* of f. Since d is a homogeneous derivation of degree 1, the cofactor of each homogeneous polynomial is a linear form. Denote by $k[X]_{(m)}$ the homogeneous component of $k[X]$ of degree m.

Lemma 1 ([11, 3.2]). *Let $n = 4$. Let $g \in k[X]_{(m)}$ be a Darboux polynomial of d with the cofactor $\lambda_1 x_1 + \cdots + \lambda_4 x_4$. Let $i \in \{1, 2, 3, 4\}$. If g is not divisible by x_i, then $\lambda_{i+1} \in \mathbb{N}$. More precisely, if*

$$g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_4) = x_{i+2}^{\beta_{i+2}} \overline{G} \quad \text{and} \quad x_{i+2} \nmid \overline{G},$$

then $\lambda_{i+1} = \beta_{i+2}$ and $\lambda_{i+3} = -C_{i+2}\lambda_{i+1}$.

Corollary 1 ([11, 3.3]). *Let $n = 4$. If $g \in k[X]$ is a strict Darboux polynomial of d, then its cofactor is a linear form with coefficients in $\mathbb{N}$.*

For any derivation $\delta : k[X] \to k[X]$ there exists exactly one derivation $\bar{\delta} : k(X) \to k(X)$ such that $\bar{\delta}_{|k[X]} = \delta$. By a *rational constant* of the derivation $\delta : k[X] \to k[X]$ we mean a constant of the corresponding derivation $\bar{\delta} : k(X) \to k(X)$. For simplicity, we write δ instead of $\bar{\delta}$. The rational constants of δ form a field. In the remainder of this section we quote some results which will be used in the two main sections.

Proposition 1 ([6, 2.2.2]). *Let $\delta : k[X] \to k[X]$ be a derivation and let f and g be nonzero relatively prime polynomials from $k[X]$. Then $\delta(f/g) = 0$ if and only if f and g are Darboux polynomials of δ with the same cofactor.*

Proposition 2 ([6, 2.2.3]). *Let δ be a homogeneous derivation of $k[X]$ and let $f \in k[X]$ be a Darboux polynomial of δ with cofactor $\Lambda \in k[X]$. Then Λ is homogeneous and each homogeneous component of f is also a Darboux polynomial of δ with the same cofactor Λ.*

Proposition 3 ([6, 2.2.1]). *Let δ be a derivation of $k[X]$. Then $f \in k[X]$ is a Darboux polynomial of δ if and only if all factors of f are Darboux polynomials of δ. Moreover, if $f = f_1 f_2$ is a Darboux polynomial, then the sum of the cofactors of f_1 and f_2 equals the cofactor of f.*

3. The field of rational constants of a generic four-variable LV derivation. Throughout this section we assume $n = 4$. Lemma 2 is a generalization of Proposition 4.5 from [8].

Lemma 2. *The field $k(X)^d$ contains a nontrivial rational monomial constant if and only if at least one of the following two conditions is fulfilled:*

(1) *$C_1, C_3 \in \mathbb{Q}$ and $C_1 C_3 = 1$,*
(2) *$C_2, C_4 \in \mathbb{Q}$ and $C_2 C_4 = 1$.*

Proof. Let $\eta = c x_1^r x_2^s x_3^t x_4^u$, where $c \in k \setminus \{0\}$ and $r, s, t, u \in \mathbb{Z}$. Then

$$d(\eta) = c x_1^r x_2^s x_3^t x_4^u \big((s - uC_4)x_1 + (t - rC_1)x_2 + (u - sC_2)x_3 + (r - tC_3)x_4\big).$$

If $d(\eta) = 0$, then we get two systems of linear equations:

$$\begin{cases} s - uC_4 = 0, \\ u - sC_2 = 0, \end{cases}$$

and

$$\begin{cases} t - rC_1 = 0, \\ r - tC_3 = 0. \end{cases}$$

Clearly $\eta \notin k$ if and only if $(r, s, t, u) \neq (0, 0, 0, 0)$, that is, if and only if at least one of the above systems has a nonzero solution. Equivalently, at least one of the conditions (1) or (2) is fulfilled. ∎

We know the ring of polynomial constants of a generic LV derivation:

Theorem 1 ([11, 5.1]). *Let $C_1, \ldots, C_4 \notin \mathbb{Q}_+$. If $C_1C_2C_3C_4 = 1$, then*

$$k[X]^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4].$$

If $C_1C_2C_3C_4 \neq 1$, then $k[X]^d = k$.

Now we describe the field of rational constants of a generic LV derivation.

Theorem 2. *Let $C_1, \ldots, C_4 \notin \mathbb{Q}$. If $C_1C_2C_3C_4 = 1$, then*

$$k(X)^d = k(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4).$$

If $C_1C_2C_3C_4 \neq 1$, then $k(X)^d = k$.

Proof. Both inclusions $\supseteq$ are straightforward. We show the inclusions $\subseteq$.

Let $\psi = f/g \in k(X)^d$, where $f, g \in k[X] \setminus \{0\}$ and $\gcd(f, g) = 1$. By Proposition 1 both f and g are Darboux polynomials of d with common cofactor. Let $d(f) = \Lambda f$ and $d(g) = \Lambda g$ for some $\Lambda \in k[X]$. Let $f = \sum f_j$ and $g = \sum g_j$, where f_j and g_j are homogeneous polynomials of degree j. Since d is homogeneous, by Proposition 2 we have $d(f_j) = \Lambda f_j$ and $d(g_j) = \Lambda g_j$ for all $j \in \mathbb{N}$.

Let $f_j = X^{\alpha_j} h_j$, where X^{α_j} is a monomial and h_j is strict (analogously we proceed for g_j). By Proposition 3 both X^{α_j} and h_j are Darboux polynomials of d. We describe all strict Darboux polynomials of d. Let

$$\lambda = \lambda_1x_1 + \cdots + \lambda_4x_4$$

be the cofactor of h_j. By Lemma 1 we have

$$\lambda_{i+3} = -C_{i+2}\lambda_{i+1}$$

for all i in the cyclic sense. However Corollary 1 gives $\lambda_i \in \mathbb{N}$ for $i = 1, \ldots, 4$. Since $C_1, \ldots, C_4 \notin \mathbb{Q}$, we have $\lambda_1 = \cdots = \lambda_4 = 0$ and the only strict Darboux polynomials of d are constants of d. In view of Theorem 1 we have $h_j \in k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4]$ or $h_j \in k$, respectively. Furthermore, by Proposition 3, the cofactor of X^{α_j} is equal to Λ, since the cofactor of h_j equals 0.

Analogously, $g_j = X^{\beta_j} l_j$, where $l_j \in k[X]^d$ and X^{β_j} is a Darboux monomial with the cofactor Λ. Then $X^{\alpha_j}/X^{\beta_i} \in k(X)^d$, by Proposition 1. In view of Lemma 2, $X^{\alpha_j}/X^{\beta_i} \in k$. Similarly, $X^{\beta_j}/X^{\beta_i} \in k$. Thus we may express ψ as a rational function, where both the numerator and the denominator are linear combinations of strict polynomials belonging to $k[X]^d$. Thus $\psi \in k(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)$ or $\psi \in k$, respectively. ■

Denote by D_0 the field of fractions of an integral domain D.

Corollary 2. *If $C_1, \ldots, C_4 \notin \mathbb{Q}$, then $k(X)^d = (k[X]^d)_0$.*

This is not true for an arbitrary Lotka–Volterra derivation.

Example 1. Let $d : k[x_1, \ldots, x_4] \to k[x_1, \ldots, x_4]$ be the derivation defined by

$$d(x_i) = x_i(x_{i-1} + x_{i+1})$$

for $i = 1, \ldots, 4$. Then $x_1/x_3 \in k(x_1, \ldots, x_4)^d$, although neither x_1 nor x_3 is in $k[x_1, \ldots, x_4]^d$, which by Theorem 1 coincides with $k[x_1 - x_2 + x_3 - x_4]$.

4. Monomial derivations. We say that a derivation $\delta : k(X) \to k(X)$ is *monomial* if

$$\delta(x_i) = x_1^{\beta_{i1}} \cdots x_n^{\beta_{in}}$$

for $i = 1, \ldots n$, where each β_{ij} is an integer. Let

$$\omega_\delta = \det \begin{bmatrix} \beta_{11} - 1 & \beta_{12} & \ldots & \beta_{1n} \\ \beta_{21} & \beta_{22} - 1 & \ldots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \ldots & \beta_{nn} - 1 \end{bmatrix}.$$

Theorem 3 ([7, 4.8]). *Let $\delta_1 : k(X) \to k(X)$ be a monomial derivation such that $\omega_{\delta_1} \neq 0$, and let $\delta_2 : k[X] \to k[X]$ be the factorizable derivation associated with δ_1. Then δ_1 has a nontrivial rational constant if and only if δ_2 has a nontrivial rational constant.*

From now on, $n = 4$. For arbitrary $C_1, C_2, C_3, C_4 \in k$ we may consider the three sentences:

$s_1 :$ $C_1C_2C_3C_4 = 1$.

$s_2 :$ $C_1, C_3 \in \mathbb{Q}_+$ and $C_1C_3 = 1$.

$s_3 :$ $C_2, C_4 \in \mathbb{Q}_+$ and $C_2C_4 = 1$.

In case s_2 let $C_1 = p/q$, where $p, q \in \mathbb{N}_+$ and $\gcd(p, q) = 1$. In case s_3 let $C_2 = r/t$, where $r, t \in \mathbb{N}_+$ and $\gcd(r, t) = 1$. Denote by $\neg s_i$ the negation of the sentence s_i. Theorem 4 below is a generalization of Theorem 1.

THEOREM 4 ([10, 5.1]). *Let $d : k[X] \to k[X]$ be a derivation of the form*

$$d = \sum_{i=1}^{4} x_i(x_{i-1} - C_i x_{i+1}) \frac{\partial}{\partial x_i},$$

where $C_1, C_2, C_3, C_4 \in k$. Then the ring of constants of d is always finitely generated over k with at most three generators. In each case it is a polynomial ring, more precisely:

(1) *if $s_1 \wedge \neg s_2 \wedge \neg s_3$, then $k[X]^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4]$,*
(2) *if $\neg s_1 \wedge \neg s_2 \wedge \neg s_3$, then $k[X]^d = k$,*
(3) *if $\neg s_1 \wedge \neg s_2 \wedge s_3$, then $k[X]^d = k[x_2^t x_4^r]$,*
(4) *if $\neg s_1 \wedge s_2 \wedge \neg s_3$, then $k[X]^d = k[x_1^q x_3^p]$,*
(5) *if $s_1 \wedge \neg s_2 \wedge s_3$, then $k[X]^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4, x_2^t x_4^r]$,*
(6) *if $s_1 \wedge s_2 \wedge \neg s_3$, then $k[X]^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4, x_1^q x_3^p]$,*
(7) *if $s_2 \wedge s_3$, then $k[X]^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4,\ x_1^q x_3^p,\ x_2^t x_4^r]$.*

THEOREM 5. *Let $s_1, \ldots, s_4 \in \mathbb{N}_+$, where $(s_1, s_3) \neq (1,1)$ and $(s_2, s_4) \neq (1,1)$. Let $D : k(X) \to k(X)$ be a derivation of the form*

$$D(x_i) = x_{i-1}^{s_{i-1}+1} x_i^{s_i+1} x_{i+2}^{s_{i+2}}$$

for $i = 1, \ldots, 4$ (in the cyclic sense). Then $k(X)^D = k$.

Proof. Let $D_2 : k(X) \to k(X)$ be the derivation defined by

$$D_2 = x_1^{-s_1} x_2^{-s_2} x_3^{-s_3} x_4^{-s_4} D.$$

Clearly $k(X)^{D_2} = k(X)^D$. Then

$$D_2(x_i) = x_{i-1} x_i x_{i+1}^{-s_{i+1}}$$

for $i = 1, \ldots, 4$. We have

$$\omega_{D_2} = \det \begin{bmatrix} 0 & -s_2 & 0 & 1 \\ 1 & 0 & -s_3 & 0 \\ 0 & 1 & 0 & -s_4 \\ -s_1 & 0 & 1 & 0 \end{bmatrix} = -1 + s_1s_3 + s_2s_4 - s_1s_2s_3s_4.$$

Therefore

$$\omega_{D_2} = -(1 - s_1s_3)(1 - s_2s_4) \neq 0.$$

Let $\Delta : k(y_1, \ldots, y_4) \to k(y_1, \ldots, y_4)$ be the factorizable derivation associated with D_2. In view of Theorem 3 we have $k(X)^{D_2} = k$ if and only if $k(y_1, \ldots, y_4)^{\Delta} = k$. The construction of the factorizable derivation associated with a given derivation is described for instance in [7]. Let $y_i = D_2(x_i)/x_i$ for $i = 1, \ldots, 4$. Then

$$\Delta(y_i) = y_i(y_{i-1} - s_{i+1}y_{i+1})$$

for $i = 1, \ldots, 4$. Put $C_i = s_{i+1}$ for $i = 1, \ldots, 4$. Hence

$$\Delta(y_i) = y_i(y_{i-1} - C_i y_{i+1})$$

for $i = 1, \ldots, 4$. For simplicity of notation, we will write x_i instead of y_i.

For the derivation Δ we have $C_1, \ldots, C_4 \in \mathbb{N}_+$, $(C_1, C_3) \neq (1, 1)$ and $(C_2, C_4) \neq (1, 1)$, therefore $\neg s_1 \wedge \neg s_2 \wedge \neg s_3$. Thus Theorem 4 implies that $k[X]^\Delta = k$.

Let $\psi = f/g \in k(X)^\Delta$, where $f, g \in k[X] \setminus \{0\}$ and $\gcd(f, g) = 1$. By Proposition 1, we have $\Delta(f) = \Lambda f$ and $\Delta(g) = \Lambda g$ for some $\Lambda \in k[X]$. Let $f = \sum f_j$ and $g = \sum g_j$, where f_j and g_j are homogeneous forms of degree j. In view of Proposition 2 we have $\Delta(f_j) = \Lambda f_j$ and $\Delta(g_j) = \Lambda g_j$ for all $j \in \mathbb{N}$.

Let $f_j = X^{\alpha_j} h_j$, where X^{α_j} is a monomial and h_j is strict. By Proposition 3 both X^{α_j} and h_j are Darboux polynomials of Δ. Let $\lambda = \lambda_1 x_1 + \cdots + \lambda_4 x_4$ be the cofactor of h_j. By Lemma 1 we have

$$\lambda_{i+3} = -C_{i+2}\lambda_{i+1} \tag{4.1}$$

for all i in the cyclic sense. However, by Corollary 1, the left-hand side of (4.1) is nonnegative, whereas the right-hand side of (4.1) is nonpositive. Therefore $\lambda_1 = \cdots = \lambda_4 = 0$ and the only strict Darboux polynomials of Δ are constants of Δ. Hence $h_j \in k$. Moreover, by Proposition 3, the cofactor of X^{α_j} is equal to Λ.

Similarly, $g_j = X^{\beta_j} l_j$, where $l_j \in k$ and X^{β_j} is a Darboux monomial with the cofactor Λ. Then $X^{\alpha_j}/X^{\beta_i} \in k(X)^\Delta$. By Lemma 2, we have $X^{\alpha_j}/X^{\beta_i} \in k$. Analogously, $X^{\beta_j}/X^{\beta_i} \in k$. Thus we may express ψ as a rational function, where both the numerator and the denominator are linear combinations of elements of k. Thus $\psi \in k$. Therefore, $k(X)^\Delta = k$, and consequently $k(X)^D = k$. ∎

Note that $k(X)^\Delta = k$ does not follow from Theorem 2. Let us also note that the case $C_1 = C_2 = C_3 = C_4 = 1$ was solved in [9]; then $k(X)^\Delta$ has three generators, namely $k(X)^\Delta = k(x_1 + x_2 + x_3 + x_4, x_1x_3, x_2x_4)$.

For a derivation δ of $k(X)$ the set $k(X)^\delta \setminus k$ coincides with the set of all rational first integrals of the corresponding system of ordinary differential equations (for more details we refer the reader to [6]). Therefore, we described all rational constants of generic (for $k = \mathbb{R}$ or $k = \mathbb{C}$) four-variable Lotka–Volterra derivations and of some monomial derivations and all rational first integrals of the corresponding systems of differential equations.

References

[1] M. A. Almeida, M. E. Magalhães and I. C. Moreira, *Lie symmetries and invariants of the Lotka–Volterra system*, J. Math. Phys. 36 (1995), 1854–1867.

[2] O. I. Bogoyavlenskiĭ, *Algebraic constructions of integrable dynamical systems—extension of the Volterra system*, Russian Math. Surveys 46 (1991), no. 3, 1–64.
[3] L. Cairó and J. Llibre, *Darboux integrability for 3D Lotka–Volterra systems*, J. Phys. A 33 (2000), 2395–2406.
[4] S. Kuroda, *Fields defined by locally nilpotent derivations and monomials*, J. Algebra 293 (2005), 395–406.
[5] A. J. Maciejewski, J. Moulin Ollagnier, A. Nowicki and J.-M. Strelcyn, *Around Jouanolou non-integrability theorem*, Indag. Math. (N.S.) 11 (2000), 239–254.
[6] A. Nowicki, *Polynomial Derivations and Their Rings of Constants*, N. Copernicus Univ. Press, Toruń, 1994.
[7] A. Nowicki and J. Zieliński, *Rational constants of monomial derivations*, J. Algebra 302 (2006), 387–418.
[8] P. Ossowski and J. Zieliński, *Polynomial algebra of constants of the four variable Lotka–Volterra system*, Colloq. Math. 120 (2010), 299–309.
[9] J. Zieliński, *The field of rational constants of the Volterra derivation*, Proc. Est. Acad. Sci. (2014), to appear.
[10] J. Zieliński, *Rings of constants of four-variable Lotka–Volterra systems*, Cent. Eur. J. Math. 11 (2013), 1923–1931.
[11] J. Zieliński and P. Ossowski, *Rings of constants of generic 4D Lotka–Volterra systems*, Czechoslovak Math. J. 63 (2013), 529–538.

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Received 20 February 2013;
received in final form 10 March 2014 (7965)