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POTENTIAL THEORY

Maximal Weak-Type Inequality for Orthogonal Harmonic Functions and Martingales

by

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Summary. Assume that u, v are conjugate harmonic functions on the unit disc of $\mathbb{C}$, normalized so that $u(0) = v(0) = 0$. Let u^*, $|v|^*$ stand for the one- and two-sided Brownian maxima of u and v, respectively. The paper contains the proof of the sharp weak-type estimate

$$\mathbb{P}(|v|^* \geq 1) \leq \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots}\, \mathbb{E}u^*.$$

Actually, this estimate is shown to be true in the more general setting of differentially subordinate harmonic functions defined on Euclidean domains. The proof exploits a novel estimate for orthogonal martingales satisfying differential subordination.

1. Introduction. Suppose that N is a fixed positive integer and D is an open connected subset of $\mathbb{R}^N$, and let u and v be real-valued harmonic functions on D. Following Burkholder [Bu2], we say that v is *differentially subordinate* to u if for all $x \in D$ we have

$$|\nabla v(x)| \leq |\nabla u(x)|. \tag{1.1}$$

The functions u, v are said to be *orthogonal* if

$$\nabla u \cdot \nabla v = 0 \quad \text{on } D, \tag{1.2}$$

where the dot $\cdot$ stands for the standard scalar product in $\mathbb{R}^N$. A classical example for which the conditions (1.1) and (1.2) hold is when $N = 2$, D is the unit disc of $\mathbb{R}^2$ and u, v satisfy the Cauchy–Riemann equations.

Fix a point $\xi \in D$ and let D_0 be a bounded connected subdomain of D satisfying $\xi \in D_0 \subset D_0 \cup \partial D_0 \subset D$. Let us impose the additional normaliza-

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tion requirement on u, v, given by

$$v(\xi) = u(\xi) = 0. \tag{1.3}$$

The conditions (1.1)–(1.3) imply many interesting estimates involving u and v. Denote by $\mu^{\xi}_{D_0}$ the harmonic measure on ∂D_0 with respect to ξ. For any $0 < p < \infty$, define the pth (quasi-)norm and the weak pth (quasi-)norm of u by

$$\|u\|_p = \Big[\sup_{D_0} \int_{\partial D_0} |u(x)|^p \, d\mu^{\xi}_{D_0}(x)\Big]^{1/p}$$

and

$$\|u\|_{p,\infty} = \sup_{\lambda>0} \lambda \Big[\sup_{D_0} \mu^{\xi}_{D_0}(\{x \in \partial D_0 : |u(x)| \geq \lambda\})\Big]^{1/p},$$

where the inner supremum is taken over all D_0 as above. If D is the unit disc of $\mathbb{R}^2$, $\xi = (0,0)$ and v is assumed to be the harmonic conjugate of u with $v(\xi) = u(\xi)$, the problem of comparing the pth norms of u and v goes back to the classical work of M. Riesz [R], who showed that for some universal c_p, $1 < p < \infty$, we have

$$\|v\|_p \leq c_p \|u\|_p. \tag{1.4}$$

The optimal choice for c_p was identified 50 years later by Pichorides [Pi] and, independently, by Cole (see Gamelin [G]): the value is $\cot(\pi/2p^*)$, where $p^* = \max\{p, p/(p-1)\}$. Bañuelos and Wang [BW] extended this result to the above general setting: namely, if u, v are given on a domain D and satisfy (1.1)–(1.3), then (1.4) holds true, with c_p equal to the Pichorides–Cole constant.

In the limit case $p = 1$, the L^p estimate does not hold with any finite constant, but one can establish a related weak-type $(1,1)$ estimate. Let us start with the setting of conjugate harmonic functions on the unit disc. Then, as proved by Kolmogorov [K], we have the following result: for some universal $c_{1,\infty} < \infty$,

$$\|v\|_{1,\infty} \leq c_{1,\infty} \|u\|_1, \tag{1.5}$$

Davis [D] proved that the optimal choice for $c_{1,\infty}$ is

$$C = \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots} \tag{1.6}$$

and Choi [C] extended this to the more general setting described above. Namely, we have the following.

Theorem 1.1. *If u, v satisfy* (1.1)–(1.3), *then the inequality* (1.5) *is valid, with $c_{1,\infty}$ equal to Davis' constant.*

The purpose of this paper is to study the appropriate maximal version of Choi's theorem. Suppose that B is an N-dimensional Brownian motion starting from ξ, defined on a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\tau = \inf\{t : B_t \notin D\}$ be the first exit-time of B from D, where, as usual, we set the infimum of an empty set to be infinity. For a given harmonic function u on D, we define the associated *one-sided* and *two-sided Brownian maximal function* of u by the formulas

$$u^* = \sup_{0\le t\le \tau} u(B_t) \quad \text{and} \quad |u|^* = \sup_{0\le t\le \tau} |u(B_t)|,$$

respectively. Note that both u^* and $|u|^*$ are random variables.

One of the main results of this paper is to establish the following statement. Let C be Davis' constant, given by (1.6) above.

THEOREM 1.2. *Suppose that u, v are harmonic functions on D with the properties* (1.1)–(1.3). *Then for any $\lambda > 0$ we have*

$$\lambda \mathbb{P}(|v|^* \ge \lambda) \le C\mathbb{E}u^*, \tag{1.7}$$

and the constant C is the best possible.

This statement can be generalized a little. Suppose that u, v satisfy (1.1) and (1.2), and assume that $v(\xi) = 0$ (with no conditions on $u(\xi)$). Then, applying the above estimate to the pair $u - u(\xi)$ and v, we get the sharp estimate

$$\lambda \mathbb{P}(|v|^* \ge \lambda) + Cu(\xi) \le C\mathbb{E}u^* \quad \text{for all } \lambda > 0.$$

This complements the aforementioned weak-type bounds as well as the results of Burkholder [Bu1], Burkholder, Gundy and Silverstein [BGS] and Petersen [Pe].

A few words about our approach and the organization of paper are in order. The proof of Theorem 1.2 will heavily depend on probabilistic arguments. More precisely, it will rest on Burkholder's method: the estimate (1.7) will be deduced from the existence of a certain special function on the plane, with appropriate majorization and harmonicity properties. See e.g. [O] for a more detailed description of this technique. The special function is introduced and studied in the next section. Section 3 is devoted to the proof of Theorem 1.2.

2. An auxiliary function and its properties. As we have announced above, the key element of the proof is a certain special function on $\mathbb{R}^2$. A similar construction appears in the work [C] of Choi, and the underlying ideas go back to the work of Baernstein [Ba] and Davis [D]. To introduce this object, we need some additional notation.

Let $\mathcal{H} = \{(x, y) : y > 0\}$ denote the upper half-plane and let $S = \mathbb{R} \times [-1, 1]$ stand for the horizontal strip in $\mathbb{R}^2$. Define $\mathcal{W} : \mathcal{H} \to \mathbb{R}$ by the

Poisson integral

$$\mathcal{W}(\alpha,\beta)=\frac{2}{\pi^2}\int_{-\infty}^{\infty}\frac{\beta(\log|t|)_+}{(\alpha-t)^2+\beta^2}\,dt=\frac{2}{\pi^2}\int_A\frac{\beta\log|t|}{(\alpha-t)^2+\beta^2}\,dt, \tag{2.1}$$

where $A=(-\infty,-1]\cup[1,\infty)$. By well-known properties of Poisson integrals (see e.g. Stein [S] or Grafakos [Gr]), this function is harmonic on $\mathcal{H}$ and satisfies the identity

$$\lim_{(\alpha,\beta)\to(t,0)}\mathcal{W}(\alpha,\beta)=\frac{2}{\pi}(\log|t|)_+,\qquad t\neq 0. \tag{2.2}$$

Consider the conformal mapping $\varphi: S\to\overline{\mathcal{H}}$ given by $\varphi(z)=ie^{\pi z/2}$, or, in real coordinates,

$$\varphi(x,y)=\Big(-e^{\pi x/2}\sin\frac{\pi y}{2},e^{\pi x/2}\cos\frac{\pi y}{2}\Big). \tag{2.3}$$

We are ready to introduce the main special function. Let $W:\mathbb{R}^2\to\mathbb{R}$ be given by

$$W(x,y)=\begin{cases}\mathcal{W}(\varphi(x,y)) & \text{if } |y|<1,\\ x_+ & \text{if } |y|\geq 1.\end{cases}$$

By virtue of (2.2), this function is continuous; furthermore, it is harmonic in the interior of S, being the composition of a harmonic function and a conformal mapping. Plugging (2.3) into (2.1), we easily check that if (x,y) lies in the interior of S, then

$$W(x,y)=\frac{2}{\pi^2}\int_{-\infty}^{\infty}\frac{\cos(\pi y/2)(\log|s|+\pi x/2)_+}{s^2+2\sin(\pi y/2)s+1}\,ds \tag{2.4}$$

(just substitute $t=se^{\pi x/2}$). In what follows, we will need some further properties of W, gathered in the lemma below. Recall that C is Davis' constant, given in (1.6).

LEMMA 2.1. *The function W satisfies the following conditions:*

(i) $W(x,y)=W(x,-y)$ *for all* $x,y\in\mathbb{R}$.

(ii) *If* (x,y) *belongs to the interior of* S, *then*

$$W_{xx}(x,y)\geq 0\quad\text{and}\quad W_{yy}(x,y)\leq 0.$$

(iii) *For any* $x\leq 0$ *and* $y\in\mathbb{R}$, *we have*

$$W(x,y)\leq W(0,0)=(2C)^{-1}. \tag{2.5}$$

(iv) *If* $|y|<1$, *then* $W_x(0,y)=1/2$.

Proof. (i) This is an immediate consequence of the following property of $\mathcal{W}$: for all $\alpha\in\mathbb{R}$ and $\beta>0$, $\mathcal{W}(\alpha,\beta)=\mathcal{W}(-\alpha,\beta)$. The latter statement can be established by substituting $t:=-t$ in (2.1).

(ii) The convexity with respect to the first variable is a consequence of (2.4) and the fact that for any nonzero s, the function $x \mapsto (\log|s| + \pi x/2)_+$ is convex. Now the inequality $W_{yy}(x,y) \leq 0$ follows at once from harmonicity of W inside the strip.

(iii) Let us first establish the formula for $W(0,0)$. We have

$$(2.6)\quad W(0,0) = \mathcal{W}(0,1) = \frac{2}{\pi^2}\int_{-\infty}^{\infty} \frac{(\log|t|)_+}{t^2+1}\,dt = \frac{4}{\pi^2}\int_1^{\infty}\frac{\log t}{t^2+1}\,dt$$

$$= \frac{4}{\pi^2}\int_0^{\infty}\frac{se^s}{e^{2s}+1}\,ds = \frac{4}{\pi^2}\int_0^{\infty} se^{-s}\sum_{k=0}^{\infty}(-e^{-2s})^k\,ds$$

$$= \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = \frac{1}{2}\cdot\frac{1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots}{1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots},$$

when in the last passage we have used the identity $\pi^2/8 = \sum_{k=0}^{\infty}(2k+1)^{-2}$. Therefore, the above chain of identities yields $W(0,0) = (2C)^{-1}$.

Now, let us turn to the majorization in (2.5). If $x \leq 0$ and $|y| \geq 1$, the bound is obvious, since $W(x,y)$ vanishes. To deal with the remaining x, y, note that for each fixed x, the function $y \mapsto W(x,y)$ is even and concave; this follows from the properties (i) and (ii) we have already proved above. Consequently, it is enough to show (2.5) under the additional assumption $y=0$. However, in this special case, the majorization follows from

$$\lim_{x\to-\infty} W(x,0) = \lim_{x\to-\infty}\frac{2}{\pi^2}\int_{-\infty}^{\infty}\frac{(\log|s|+\pi x/2)_+}{s^2+1}\,ds = 0$$

and the inequality

$$W_x(x,0) = \frac{d}{dx}\left(\frac{2}{\pi^2}\int_{-\infty}^{-e^{-\pi x/2}}\frac{\log(-s)+\pi x/2}{s^2+1}\,ds + \int_{e^{-\pi x/2}}^{\infty}\frac{\log s + \pi x/2}{s^2+1}\,ds\right)$$

$$= 1 - \frac{2}{\pi}\arctan(e^{-\pi x/2}) \geq 0$$

for $x \leq 0$.

(iv) A similar computation shows that

$$W_x(0,y) = \frac{1}{\pi}\left(\int_{-\infty}^{-1}\frac{\cos(\pi y/2)}{s^2+2\sin(\pi y/2)s+1}\,ds + \int_1^{\infty}\frac{\cos(\pi y/2)}{s^2+2\sin(\pi y/2)s+1}\,ds\right)$$

$$= \frac{1}{\pi}\left[\pi + \arctan\left(\tan(\pi y/2) - \frac{1}{\cos(\pi y/2)}\right) - \arctan\left(\tan(\pi y/2)+\frac{1}{\cos(\pi y/2)}\right)\right]$$

$$= 1/2.$$

Here in the last line we have used the identity

$$\left(\tan(\pi y/2) - \frac{1}{\cos(\pi y/2)}\right)\left(\tan(\pi y/2) + \frac{1}{\cos(\pi y/2)}\right) = -1$$

and the fact that $\arctan a - \arctan b = \pi/2$ when $ab = -1$ and $a > 0$.

This completes the proof. ■

3. Proof of Theorem 1.2. Our reasoning will depend heavily on the theory of continuous-time martingales, and we begin with a brief introduction of the necessary notions. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, equipped with a nondecreasing family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all the events of probability 0.

Let X, Y be two adapted, real-valued continuous-path martingales. Then $X^* = \sup_{t\geq 0} X_t$ and $|X|^* = \sup_{t\geq 0} |X_t|$ denote the one-sided and two-sided maximal functions of X; we will also use the notation $X_t^* = \sup_{0\leq s\leq t} X_s$ and $|X|_t^* = \sup_{0\leq s\leq t} |X_t|$ for the corresponding truncated maximal functions. The symbols $[X, X]$ and $[Y, Y]$ will stand for the square brackets of X and Y, respectively; see e.g. Dellacherie and Meyer [DM] for the definition.

Following Bañuelos and Wang [BW] and Wang [W], we say that Y is *differentially subordinate* to X if the process $([X, X]_t - [Y, Y]_t)_{t\geq 0}$ is nonnegative and nondecreasing as a function of t. Furthermore, X and Y are said to be *orthogonal* if their bracket $[X, Y]$ (defined by the polarization formula $[X, Y] = ([X+Y, X+Y] - [X-Y, X-Y])/4$) is constant.

We are ready to establish an auxiliary estimate, which can be regarded as a probabilistic version of Theorem 1.2. The statement is of independent interest; for related results, see e.g. Bañuelos and Wang [BW], Burkholder [Bu3], Janakiraman [J], Osękowski [O] and Wang [W]. In the statement below, C stands for Davis' constant defined in (1.6).

THEOREM 3.1. *Suppose that X, Y are orthogonal martingales such that $X_0 = Y_0 = 0$ and Y is differentially subordinate to X. Then*

$$\mathbb{P}(|Y|^* \geq 1) \leq C\mathbb{E}X^* \tag{3.1}$$

and the inequality is sharp.

Proof. It is convenient to split the reasoning into three parts.

Step 1. Let us start with some reductions. First, we may and do assume that the martingale Y takes values in the interval $[-1, 1]$. Indeed, if it is not the case, we replace it with the stopped martingale $Y^\tau = (Y_{\tau\wedge t})_{t\geq 0}$, where $\tau = \inf\{t : |Y_t| \geq 1\}$, which has the boundedness property (note that the orthogonality and the differential subordination to X is preserved under stopping).

Next, observe that it is enough to show that

$$\mathbb{P}(|Y_t| \geq 1) \leq C\mathbb{E}X^* \quad \text{for all } t \geq 0. \tag{3.2}$$

Indeed, having done this, we fix $\varepsilon \in (0,1)$ and apply the estimate to the stopped martingales $X^\sigma = (X_{\sigma\wedge t})_{t\geq 0}$, $Y^\sigma = (Y_{\sigma\wedge t})_{t\geq 0}$, where $\sigma = \inf\{t \geq 0 : |Y_t| \geq \varepsilon\}$. This is allowed, since orthogonality and differential subordination hold for this new pair as well. Note that $\{|Y|^* \geq 1\} \subseteq \{|Y_t| \geq \varepsilon \text{ for some } t \geq 0\}$ and hence

$$\mathbb{P}(|Y|^* \geq 1) \leq \lim_{t\to\infty} \mathbb{P}(|Y_{\sigma\wedge t}| \geq \varepsilon) \leq C\mathbb{E}(X^\sigma)^*/\varepsilon \leq C\mathbb{E}X^*/\varepsilon.$$

Therefore, letting $\varepsilon \uparrow 1$ yields (3.1).

Step 2. So, we must prove (3.2) for Y bounded in absolute value by 1. Introduce the process $Z = ((X_t, Y_t, X_t^*))_{t\geq 0}$ and let $t \geq 0$ be a fixed time parameter. Consider the continuous function $U : \mathbb{R}^3 \to \mathbb{R}$ given by

$$U(x,y,z) = 1 - 2CW(x-z,y) - Cz,$$

where W comes from the preceding section. The function W is of class C^∞ in the interior of the strip $S = \mathbb{R} \times [-1,1]$ and hence U is of class C^∞ on $\mathbb{R} \times (-1,1) \times \mathbb{R}$. Therefore, by the assumed boundedness of Y, we can apply Itô's formula to obtain

$$U(Z_t) = I_0 + I_1 + I_2 + \frac{1}{2} I_3 + I_4, \tag{3.3}$$

where

$$\begin{aligned}
I_0 &= U(Z_0),\\
I_1 &= \int_{0+}^{t} U_x(Z_s)\,dX_s + \int_{0+}^{t} U_y(Z_s)\,dY_s,\\
I_2 &= \int_{0+}^{t} U_z(Z_s)\,dX_s^*,\\
I_3 &= \int_{0+}^{t} U_{xy}(Z_s)\,d[X,Y]_s,\\
I_4 &= \int_{0+}^{t} U_{xx}(Z_s)\,d[X,X]_s + \int_{0+}^{t} U_{yy}(Z_s)\,d[Y,Y]_s.
\end{aligned}$$

Let us take a look at each term above separately. By Lemma 2.1(iii), we see that

$$U(Z_0) = U(0,0,0) = 1 - 2CW(0,0) = 0.$$

The term I_1 has zero expectation, by the properties of stochastic integrals. To deal with $I_2 = 0$, we apply the following argument. By the continuity of the trajectories of X, the times at which $X_t^*(\omega)$ increases satisfy $X_t(\omega) = X_t^*(\omega)$. However, the integrand then vanishes: indeed, by Lemma 2.1(iv), we have $U_z(z, y, z) = 2CW_x(0, y) - C = 0$; consequently, $I_2 = 0$. Next, we see that the term I_3 also vanishes, in view of the orthogonality of X and Y. Finally, differential subordination combined with Lemma 2.1(ii) imply

$$I_4 \leq \int_0^t U_{xx}(Z_s)\, d[X, X]_s + \int_0^t U_{yy}(Z_s)\, d[X, X]_s = 0,$$

since $U_{xx}(x, y, z) + U_{yy}(x, y, z) = 0$ for $|y| < 1$. Putting all the above facts together and integrating both sides of (3.3) gives

$$\mathbb{E}U(Z_t) = \mathbb{E}U(X_t, Y_t, X_t^*) \leq 0.$$

Step 3. Now we will deduce (3.2) from the latter bound. Observe that for any $x \leq z$ and any $y \in \mathbb{R}$ we have

$$U(x, y, z) \geq 1_{\{|y| \geq 1\}} - Cz. \tag{3.4}$$

Indeed, if $|y| \geq 1$, then $U(x, y, z) = 1 - 2CW(x - z, y) - Cz = 1 - Cz$; on the other hand, if $|y| < 1$, Lemma 2.1(iii) gives

$$U(x, y, z) = 1 - 2CW(x - z, y) - Cz \geq -Cz.$$

Thus, the inequality (3.4) is valid, and hence the bound $\mathbb{E}U(X_t, Y_t, X_t^*) \leq 0$, proved in the previous step, yields

$$\mathbb{P}(|Y_t| \geq 1) - C\mathbb{E}X_t^* \leq 0.$$

This of course implies (3.2) and completes the proof of (3.1). Sharpness will follow at once from the optimality of C in (1.7); see the proof below. ■

Finally, we are ready to proceed with Theorem 1.2.

Proof of (1.7). Let u, v be harmonic functions as in the statement of the theorem and let B be a Brownian motion starting from ξ. Let τ be the corresponding exit-time defined in Section 1. By homogeneity, we will be done if we establish the estimate in the particular case $\lambda = 1$. Introduce the processes

$$X = (X_t)_{t\geq 0} = (u(B_{\tau\wedge t}))_{t\geq 0}, \qquad Y = (Y_t)_{t\geq 0} = (v(B_{\tau\wedge t}))_{t\geq 0}.$$

As compositions of harmonic functions with Brownian motion, these processes are martingales. Furthermore, Y is orthogonal and differentially subordinate to X, as follows immediately from the identities

$$[X, X]_t = u(\xi)^2 + \int_0^{\tau\wedge t} |\nabla u(B_s)|^2\, ds,$$

$$[Y,Y]_t = v(\xi)^2 + \int_0^{\tau\wedge t} |\nabla v(B_s)|^2\, ds,$$

$$[X,Y]_t = u(\xi)v(\xi) + \int_0^{\tau\wedge t} \nabla u(B_s)\cdot\nabla v(B_s)\, ds$$

and the assumptions (1.1)–(1.3). Consequently, by the previous theorem, we get

$$\mathbb{P}(|v|^* \geq 1) = \mathbb{P}(|Y|^* \geq 1) \leq C\mathbb{E}X^* \leq C\mathbb{E}u^*,$$

which is (1.7).

To see that the estimate is sharp, we may again restrict ourselves to the case $\lambda = 1$. Take D to be the strip $\mathbb{R}\times[-1,1]$, let $\xi = 0$ and consider the harmonic functions $u(x,y) = x$, $v(x,y) = y$. Clearly, these functions satisfy the Cauchy–Riemann equations and the normalization requirement $u(0) = v(0) = 0$. Observe that the exit-time τ satisfies

$$\tau = \inf\{t \geq 0 : |B_t^{(2)}| = 1\} \tag{3.5}$$

with probability 1 (where $B^{(2)}$ denotes the second coordinate of the Brownian motion B). Hence $|v|^* = |B_t^{(2)}|^*$ is equal to 1 almost surely and thus the left-hand side of (1.7) is 1. Furthermore, by (3.5), the stopping time τ is independent of $B^{(1)}$, the first coordinate of B. Consequently,

$$\mathbb{E}u^* = \mathbb{E}(B_\tau^{(1)})^* = \mathbb{E}[\mathbb{E}((B_\tau^{(1)})^* \,|\, \tau)] = \mathbb{E}[\mathbb{E}|B_\tau^{(1)}| \,|\, \tau] = \mathbb{E}|B_\tau^{(1)}|,$$

where in the third equality we have used the above independence and the well-known fact that for any t, the variables $(B_t^{(1)})^*$ and $|B_t^{(1)}|$ have the same distribution. Now, we exploit the function W of Section 2: by Itô's formula, we obtain

$$\mathbb{E}|B_\tau^{(1)}| = 2\mathbb{E}[W(B_\tau) - B_\tau^{(1)}] = 2\mathbb{E}W(B_\tau) = 2\mathbb{E}W(0,0) = \frac{1}{C}.$$

Consequently, the two sides of (1.7) are equal and the proof of the theorem is complete. Finally, let us mention that this immediately implies that the estimate (3.1) is sharp: indeed, if the constant C could be decreased there, this would lead to the corresponding improvement of (1.7). ∎

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