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GENERAL TOPOLOGY

Remarks on Star-Hurewicz Spaces

by

Yan-Kui SONG

Presented by Czesław BESSAGA

Summary. A space X is *star-Hurewicz* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and for each $x \in X$, $x \in \mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many n. We investigate the relationship between star-Hurewicz spaces and related spaces, and also study topological properties of star-Hurewicz spaces.

1. Introduction. By a space, we mean a topological space. Let $\mathbb{N}$ denote the set of positive integers. Let X be a space and $\mathcal{U}$ be a collection of subsets of X. For $A \subseteq X$, let

$$\mathrm{St}(A, \mathcal{U}) = \bigcup \{U \in \mathcal{U} : U \cap A \neq \emptyset\}.$$

As usual, we write $\mathrm{St}(x, \mathcal{U})$ instead of $\mathrm{St}(\{x\}, \mathcal{U})$.

Let us recall that a space X is *countably compact* if every countable open cover of X has a finite subcover. Van Douwen et al. [5] defined a space X to be *strongly starcompact* if for every open cover $\mathcal{U}$ of X there exists a finite subset F of X such that $\mathrm{St}(F, \mathcal{U}) = X$. They proved that every countably compact space is strongly starcompact and every strongly starcompact T_2-space is countably compact; however, the latter does not hold for T_1-spaces (see [12, Example 2.5]).

Van Douwen et al. [5] defined a space X to be *starcompact* if for every open cover $\mathcal{U}$ of X there exists a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{St}(\bigcup \mathcal{V}, \mathcal{U}) = X$.

In [7], a strongly starcompact space is called starcompact, and in [11], a starcompact space is called $1\frac{1}{2}$-starcompact.

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In 1925, Hurewicz [8] (see also [9]) introduced the *Hurewicz covering property* for a space X in the following way:

H: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and for each $x \in X$, $x \in \bigcup \mathcal{V}_n$ for all but finitely many n.

Cammaroto et al. [3] introduced and investigated a generalization of the Hurewicz covering property. Bonanzinga et al. [1] introduced star selection hypotheses similar to the above (see also [10]):

SH: A space X has the *star-Hurewicz property* (or is a *star-Hurewicz space*) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and for each $x \in X$, $x \in \mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many n.

SSH: A space X has the *strongly star-Hurewicz property* (or is a *Strongly star-Hurewicz space*) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that for each $x \in X$, $x \in \mathrm{St}(A_n, \mathcal{U}_n)$ for all but finitely many n.

From the above definitions, we have the following diagram:

$$\begin{array}{ccc}
\text{strongly starcompact} & \longrightarrow & \text{strongly star-Hurewicz} \\
\downarrow & & \downarrow \\
\text{starcompact} & \longrightarrow & \text{star-Hurewicz}
\end{array}$$

Bonanzinga et al. [1] studied star-Hurewicz and related spaces. The purpose of this paper is to investigate the relationships between star-Hurewicz spaces and related spaces, and also to study topological properties of star-Hurewicz spaces.

Throughout this paper, the *extent* $e(X)$ of a space X is the smallest infinite cardinal κ such that every discrete closed subset of X has cardinality at most κ. Let ω denote the first infinite cardinal, ω_1 the first uncountable cardinal and $\mathfrak{c}$ the cardinality of the set of all real numbers. For a cardinal κ, let κ^+ be the smallest cardinal greater than κ. For any ordinals α, β with $\alpha < \beta$, we write $[\alpha, \beta)$, (α, β) etc. for the usual ordinal intervals. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [6].

2. Main results. First we give some examples showing relationships between strongly star-Hurewicz spaces and related spaces.

Example 2.1. *There exists a Tychonoff strongly star-Hurewicz (hence star-Hurewicz) space X which is not starcompact (hence not strongly starcompact).*

Proof. Consider the subspace

$$X = ([0,\omega] \times [0,\omega]) \setminus \{\langle \omega,\omega \rangle\}$$

of the product space $[0,\omega] \times [0,\omega]$. Clearly, X is a Tychonoff space.

First we show that X is strongly star-Hurewicz. To this end, let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. For each $n \in \mathbb{N}$, let

$$K_n = ([0,\omega] \times [0,n-1]) \cup ([0,n-1] \times [0,\omega]).$$

Then K_n is the union of finite compact subsets of X. We can find a finite subset A_n of K_n such that $K_n \subseteq \mathrm{St}(A_n, \mathcal{U}_n)$. Note that $\bigcup_{n\in\mathbb{N}} K_n = X$. For any $x \in X$, there exists $n_0 \in \mathbb{N}$ such that $x \in K_{n_0}$, thus $x \in \mathrm{St}(A_n, \mathcal{U}_n)$ for each $n > n_0$, which shows that X is strongly star-Hurewicz.

Next we show that X is not starcompact. For each $n \in \omega$, let

$$U_n = \{n\} \times [0,\omega] \quad \text{and} \quad V_n = (n,\omega] \times \{n\}.$$

Let us consider the open cover

$$\mathcal{U} = \{U_n : n \in \omega\} \cup \{V_n : n \in \omega\}$$

of X. It suffices to show that $\mathrm{St}(\bigcup \mathcal{V}, \mathcal{U}) \neq X$ for any finite subset $\mathcal{V}$ of $\mathcal{U}$. To see this, let $\mathcal{V}$ be any finite subset of $\mathcal{U}$. There exist $n_1 \in \omega$ and $n_2 \in \omega$ such that $U_n \notin \mathcal{V}$ for each $n > n_1$ and $V_n \notin \mathcal{V}$ for each $n > n_2$. Pick $n' > \max\{n_1, n_2\}$; then $V_{n'} \cap (\bigcup \mathcal{V}) = \emptyset$, hence $\langle n', \omega \rangle \notin \mathrm{St}(\bigcup \mathcal{V}, \mathcal{U})$, since $V_{n'}$ is the only element of $\mathcal{U}$ containing $\langle n', \omega \rangle$. This shows that X is not starcompact. ∎

Example 2.2. *There exists a Tychonoff star-Hurewicz space which is not strongly star-Hurewicz.*

Proof. Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let $D^* = D \cup \{d^*\}$ be one-point compactification of D, where $d^* \notin D$.

Consider the subspace

$$X = (D^* \times [0,\mathfrak{c}^+]) \setminus \{\langle d^*, \mathfrak{c}^+ \rangle\}$$

of the product space $D^* \times [0,\mathfrak{c}^+]$. Then X is Tychonoff.

To show that X is star-Hurewicz, it suffices to show that X is starcompact, since every starcompact space is star-Hurewicz.

To this end, let $\mathcal{U}$ be an open cover of X. For each $\alpha < \mathfrak{c}$, there exists $U_\alpha \in \mathcal{U}$ such that $\langle d_\alpha, \mathfrak{c}^+ \rangle \in U_\alpha$, hence we can find $\beta_\alpha < \mathfrak{c}^+$ such that $\{d_\alpha\} \times (\beta_\alpha, \mathfrak{c}^+] \subseteq U_\alpha$. Let $\beta = \sup\{\beta_\alpha : \alpha < \mathfrak{c}\}$. Then $\beta < \mathfrak{c}^+$. Let $K = D^* \times \{\beta\}$. Then K is compact and $U_\alpha \cap K \neq \emptyset$ for each $\alpha < \mathfrak{c}$. Since $\mathcal{U}$

covers K, there exists a finite subset $\mathcal{U}'$ of $\mathcal{U}$ such that $K \subseteq \bigcup \mathcal{U}'$. Thus

$$D \times \{\mathfrak{c}^+\} \subseteq \mathrm{St}\Big(\bigcup \mathcal{U}', \mathcal{U}\Big).$$

On the other hand, since $D^* \times [0, \mathfrak{c}^+)$ is countably compact, it is strongly starcompact (see [5, 11]), so we can find a finite subset $\mathcal{U}''$ of $\mathcal{U}$ such that

$$D^* \times [0, \mathfrak{c}^+) \subseteq \mathrm{St}\Big(\bigcup \mathcal{U}'', \mathcal{U}\Big).$$

If we put $\mathcal{V} = \mathcal{U}' \cup \mathcal{U}''$, then $\mathcal{V}$ is a finite subset of $\mathcal{U}$ such that $X = \mathrm{St}(\bigcup \mathcal{V}, \mathcal{U})$, which shows that X is starcompact.

Next we show that X is not strongly star-Hurewicz. For each $\alpha < \mathfrak{c}$, let

$$U_\alpha = \{d_\alpha\} \times [0, \mathfrak{c}^+].$$

For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{D^* \times [0, \mathfrak{c}^+)\}.$$

Then $\mathcal{U}_n$ is an open cover of X. Let $(A_n : n \in \mathbb{N})$ be any sequence of finite subsets of X. It suffices to show that there exists $x \in X$ such that $x \notin \mathrm{St}(A_n, \mathcal{U}_n)$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, since A_n is finite, there exists $\alpha_n < \mathfrak{c}$ such that

$$A_n \cap U_\alpha = \emptyset \quad \text{for each } \alpha > \alpha_n.$$

Let $\beta = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\beta < \mathfrak{c}$ and

$$\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) \cap U_\alpha = \emptyset \quad \text{for each } \alpha > \beta.$$

If we pick $\alpha' > \beta$, then $\langle d_{\alpha'}, \mathfrak{c}^+\rangle \notin \mathrm{St}(A_n, \mathcal{U}_n)$ for each $n \in \mathbb{N}$, since $U_{\alpha'}$ is the only element of $\mathcal{U}_n$ containing $\langle d_{\alpha'}, \mathfrak{c}^+\rangle$; this shows that X is not strongly star-Hurewicz. ∎

REMARK 2.1. Example 2.2 also shows that there exists a Tychonoff starcompact space that is not strongly starcompact.

Recall that the *Alexandorff duplicate* $A(X)$ of a space X is constructed in the following way: The underlying set of $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of $\langle x, 0\rangle \in X \times \{0\}$ is of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X.

It is well known that a space X is countably compact if and only if so is $A(X)$. In the following, we give two examples to show that this result cannot be generalized to star-Hurewicz spaces.

EXAMPLE 2.3. *There exists a Tychonoff star-Hurewicz space X such that $A(X)$ is not star-Hurewicz.*

Proof. Let $X = (D^* \times [0, \mathfrak{c}^+]) \setminus \{\langle d^*, \mathfrak{c}^+\rangle\}$ be the space of Example 2.2. Then X is a star-Hurewicz space with $e(X) = \mathfrak{c}$.

However, $A(X)$ is not star-Hurewicz. In fact, $(D \times \{\mathfrak{c}\}) \times \{1\}$ is an open and closed subset of X with $|(D \times \{\mathfrak{c}\}) \times \{1\}| = \mathfrak{c}$, and each point $\langle\langle d_\alpha, \mathfrak{c}\rangle, 1\rangle$ is isolated for each $\alpha < \mathfrak{c}$. Hence $A(X)$ is not star-Hurewicz, since every open and closed subset of a star-Hurewicz space is star-Hurewicz, and $(D \times \{\mathfrak{c}\}) \times \{1\}$ is not star-Hurewicz. ∎

To show the next result, we need a lemma from [4].

LEMMA 2.4. *For a T_1-space X, $e(X) = e(A(X))$.*

From the proof of Example 2.3, it is not difficult to deduce the following result.

THEOREM 2.5. *If X is a T_1-space and $A(X)$ is a star-Hurewicz space, then $e(A(X)) < \omega_1$.*

Proof. By Lemma 2.4, we need to show that $e(X) < \omega_1$. Suppose that $e(X) \geq \omega_1$. Then there exists a discrete closed subset B of X such that $|B| \geq \omega_1$. Hence $B \times \{1\}$ is an open and closed subset of $A(X)$ and every point of $B \times \{1\}$ is isolated. Thus $A(X)$ is not star-Hurewicz, since every open and closed subset of a star-Hurewicz space is star-Hurewicz, and $B \times \{1\}$ is not star-Hurewicz. ∎

REMARK 2.2. The author does not know if the Alexandorff duplicate $A(X)$ of a star-Hurewicz T_1-space X with $e(X) < \omega_1$ is star-Hurewicz.

It is not difficult to prove the following result.

THEOREM 2.6. *Every continuous image of a star-Hurewicz space is star-Hurewicz.*

Proof. Let $f : X \to Y$ be a continuous mapping from a star-Hurewicz space X onto a space Y. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$. It is an open cover of X. Since X is star-Hurewicz, there exists a sequence $(\mathcal{V}'_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}'_n$ is a finite subset of $\mathcal{U}_n$, and for each $x \in X$, $x \in \mathrm{St}(\bigcup \mathcal{V}'_n, \mathcal{U}_n)$ for all but finitely many n. For each $n \in \mathbb{N}$, let $\mathcal{U}'_n = \{f(U) : U \in \mathcal{V}'_n\}$. Then $\mathcal{U}'_n$ is a finite subset of $\mathcal{U}_n$.

We will show that each $y \in Y$ is in $\mathrm{St}(\bigcup \mathcal{U}'_n, \mathcal{U}_n)$ for all but finitely many n. In fact, let $y \in Y$. Then there is $x \in X$ such that $f(x) = y$. Hence $x \in \mathrm{St}(\bigcup \mathcal{V}'_n, \mathcal{V}_n)$ for all but finitely many n. Thus $y = f(x) \in \mathrm{St}(\bigcup\{f(U) : U \in \mathcal{V}'_n\}, \{f(U) : U \in \mathcal{V}_n\}) = \mathrm{St}(\bigcup \mathcal{U}'_n, \mathcal{U}_n)$ for all but finitely many n, which shows that Y is star-Hurewicz. ∎

Next we turn to preimages. We show that the preimage of a star-Hurewicz space under a closed 2-to-1 continuous map need not be star-Hurewicz.

EXAMPLE 2.7. *There exists a closed 2-to-1 continuous map $f : X \to Y$ such that Y is star-Hurewicz, but X is not.*

Proof. Just consider the projection map $A(X) \to X$, where X is the space of Example 2.3. ∎

Now we give a positive result:

THEOREM 2.8. *If f is an open perfect continuous map from a space X onto a star-Hurewicz space Y, then X is star-Hurewicz.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and let $y \in Y$. For each $n \in \mathbb{N}$, since $f^{-1}(y)$ is compact, there exists a finite subcollection $\mathcal{U}_{n_y}$ of $\mathcal{U}_n$ such that $f^{-1}(y) \subseteq \bigcup \mathcal{U}_{n_y}$ and $U \cap f^{-1}(y) \neq \emptyset$ for each $U \in \mathcal{U}_{n_y}$. Since f is closed, there exists an open neighborhood V_{n_y} of y in Y such that

$$f^{-1}(V_{n_y}) \subseteq \bigcup \{U : U \in \mathcal{U}_{n_y}\}.$$

Since f is open, we can assume that

$$V_{n_y} \subseteq \bigcap \{f(U) : U \in \mathcal{U}_{n_y}\}. \tag{1}$$

Since $\mathcal{V}_n = \{V_{n_y} : y \in Y\}$ is an open cover of Y. Y is star-Hurewicz, there exists a sequence $(\mathcal{V}'_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{V}_n$, and for each $y \in Y$, $y \in \mathrm{St}(\bigcup \mathcal{V}'_n, \mathcal{V}_n)$ for all but finitely many n. Without loss of generality, we can assume that $\mathcal{V}'_n = \{V_{n_{y_i}} : i < n'\}$. Let $\mathcal{U}'_n = \bigcup_{i<n'} \mathcal{U}_{n_{y_i}}$. Then $\mathcal{U}'_n$ is a finite subset of $\mathcal{U}_n$ and

$$f^{-1}\Big(\bigcup \mathcal{V}'_n\Big) \subseteq \bigcup \mathcal{U}'_n. \tag{2}$$

Next we show that each $x \in X$ is in $\mathrm{St}(\bigcup \mathcal{U}'_n, \mathcal{U}_n)$ for all but finitely many n. Let $x \in X$. Then $y = f(x) \in \mathrm{St}(\bigcup \mathcal{V}'_n, \mathcal{V}_n)$ for all but finitely many n. For $n \in \mathbb{N}$, if $y = f(x) \in \mathrm{St}(\bigcup \mathcal{V}'_n, \mathcal{V}_n)$, then there exist $i < n'$ and $y' \in Y$ such that $V_{n_{y_i}} \cap V_{n_{y'}} \neq \emptyset$ and $y \in V_{n_{y'}}$. Since

$$x \in f^{-1}(V_{n_{y'}}) \subseteq \bigcup \{U : U \in \mathcal{U}_{n_{y'}}\},$$

we can choose $U \in \mathcal{U}_{n_{y'}}$ with $x \in U$. Then $V_{n_{y'}} \subseteq f(U)$ by (1). Hence $U \cap f^{-1}(\bigcup \mathcal{V}'_n) \neq \emptyset$. Thus $x \in \mathrm{St}(f^{-1}(\bigcup \mathcal{V}'_n), \mathcal{U}_n)$. Therefore $x \in \mathrm{St}(\bigcup \mathcal{U}'_n, \mathcal{U}_n)$ by (2), which shows that each $x \in X$ is in $\mathrm{St}(\bigcup \mathcal{U}'_n, \mathcal{U}_n)$ for all but finitely many n. Consequently, X is star-Hurewicz. ∎

We have the following corollary from Theorem 2.8.

COROLLARY 2.9 ([1]). *The product of a star-Hurewicz space and a compact space is star-Hurewicz.*

However, the product of two star-Hurewicz spaces need not be star-Hurewicz. In fact, the following well-known example shows that the product of two countably compact (hence star-Hurewicz) spaces need not be star-Hurewicz. We sketch the proof for completeness.

EXAMPLE 2.10. *There exist two Tychonoff countably compact (hence star-Hurewicz) spaces X and Y such that $X \times Y$ is not star-Hurewicz.*

Proof. Let D be a discrete space of cardinality $\mathfrak{c}$. We define $X = \bigcup_{\alpha<\omega_1} E_\alpha$ and $Y = \bigcup_{\alpha<\omega_1} F_\alpha$, where E_α and F_α are subsets of βD which are defined inductively so as to satisfy the following conditions:

(1) $E_\alpha \cap F_\beta = D$ if $\alpha \neq \beta$;
(2) $|E_\alpha| \leq \mathfrak{c}$ and $|F_\beta| \leq \mathfrak{c}$;
(3) every infinite subset of E_α (resp., F_α) has an accumulation point in $E_{\alpha+1}$ (resp., $F_{\alpha+1}$).

The sets E_α and F_α are well-defined since every infinite closed set in βD has cardinality $2^{\mathfrak{c}}$ (see [13]). The diagonal $\{\langle d,d\rangle : d \in D\}$ is an open and closed subset of $X \times Y$ with cardinality $\mathfrak{c}$ and $\{\langle d,d\rangle\}$ is isolated for each $d \in D$. Thus $X \times Y$ is not star-Hurewicz, since open and closed subsets of star-Hurewicz spaces are star-Hurewicz, and the diagonal $\{\langle d,d\rangle : d \in D\}$ is not star-Hurewicz. ∎

In [5, Example 3.3.3], van Douwen et al. gave an example of a countably compact (and hence star-Hurewicz) space X and a Lindelöf space Y such that $X \times Y$ is not strongly star-Lindelöf. Now, we shall show that $X \times Y$ is not star-Hurewicz either. We need the following lemma.

LEMMA 2.11 ([2]). *A space X is a star-Hurewicz space iff for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a finite $\mathcal{V}_n \subseteq \mathcal{U}_n$ $(n \in \mathbb{N})$ such that for every $x \in X$, $\mathrm{St}(x,\mathcal{U}_n) \cap (\bigcup \mathcal{V}_n) \neq \emptyset$ for all but finitely many $n \in \mathbb{N}$.*

EXAMPLE 2.12. *There exist a countably compact (hence star-Hurewicz) space X and a Lindelöf space Y such that $X \times Y$ is not star-Hurewicz.*

Proof. Let $X = [0,\omega_1)$ with the usual order topology and $Y = [0,\omega_1]$ with the following topology: each point α with $\alpha < \omega_1$ is isolated and a set U containing ω_1 is open if and only if $Y \setminus U$ is countable. Then X is countably compact and Y is Lindelöf.

Now, we show that $X \times Y$ is not star-Hurewicz. For each $\alpha < \omega_1$, let

$$U_\alpha = [0,\alpha] \times [\alpha,\omega_1] \quad \text{and} \quad V_\alpha = (\alpha,\omega_1) \times \{\alpha\}.$$

Then

$$U_\alpha \cap V_{\alpha'} = \emptyset \quad \text{for any } \alpha < \omega_1 \text{ and } \alpha' < \omega_1,$$

and

$$V_\alpha \cap V_{\alpha'} = \emptyset \quad \text{if } \alpha \neq \alpha'.$$

For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\alpha : \alpha < \omega_1\}.$$

Then $\mathcal{U}_n$ is an open cover of $X \times Y$. By Lemma 2.11 it suffices to show that for any sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that $\mathcal{V}_n \subseteq \mathcal{U}_n$ for each $n \in \mathbb{N}$, there exists a point $a \in X \times Y$ such that $\mathrm{St}(a, \mathcal{U}_n) \cap (\bigcup \mathcal{V}_n) = \emptyset$ for all $n \in \mathbb{N}$.

Let $(\mathcal{V}_n : n \in \mathbb{N})$ be any sequence such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$. For each $n \in \mathbb{N}$, since $\mathcal{V}_n$ is finite, there exists $\alpha_n < \omega_1$ such that

$$V_\alpha \notin \mathcal{V}_n \quad \text{for each } \alpha > \alpha_n.$$

Let $\beta = \sup\{\alpha_n : n \in \mathbb{N}\}$. Clearly, $\beta < \omega_1$. Thus we have

$$V_\alpha \notin \mathcal{V}_n \quad \text{for each } \alpha > \beta \text{ and } n \in \mathbb{N}.$$

Pick $\alpha > \beta$. Since V_α is the only element of $\mathcal{U}_n$ containing the point $\langle \alpha+1, \alpha\rangle$, and $V_\alpha \cap V = \emptyset$ for each $V \in \mathcal{V}_n$ and $n \in \mathbb{N}$, it follows that $\mathrm{St}(\langle \alpha+1, \alpha\rangle, \mathcal{U}_n) = V_\alpha$ and $V_\alpha \cap (\bigcup \mathcal{V}_n) = \emptyset$ for $n \in \mathbb{N}$. ∎

Next we give a condition under which paraLindelöfness implies Lindelöfness. Recall that a space X is *paraLindelöf* if every open cover $\mathcal{U}$ of X has a locally countable open refinement.

Theorem 2.13. *Every paraLindelöf star-Hurewicz space is Lindelöf.*

Proof. Let X be a paraLindelöf star-Hurewicz space and $\mathcal{U}$ be an open cover of X. Then there exists a locally countable open refinement $\mathcal{V}$ of $\mathcal{U}$. For each $x \in X$, there exists an open neighborhood V_x of x such that $V_x \subseteq V$ for some $V \in \mathcal{V}$ and $\{V \in \mathcal{V} : V_x \cap V \neq \emptyset\}$ is countable. Let $\mathcal{V}' = \{V_x : x \in X\}$. Then $\mathcal{V}'$ is an open refinement of $\mathcal{V}$. Since X is star-Hurewicz, there exists a sequence $(\mathcal{V}'_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}'_n$ is a finite subset of $\mathcal{V}'$, and for each $x \in X$, $x \in \mathrm{St}(\bigcup \mathcal{V}'_n, \mathcal{V}')$ for all but finitely many n.

For each $n \in \mathbb{N}$, let

$$\mathcal{V}_n = \{V \in \mathcal{V} : \text{there exists some } V' \in \mathcal{V}'_n \text{ such that } V \cap V' \neq \emptyset\}.$$

Then $\mathcal{V}_n$ is a countable subset of $\mathcal{V}$. Let $\mathcal{W} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$. Then $\mathcal{W}$ is a countable open cover of X. For each $V \in \mathcal{W}$, choose $U_V \in \mathcal{U}$ such that $V \subseteq U_V$. Then $\{U_V : V \in \mathcal{W}\}$ is a countable subcover of $\mathcal{U}$, which shows that X is Lindelöf. ∎

Since every paracompact space is paraLindelöf, the following corollary follows from Theorem 2.13.

Corollary 2.14. *A paracompact star-Hurewicz space X is Lindelöf.*

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Yan-Kui Song
Institute of Mathematics
School of Mathematical Sciences
Nanjing Normal University
Nanjing 210023, P.R. China
E-mail: songyankui@njnu.edu.cn

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