BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 61, No. 3-4, 2013

ALGEBRAIC TOPOLOGY

Fixed Points of Maps of Sets of Polyhedra into the Disc

by

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Summary. We prove that Platonic and some Archimedean polyhedra have the fixed point property in a non-classical sense.

1. Introduction. First we define a property of polyhedra $P \subset \mathbb{R}^n$ which will be studied in this paper. Denote by $\mathbb{B}^n$ the closed unit ball in $\mathbb{R}^n$ with respect to the Euclidean metric d. Fix a real positive number $\epsilon < 1$. Let $P(\epsilon)$ be a polyhedron similar to P of diameter ϵ. Consider the family $F(P, \epsilon)$ of all $A \subset \mathbb{B}^n$ that are isometric to $P(\epsilon)$ by an orientation preserving isometry (i.e. a rotation composed with a translation).

DEFINITION 1. We say that the pair (P, ϵ) has the *Brouwer property* in dimension n if for every Hausdorff continuous mapping $f : F(P, \epsilon) \to \mathbb{B}^n$ there is a polyhedron $A \in F(P, \epsilon)$ such that $f(A) \in A$; we then say that A is a *fixed point* of f.

REMARKS. 1. The above property a priori depends on the number ϵ. We have no example of real dependence, but we are also unable to prove independence. In all our results independence occurs, so we fix an ϵ and say that P has (or does not have) the Brouwer property in dimension n. In what follows, $n = 3$. The case $n = 2$ in a slightly different setting is considered in [12].

2. Recall the notion of the Hausdorff metric d_{H} in the space $\exp(Y)$ of all nonempty closed bounded subsets of the metric space (Y, d):

$$\forall_{A,B} \quad d_{\mathrm{H}}(A, B) = \inf\{\delta > 0 : A \subset O_\delta(B) \text{ and } B \subset O_\delta(A)\},$$
$$O_\delta(A) = \{x \in Y : \exists_{a \in A} d(x, a) < \delta\}.$$

In $F(P, \epsilon)$ we consider the topology of a subspace of $(\exp(\mathbb{R}^n), d_{\mathrm{H}})$.

2010 *Mathematics Subject Classification*: 55M20, 52B10.
Key words and phrases: fixed points, Brouwer's theorem, Platonic solids.

DOI: 10.4064/ba61-3-8

3. Some polyhedra have the Brouwer property (e.g. the cube) and some other do not (e.g. the pyramid with the square base). The goal of this paper is to prove the Brouwer property for Platonic solids and for regular polyhedra obtained from them by truncation of vertices.

4. The idea of our proof that a solid P has the Brouwer property comes from one of the proofs of the classical Brouwer fixed point theorem: the assumption that f has no fixed points results in the construction of a *retraction-like* mapping onto the sphere $\mathbb{S}^2$, which cannot exist for some algebraic topology reasons.

5. Is studying the Brouwer property somehow motivated? In my opinion a sufficient motivation could be an application of wonderful mathematics. Nevertheless we can imagine a crystal moving in the disc and ...—I can only dream that these results will turn out to be suitable for some physical or chemical descriptions.

1.1. Scheme of construction. Denote by P_i, $i = 4, 6, 8, 12, 20$, one of Platonic polyhedra: tetrahedron, cube, octahedron, dodecahedron, icosahedron. Consider

$$\begin{aligned} X &:= F(P_i, \epsilon), \\ X_0 &:= \{A \in X : A \text{ is centred at } O = 0 \in \mathbb{R}^3\}, \\ X_B &:= \{A \in X : \text{ one of the vertices of } A, \text{ say } V, \text{ is in } \mathbb{S}^2 \text{ and } A \text{ is} \\ &\qquad\quad \text{centred at a point of the segment } OV\}. \end{aligned}$$

For fixed $A_0 \in X_0$ and a vertex V_0 of A_0 consider

$$G_i := \{g \in SO(3) : g[A_0] = A_0\}, \text{ the } \textit{rotation group of } P_i,$$

and its cyclic subgroup of rank $m(i)$,

$$Z_{m(i)} := \{g \in G_i : g(V_0) = V_0\},$$

where $m(i)$ is the number of edges of A_0 emanating from V_0.

We find homeomorphisms $\phi : SO(3)/G_i \to X_0$ and $\psi : SO(3)/\mathbb{Z}_{m(i)} \to X_B$, which together with the parallel displacement $X_B \to X_0$ and the natural projection $\eta : SO(3)/\mathbb{Z}_{m(i)} \to SO(3)/G_i$ make the left rectangle in the diagram

$$\begin{tikzcd} SO(3)/\mathbb{Z}_{m(i)} \arrow[r] \arrow[d] & X_B \arrow[d] \arrow[dr] & \\ & & X \\ SO(3)/G_i \arrow[r] & X_0 \arrow[ur] & \end{tikzcd}$$

commutative, while the inclusions $X_0 \subset X$, $X_B \subset X$ make the right triangle homotopy commutative.

Assume that there exists a fixed point free map $f : X \to \mathbb{B}^3$. We will show that it induces a map $r : X \to \mathbb{S}^2$ with the property that the composition $SO(3)/\mathbb{Z}_{m(i)} \to X_B \to X \to S^2$ is homotopic to the natural projection $\pi : SO(3)/\mathbb{Z}_{m(i)} \to \mathbb{S}^2$. This gives a homotopy commutative diagram

$$\begin{tikzcd} SO(3)/\mathbb{Z}_{m(i)} \arrow[dd] \arrow[dr] & \\ & \mathbb{S}^2 \\ SO(3)/G_i \arrow[ur] & \end{tikzcd}$$

In fact the map r is multivalued, but its values are acyclic, so it induces a homomorphism of cohomology groups. Thus if for the given group G_i, the last diagram is not possible then there is no fixed point free map $X \to \mathbb{B}^3$. This implies the following

THEOREM 1. *If there does not exist a homomorphism λ—a lift of $\pi^\star$ with respect to $\eta^\star$ for π, η the natural projections (as in the diagram below),*

$$\begin{tikzcd} & H^2(SO(3/G_i)) \arrow[d, "\eta^\star"] \\ H^2(\mathbb{S}^2) \arrow[ur, dashed, "(\neg\exists)\lambda"] \arrow[r, "\pi^\star"] & H^2(SO(3)/\mathbb{Z}_{m(i)}) \end{tikzcd}$$

then the Platonic polyhedron P_i has the Brouwer property.

2. Details of construction. Proof of Theorem 1. We explain here the details of the constructions announced in the previous subsection. Let us fix a Platonic polyhedron P_i.

Construction of homeomorphisms

$$\phi : SO(3)/G_i \to X_0, \qquad \psi : SO(3)/\mathbb{Z}_{m(i)} \to X_B.$$

Fix a polyhedron A_0 in X_0 and its vertex V_0. Recall that $G_i = \{g \in SO(3) : g[A_0] = A_0\}$. Every $A \in X_0$ equals $g[A_0]$ for some $g \in SO(3)$. Since $g[A_0] = h[A_0]$ if and only if $h^{-1}g \in G_i$, $\phi(gG_i) := g[A_0]$ is well defined.

To define ψ, we answer the question: how are elements of X_B built from the pair (A_0, V_0)? A recipe could be this: take any $g \in SO(3)$, determine $g[A_0]$ with its vertex $W := g(V_0)$, finally translate $g[A_0]$ in the direction $\vec{OW}$ with W mapped to a point V in $\mathbb{S}^2$. In this way $g[A_0]$ will be transferred to an element A_g of X_B. Every $A \in X_B$ can be obtained in this way, and $A_g = A_h$ if and only if $g[A_0] = h[A_0]$ and $g(V_0) = h(V_0)$. Because $\mathbb{Z}_{m(i)} = \{g \in G_i : g(V_0) = V_0\}$, $\psi(g\mathbb{Z}_{m(i)}) := A_g$ is well defined.

Now, towards a contradiction, *assume that there exists a fixed point free Hausdorff continuous mapping* $f : F(P_i, \epsilon) \to \mathbb{B}^3$, that is, $f(A) \notin A$ for every $A \in F(P_i, \epsilon)$.

Every ray from $f(A)$ through $a \in A$ intersects $\mathbb{S}^2$ in a point $r(A, a)$. It is easily seen that the set $r(A) := \{r(A, a) \mid a \in A\}$ is contractible. Moreover, the mapping $r : F(P_i, \epsilon) \to \exp(\mathbb{S}^2)$, $A \mapsto r(A)$, is continuous with respect to Hausdorff metrics in $F(P_i, \epsilon)$ and $\exp(\mathbb{S}^2)$.

Consider the inclusion $j : X_B \to X$ and the continuous mapping $v : X_B \to \mathbb{S}^2$, where $v(A) = V$ is the unique vertex V of A in $\mathbb{S}^2$. It follows from the definition of r that

$$\forall_{A \in X_B} \quad v(A) \in r \circ j(A),$$

so v is a continuous single-valued selector of the set-valued mapping $r \circ j$. This property for the concrete v given above motivated us to call r a *retraction-like* mapping.

We stress that the construction of the retraction-like mapping r was possible only because we have assumed that there exists an $f : X \to \mathbb{B}^3$ which is fixed point free. To get a contradiction we should prove that r does not exist. This will be done in the next section; now we collect some useful facts.

Since the values $r(A)$ are contractible sets, they are also acyclic—not distinguishable from points by the cohomology functors $H^\star$ (singular) and $\check{H}^\star$ (Čech). Acyclicity with respect to the Čech functor is important in what follows. All cohomologies in this paper have integer coefficients. Since r is Hausdorff continuous with compact values, it is also an upper semicontinuous mapping. It is a standard fact that such an $r : X \to \exp(\mathbb{S}^2)$ induces a homomorphism $r^\star : H^\star(\mathbb{S}^2) \to H^\star(X)$, [7] (1). For completeness we recall the definition of $r^\star$. Consider the graph $\Gamma = \{(x, y) \in X \times \mathbb{S}^2 : y \in r(x)\}$. In the diagram

$$H^\star(\mathbb{S}^2) \cong \check{H}^\star(\mathbb{S}^2) \xrightarrow{q^\star} \check{H}^\star(\Gamma) \xleftarrow{p^\star} \check{H}^\star(X) \cong H^\star(X)$$

with $q(x, y) = y$, $p(x, y) = x$, $p^\star$ is an isomorphism by the Vietoris–Begle Theorem [10]. We define $r^\star := p^{\star -1} \circ q^\star$. Directly from this definition, we obtain the implication

$$v(\cdot) \in r \circ j(\cdot) \;\Rightarrow\; v^\star = j^\star r^\star.$$

(1) L. Górniewicz and his school studied consequences of general versions of this fact for fixed point theory, set-valued analysis and differential inclusions.

We shall rewrite the above cohomology equality with the help of the identifications ϕ and ψ. Of course, the map $\rho : X \to X_0$, translating every $A \in X$ into $\rho(A)$ centred at O, is a strong deformation retraction. It follows immediately from our definitions that the standard projection $\eta : SO(3)/\mathbb{Z}_{m(i)} \to SO(3)/G_i$ satisfies $\phi \circ \eta = \rho \circ j \circ \psi$. In this way,

$$\psi^\star v^\star = \psi^\star j^\star r^\star = \psi^\star j^\star \rho^\star (\rho^\star)^{-1} r^\star = \eta^\star \phi^\star (\rho^\star)^{-1} r^\star.$$

To complete the algebraization of the *Retraction-like Non-Existence Problem* we still need one element of our puzzle: we shall apply a standard model of $SO(3)$ to iterpret the mapping $v \circ \psi$. Fix a point V_0 in $\mathbb{S}^2$. Every rotation $g \in SO(3)$ sends V_0 to a point $g(V_0) \in \mathbb{S}^2$. Given $V \in \mathbb{S}^2$, the set $C := \{g \in SO(3) : g(V_0) = V\}$ is parametrized by the group H of all rotations around the OV-axis: for any chosen $g_0 \in C$, $C = Hg_0$. The group H acts on the unit circle S in the plane orthogonal to the OV-axis and centred at V. We may identify H with S, provided an orientation of S and a point in S corresponding to the identity in H are chosen. In this way also the set C of some rotations is identified with the circle S. Locally all our choices leading to this identification may be made in a continuous way, and an orientation even globally. Because $SO(3)$ is the union of all possible sets C, the above construction makes $SO(3)$ the space of a locally trivial fibration over $\mathbb{S}^2$ with standard fibre $\mathbb{S}^1$ and with projection π_0 sending the fibre C to V. We easily recognize this fibration as the bundle of unit circles in the tangent bundle $T\mathbb{S}^2$. Now, $\mathbb{Z}_{m(i)}$ acts in an obvious way on fibres of this bundle. The orbit space of this action, $SO(3)/\mathbb{Z}_{m(i)}$, is the space of another $\mathbb{S}^1$-bundle over $\mathbb{S}^2$ equipped with the projection

$$\pi = v \circ \psi.$$

This space is known as a *lens space*. We are ready to prove Theorem 1:

Proof of Theorem 1. We have seen that if P_i does not have the Brouwer property then the homomorphism $\lambda := \phi^\star(\rho^\star)^{-1} r^\star$ satisfies $\eta^\star \lambda = \pi^\star$. ∎

3. Verifying the assumptions of Theorem 1. We would like to determine the cohomology groups involved in Theorem 1. It is interesting that the necessary information on these groups is contained in the corresponding fundamental groups. At this moment, coverings come in useful.

If G is a finite subgroup of $SO(3)$ then we have the following covering, a composition of two other coverings:

$$\mathbb{S}^3 \xrightarrow{p} SO(3) \xrightarrow{q} SO(3)/G.$$

The sphere $\mathbb{S}^3$ is the multiplicative group of quaternions $x = a + bi + cj + dk$ of norm $|x| = 1$. The covering p, which additionally is a group homomorphism, is described in [2]. Since $p(x) = p(-x)$ and $p(y) \neq p(x)$ for

$y \neq \pm x$, $SO(3)$ is a 3-dimensional real projective space $\mathbb{RP}^3$. For the covering $c : E \to B$ we denote by $K(c)$ the group of all covering mappings of c. Of course,

$$K(q) = G, \quad K(p) = \mathbb{Z}_2, \quad G_b := K(q \circ p) = p^{-1}[G], \quad |G_b| = 2|G|,$$

where $|G|$ is the number of elements of G. Since $q \circ p$ is a universal covering,

$$\pi_1(SO(3)/G) = G_b, \quad \text{a subgroup of } \mathbb{S}^3.$$

All finite subgroups of $SO(3)$ and of $\mathbb{S}^3$ are classified [16], [14]. Looking at these classifications we easily recognize groups G_b for $G = G_i$ as the *binary polyhedral* groups which are described by means of generators and relations in [3]. If $[l, m, n]$, for some integer $l, m, n \geq 2$, is the group on generators R, S, T with relations

$$R^l = S^m = T^n = RST,$$

then

$$G_{4b} = [2, 3, 3], \quad G_{6b} = G_{8b} = [2, 3, 4], \quad G_{12b} = G_{20b} = [2, 3, 5].$$

Moreover, in these cases the fourth generator $Z := RST$ satisfies $Z^2 = 1 \neq Z$, [3]. Adding the relation $Z = 1$ corresponds to the projection $G_{ib} \to G_{ib}/\mathbb{Z}_2 = G_i$.

We were not sure if the equality $\mathbb{Z}_{mb} = \mathbb{Z}_{2m}$ followed from the given classifications, so we proved it directly, determining $p^{-1}[\mathbb{Z}_m]$ by the explicit formula for p from [2]. We omit here this straightforward but not very short calculation. Thus $H_1(SO(3)/\mathbb{Z}_{m(i)}) = \mathbb{Z}_{2m(i)}$.

Following [3] note that another presentation of the group $[2, 3, n]$ on generators S, T is defined by the relations

$$S^3 = T^n = (ST)^2.$$

The abelianization of this group has commuting generators S, T and two relations: $S = T^2$, $T^6 = T^n$, which is equivalent [5] to the presentation with one generator T and one relation $T^{6-n} = 1$. In this way, $H_1(SO(3)/G_i) = \mathbb{Z}_{6-n}$ when $G_{ib} = [2, 3, n]$.

Lemma 1.

$$H^2(SO(3)/\mathbb{Z}_{m(i)}) = \mathbb{Z}_{2m(i)},$$

$$H^2(SO(3)/G_i) = \begin{cases} \mathbb{Z}_3 & \text{for } i = 4, \\ \mathbb{Z}_2 & \text{for } i = 6, 8, \\ \{0\} & \text{for } i = 12, 20. \end{cases}$$

Proof. The result follows from the Poincaré duality for $SO(3)/G$. We can apply it, because this 3-manifold is orientable, being regularly covered by the orientable manifold $SO(3)$ equipped with orientation preserving covering mappings (homotopic to identity) [6, VIII.2.22, Exercise 3]. ■

To verify the assumption of Theorem 1 we only need

LEMMA 2. *$\pi^\star : H^2(\mathbb{S}^2) \to H^2(SO(3)/\mathbb{Z}_m)$ is an epimorphism.*

We will prove Lemma 2 after proving the following

THEOREM 2. *Every Platonic polyhedron has the Brouwer property.*

Proof. Consider P_i. By Lemma 2, the image of $\pi^\star$ in Theorem 1 has $2m(i) \geq 6$ elements. By Lemma 1, the image of $\eta^\star \circ \lambda$ has at most three elements. Hence $\pi^\star = \eta^\star \circ \lambda$ is impossible. Finally, we apply Theorem 1. ∎

The rest of this section is devoted to the proof of Lemma 2. Although this lemma deals with standard topological objects, we have not found a suitable reference in the literature. We begin with the case $m = 1$ (part I) and then show what should be changed for $m > 1$ (part II).

Proof of Lemma 2, part I. We prove here that $\pi^\star : H^2(\mathbb{S}^2) \to H^2(SO(3))$ is an epimorphism. Let U, V be half-spheres of $\mathbb{S}^2$ such that $U \cup V = \mathbb{S}^2$ and $U \cap V = \mathbb{S}^1$. Let $E = SO(3)$ and E_U, E_V, E_S be the spaces of restrictions of the $\mathbb{S}^1$-bundle $\pi : E \to \mathbb{S}^2$ to $U, V, U \cap V$, respectively. Consider the following commutative diagram with rows being fragments of Mayer–Vietoris exact sequences:

$$\begin{array}{ccccccc}
H^1U \oplus H^1V & \to & H^1\mathbb{S}^1 & \xrightarrow{i} & H^2\mathbb{S}^2 & \to & H^2U \oplus H^2V \\
\downarrow & & \downarrow h & & \downarrow \pi^\star & & \downarrow \\
H^1E_U \oplus H^1E_V & \xrightarrow{f} & H^1E_S & \xrightarrow{g} & H^2E & \to & H^2E_U \oplus H^2E_V
\end{array}$$

Let us look for zeros in this diagram: $H^1U = H^1V = H^2U = H^2V = 0$ since U, V are contractible, $H^2E_U = H^2E_V = 0$ since $E_U \simeq E_V \simeq \mathbb{S}^1$.

The remaining groups are: $H^1\mathbb{S}^1 = H^2\mathbb{S}^2 = \mathbb{Z}$, $H^1E_U = H^1E_V = \mathbb{Z}$, $H^1E_S = H^1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z}^2$ since the $\mathbb{S}^1$-bundle $E_S \to \mathbb{S}^1$ is trivial as a restriction of the trivial bundle $E_U \to U$, and $H^2E = H_1\mathbb{RP}^3 = \mathbb{Z}_2$, [8].

To determine the matrix of f we fix the generators of the corresponding homology groups. Let us specify that $U = \{(x^1, x^2, x^3) \in \mathbb{S}^2 : x^3 \geq 0\}$ and $V = \{x \in \mathbb{S}^2 : x^3 \leq 0\}$. Then choose

$$\begin{aligned}
\alpha(t) &= (\cos(t), \sin(t), 0; 0, 0, -1), \qquad t \in [0, 2\pi], \\
\beta(t) &= (0, -1, 0; \cos(t), 0, \sin(t)), \qquad t \in [0, 2\pi],
\end{aligned}$$

as generators of $H_1(E_S)$ with $E_S \subset T\mathbb{S}^2|\mathbb{S}^1$; and

$$\gamma := \beta, \qquad \delta := \beta,$$

as generators of H_1E_U and of H_1E_V, respectively.

We know that in the homology Mayer–Vietoris sequence the homomorphism $F : H_1E_S \to H_1E_U \oplus H_1E_V$ satisfies the condition

$$F(x) = j_\star(x) + k_\star(x)$$

with inclusions $j : E_S \to E_U$, $k : E_S \to E_V$.

Of course $F(\beta) = \gamma + \delta$. To compare $j^\star(\alpha)$ with γ we will find cycles α_N, β_N homologically equivalent to α, γ lying in the tangent plane to $\mathbb{S}^2$ at the North Pole $N = (0,0,1)$. The cycles α_N, β_N are images of α, β under parallel transport of all tangent planes in $TU \subset T\mathbb{S}^2$ along meridians to the North Pole. We have

$$\alpha_N(t) = (0,0,1;\cos(t),\sin(t),0), \qquad \beta_N(t) = (0,0,1;\cos(t),\sin(t),0).$$

In this way, $j^\star(\alpha) = \gamma$. Similarly, we determine $k^\star(\alpha)$ by parallel transport along meridians to the South Pole $S = (0,0,-1)$:

$$\begin{aligned}\alpha_S(t) &= (0,0,-1;\cos(t+\pi),\sin(t+\pi),0),\\ \beta_S(t) &= (0,0,-1;\cos(t),-\sin(t),0),\end{aligned}$$

so $k^\star(\alpha) = -\delta$. We conclude that $F(\alpha) = \gamma - \delta$ and the matrix $[F]$ of F in the given bases is

$$[F] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Taking for cohomology the dual bases to those considered above we conclude that $[f] = [F]^T$, so $f(x,y) = (x-y, x+y)$.

For an epimorphism $g : \mathbb{Z}^2 \to \mathbb{Z}_2$, we have only three possibilities:

1. $g(1,0) = 1$, $g(0,1) = 1$, $g(x,y) = x + y \pmod 2$, $g(f(x,y)) = 0$.
2. $g(1,0) = 1$, $g(0,1) = 0$, $g(f(1,0)) = 1$.
3. $g(1,0) = 0$, $g(0,1) = 1$, $g(f(1,0)) = 1$.

The exactness of the Mayer–Vietoris sequence implies that case 1 is true. The isomorphism i can be considered to be the identity. By the Künneth formula [8] the functor H^1 respects products and $h(x) = (x,0)$. Summarizing,

$$\pi^\star(x) = g(h(x)) = g(x,0) = x \pmod 2,$$

which is an epimorphism.

Proof of Lemma 2, part II. We prove that $\pi^\star : H^2(\mathbb{S}^2) \to H^2(SO(3)/\mathbb{Z}_m)$ is an epimorphism for $m \geq 2$. The setting is almost the same as for the previous one, in particular we consider *the same Mayer–Vietoris diagram.* The differences are:

1. $E = SO(3)/\mathbb{Z}_m$, $H^2E = \mathbb{Z}_{2m}$.
2. The domain of the cycle β is $[0, 2\pi/m]$.
3. $j^\star(\alpha) = m\gamma$, $k^\star(\alpha) = -m\delta$, $F(\alpha) = m\gamma - m\delta$,

$$[F] = \begin{bmatrix} m & 1 \\ -m & 1 \end{bmatrix},$$

$f(x,y) = (mx - my, x+y)$, $f[\mathbb{Z}^2] = \{(mc,d) : c,d \in \mathbb{Z},\ c \equiv d \pmod 2\}$.

The homomorphisms i and h are the same as in part I but g makes the further proof different. We have

$$g(1,0)=a,\quad g(0,1)=b,\quad a,b\in\{0,1,\dots,2m-1\},$$
$$g(x,y)=xa+yb \pmod{2m}.$$

Since g is an epimorphism, there are $x,y\in\mathbb{Z}$ such that $xa+yb\equiv 1 \pmod{2m}$, i.e.

$$\exists_{x,y,z\in\mathbb{Z}}\quad xa+yb=1+2mz,\qquad \mathrm{GCD}(a,b)\,|\,(1+2mz).$$

Moreover,

$$g[f[\mathbb{Z}^2]]=\{0\}\ \Rightarrow\ (c\equiv d \pmod 2\ \Rightarrow\ mca+db\equiv 0 \pmod{2m}).$$

In particular, for $(c,d)=(0,2)$, we have $2b\equiv 0 \pmod{2m}$, $b\equiv 0 \pmod m$, $b=0$ or $b=m$.

For $(c,d)=(1,1)$, we have $ma+b\equiv 0 \pmod{2m}$. This excludes $b=0$, because in this case $ma\equiv 0 \pmod{2m}$, a is even, $a=\mathrm{GCD}(a,0)\,|\,(1+2mz)$, a contradiction.

Thus we have $b=m$, $ma+m\equiv 0 \pmod{2m}$, and a is odd. Moreover, $\mathrm{GCD}(a,m)\,|\,(1+2mz)$, so $\mathrm{GCD}(a,m)=1$. We conclude that

$$\pi^\star(x)=g(h(x))=g(x,0)=xa \pmod{2m}$$

is an epimorphism $\mathbb{Z}\to\mathbb{Z}_{2m}$, because $\mathrm{GCD}(a,2m)=1$. The proof of Lemma 2 is finished.

4. Truncated Platonic polyhedra. Besides Platonic P_i, another family of solids is formed by the truncated Platonic polyhedra P_i^T, $i=4,6,8,12,20$. The set $P_i^T\subset P_i$ is obtained by removing from P_i every vertex V with its open neighbourhood U_V. Every U_V is separated from $P_i\setminus \bar{U}_V$ by a plane σ_V which is orthogonal to the line joining V to the centre of P_i. We assume that the distance d_V of V from σ_V does not depend on V.

Every polygon $\sigma_V\cap P_i$ is regular and has $m(i)$ edges. These polygons are called *new* faces of P_i^T. The other faces of P_i^T are *old*. Each old face of P_i^T has $2e(i)$ edges, while each face of P_i has $e(i)$ edges. Even the regular old face of P_i^T has two kinds of alternating edges: (i) old edges bordering other old faces, and (ii) new edges lying in the boundary of new faces. If an orientation of the old face F is fixed, the edges of F become arrows and we can unambiguously define a cycle $(V_1,\dots,V_{e(i)})$ of vertices of F (up to a cyclic permutation) which are arrowheads of old edges. We call this cycle the F-*cycle* of length $e(i)$.

In this setting we will generalize the results of Section 2. The rotation group of P_i^T is still G_i. Similarly, $X=F(P_i^T,\epsilon)\simeq X_0=SO(3)/G_i$. For X_B we consider three different definitions:

(1) X_B^v with the same definition as for X_B in Section 1.1. But now every vertex is common to two old faces and one new face, getting only the trivial symmetry. Consequently, $X_B^v = SO(3)$ and $H^2(SO(3)) = \mathbb{Z}_2$.

(2) X_B^n is formed by all polyhedra $A \in X$ having all $m(i)$ vertices of one of their new faces in $\mathbb{S}^2$. We have $X_B^n = SO(3)/\mathbb{Z}_{m(i)}$.

(3) X_B^o is formed by all polyhedra $A \in X$ having all $2e(i)$ vertices of one of their *old* faces in $\mathbb{S}^2$. Then $X_B^o = SO(3)/\mathbb{Z}_{e(i)}$, because each rotation of the old face which extends to a rotation of the whole solid should preserve the structure of its old and new edges.

It will be convenient to treat (2) and (3) as special cases of

(4) $X_B^f(l,k)$ is formed by all polyhedra $A \in X$ with a face F inscribed in $\mathbb{S}^2$ such that k is the rank of the cyclic group of all rotations of F extendable to rotations of A, and l is the length of a well-defined F-cycle. Then $X_B^f(l,k) = SO(3)/\mathbb{Z}_k$, $H^2(SO(3)/\mathbb{Z}_k) = \mathbb{Z}_{2k}$ and

$$X_B^n = X_B^f(m(i), m(i)), \qquad X_B^o = X_B^f(e(i), e(i)).$$

Constructions (1)–(3) will be applied along parallel lines marked out generally in Section 1.1. If at least one of them makes the corresponding commutative diagram impossible then there is no fixed point free map. Our final choice of the kind of X_B will depend on i (i.e. on P_i^T).

For X_B^v we know how to proceed. Let $X_B^f := X_B^f(l,k)$. The main difference with the previous case is that a polyhedron $A \in X_B^f$ has more than one vertex in $\mathbb{S}^2$ and we cannot continuously choose one of them. Happily, with a given orientation of $\mathbb{S}^2$, the mapping

$$v : X_B^f \to SP^l(\mathbb{S}^2, \mathbb{Z}_l), \qquad v(A) = \tau(V_1, \ldots, V_l),$$

is well-defined, where $(V_1, \ldots, V_l)$ is an F-cycle of some vertices of this face F of A which is inscribed in $\mathbb{S}^2$. We denote by $SP^l(Y, H)$ the lth symmetric product of the topological space Y with respect to a subgroup H of the l-th symmetric group S_l, and $\tau : Y^l \to SP^l(Y, H)$ is the projection which sends the point $(y_1, \ldots, y_l)$ to its orbit under the H-action on Y^l:

$$(y_1, \ldots, y_l)\sigma := (y_{\sigma(1)}, \ldots, y_{\sigma(l)})$$

for every permutation $\sigma \in H$. Since $Y = \mathbb{S}^2$ and H is the cyclic group $\mathbb{Z}_l$, we write $SP := SP^l(\mathbb{S}^2, \mathbb{Z}_l)$.

The retraction-like mapping $r : X \to \exp(\mathbb{S}^2)$ defines a continuous mapping $\bar{r} : X \to \exp(SP)$ with contractible values by

$$\bar{r}(A) := \tau[r(A) \times \cdots \times r(A)] = \tau[(r(A))^l].$$

Of course, v is a continuous selector of $\bar{r} \circ j$ with $j : X_B^f \to X$ the inclusion and

$$v^\star = j^\star \bar{r}^\star : H^\star SP \to H^\star X_B^f$$

in cohomology. From [11], [13] we adopt the idea of considering the homomorphism

$$H^\star\mathbb{S}^2 \overset{\pi_1^\star}{\to} H^\star(\mathbb{S}^2)^l \overset{\mu^\star}{\to} H^\star SP,$$

where $\pi_1(y_1,\dots,y_l) = y_1$ and $\mu^\star$, called the transfer [1], has the property that

$$\tau^\star\mu^\star = \sum_{\sigma\in\mathbb{Z}_l}\sigma^\star.$$

By the previous equality,

$$v^\star\mu^\star\pi_1^\star = j^\star\bar{r}^\star\mu^\star\pi_1^\star.$$

Another mapping $u : X_B^f \to SP$ is defined as follows. For $A \in X_B^f$ we take its face F equipped with the F-cycle $(V_1,\dots,V_l)$ and find the point $w(A) \in \mathbb{S}^2$ lying in the intersection of $\mathbb{S}^2$ with the line from O through the centre of F. We have $\varDelta : \mathbb{S}^2 \to (\mathbb{S}^2)^l$, $\varDelta(x) = (x,\dots,x)$, and $u := \tau\circ\varDelta\circ w$. Since points of the F-cycle may be moved to $w(A)$ on the arcs of great circles of $\mathbb{S}^2$, $v \simeq u$ and

$$v^\star\mu^\star\pi_1^\star = u^\star\mu^\star\pi_1^\star = w^\star\varDelta^\star\tau^\star\mu^\star\pi_1^\star = \sum_{\sigma\in\mathbb{Z}_l} w^\star\varDelta^\star\sigma^\star\pi_1^\star = l\cdot w^\star,$$

because $\pi_1\circ\sigma\circ\varDelta = \mathrm{id}$ for every σ. As in Section 2, the equality

$$l\cdot w^\star = j^\star\bar{r}^\star\mu^\star\pi_1^\star$$

takes the form

$$l\cdot\pi^\star = l\cdot\psi^\star w^\star = \eta^\star\phi^\star(\rho^\star)^{-1}\bar{r}^\star\mu^\star\pi_1^\star,$$

for some $\psi : SO(3)/\mathbb{Z}_k \approx X_B^f$, $\phi : SO(3)/G_i \approx X_0$, $\eta : SO(3)/\mathbb{Z}_k \to SO(3)/G_i$ and $\pi = w\circ\psi : SO(3)/\mathbb{Z}_k \to \mathbb{S}^2$.

We will use these results for $l = k \in \{m(i), e(i)\}$. By Lemma 2, $\pi^\star : H^2\mathbb{S}^2 \to H^2(SO(3)/\mathbb{Z}_k) = \mathbb{Z}_{2k}$ is an epimorphism, so

$$l\cdot\pi^\star[H^2\mathbb{S}^2] = k\cdot\mathbb{Z}_{2k} = \{0,k\} = \mathbb{Z}_2 \subset \mathbb{Z}_{2k}.$$

Our efforts can be summarized in the following

THEOREM 3. *If*

$$\exists_{k\in\{1,m(i),e(i)\}} \quad 0 = \eta^\star : H^2(SO(3)/G_i) \to H^2(SO(3)/\mathbb{Z}_k)$$

then P_i^T has the Brouwer property.

Proof. For $k = 1, m(i), e(i)$ we apply the above results with $X_B = X_B^v, X_B^n, X_B^o$ respectively and conclude that there is no homomorphism $\lambda : H^2\mathbb{S}^2 \to H^2(SO(3)/G_i)$ such that $\mathbb{Z}_2 = k\cdot\pi^\star[H^2\mathbb{S}^2] = \eta^\star\lambda[H^2\mathbb{S}^2]$. If P_i^T did not have the Brouwer property then λ satisfying this equality could be defined exactly as in the proof of Theorem 1 for X_B^v and $\lambda := \phi^\star(\rho^\star)^{-1}\bar{r}^\star\mu^\star\pi_1^\star$ for X_B^n, X_B^o. ∎

Lemma 3.

$$0 = \eta^\star : H^2(SO(3)/G_8) \to H^2(SO(3)/\mathbb{Z}_3).$$

Proof. First note that $\eta : SO(3)/\mathbb{Z}_3 \to SO(3)/G_8$ is well-defined because faces of the octahedron are triangles. Since η is a covering, $\eta_\sharp : \pi_1(SO(3)/\mathbb{Z}_3) \to \pi_1(SO(3)/G_8)$ is a monomorphism, [8]. Consequently, since the element $1 \in \mathbb{Z}_6 = \pi_1(SO(3)/\mathbb{Z}_3)$ is of order 6, $\eta_\sharp(1) \in G_{8b} = [2,3,4]$ is also of order 6.

From [3] we see that the group $[2,3,4]$ with generators S, $T = \frac{1}{\sqrt{2}}(1+i)$ and relations $S^3 = T^4 = (ST)^2$ has eight elements of order 6. These are:

$$\begin{aligned}
S &= \tfrac{1}{2}(1+i+j+k), & S^{-1} &= \tfrac{1}{2}(1-i-j-k),\\
TST^{-1} &= \tfrac{1}{2}(1+i-j+k), & TS^{-1}T^{-1} &= \tfrac{1}{2}(1-i+j-k),\\
T^2ST^{-2} &= \tfrac{1}{2}(1+i-j-k), & T^2S^{-1}T^{-2} &= \tfrac{1}{2}(1-i+j+k),\\
T^3ST^{-3} &= \tfrac{1}{2}(1+i+j-k), & T^3S^{-1}T^{-3} &= \tfrac{1}{2}(1-i-j+k).
\end{aligned}$$

Thus $\eta_\sharp(1) = T^aS^eT^{-a}$ for some $a \in \{0,1,2,3\}$, $e = \pm 1$. For the abelianization:

$$\eta_\star : H_1(SO(3)/\mathbb{Z}_3) \to H_1(SO(3)/G_8), \qquad \eta_\star(1) = \pm S = 0,$$

because $S = T^2$, $T^4 = T^6$ and $T^2 = 1$ (zero in the additive language).

By [6, VII.1.(1.7)], we have the following commutative diagram with exact rows and $M := SO(3)/G_8$, $N := SO(3)/\mathbb{Z}_3$:

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Ext}(H_1M,\mathbb{Z}) & \to & H^2M & \to & \mathrm{Hom}(H_2M,\mathbb{Z}) & \to & 0\\
 & & \downarrow{\scriptstyle \mathrm{Ext}(\eta_\star,\mathrm{id})} & & \downarrow{\scriptstyle \eta^\star} & & \downarrow{\scriptstyle \mathrm{Hom}(\eta_\star,\mathrm{id})} & & \\
0 & \to & \mathrm{Ext}(H_1N,\mathbb{Z}) & \to & H^2N & \to & \mathrm{Hom}(H_2N,\mathbb{Z}) & \to & 0
\end{array}$$

Since $H^2M = \mathbb{Z}_2$ and $H^2N = \mathbb{Z}_6$ are finite groups, H_2M and H_2N have no infinite cyclic group in their decompositions into cyclic components, so $\mathrm{Hom}(H_2M,\mathbb{Z}) = 0 = \mathrm{Hom}(H_2N,\mathbb{Z})$. We know that $0 = \eta_\star$ in $\mathrm{Ext}(\cdot,\mathrm{id})$, so $\eta^\star = 0$. ∎

Theorem 4. *Every truncated Platonic polyhedron has the Brouwer property.*

Proof. For P_4^T we use Theorem 3 with $k = 1$: the homomorphism $\eta^\star : \mathbb{Z}_3 \to \mathbb{Z}_2$ equals zero. For P_{12}^T and P_{20}^T the same statement holds for $k = 1$, because the domain of $\eta^\star$ is $\{0\}$. Of course $G_6 = G_8$ and $m(6) = 3$, $e(8) = 3$, so our assertion for P_6^T and P_8^T follows from Theorem 3 combined with Lemma 3. ∎

Remark. The author hopes that similar methods could be applied to prove that all 13 Archimedean polyhedra, their medials and duals ([4], [9], [15]) have the Brouwer property.

Acknowledgments. I would like to thank the referee and Jerzy Jezierski for remarks and suggestions shaping the final version of this paper.

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Received 20 February 2013;
received in final form 26 November 2013 (7921)