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OPERATOR THEORY

Generalized Analytic and Quasi-Analytic Vectors

by

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Summary. For every sequence (a_n) of positive real numbers and an operator acting in a Banach space, we introduce the families of (a_n)-analytic and (a_n)-quasi-analytic vectors. We prove various properties of these families.

Introduction. Let E be a Banach space, and A an operator (bounded or unbounded) acting in E. Various sets of vectors, members of E, can be associated with A. The simplest examples include the domain $D(A)$ of A, and the set

$$C^\infty(A) = \bigcap_{n=1}^{\infty} D(A^n)$$

of C^∞-vectors for A.

If a given operator has some geometric properties (for example, is symmetric acting in Hilbert space) and has a sufficiently large (say, dense) set of vectors of a special class, then the operator often has useful properties: it is essentially self-adjoint or generates a strongly continuous group or semigroup.

Important "classical" classes of vectors are those of *analytic vectors*, *quasi-analytic vectors*, *semi-analytic vectors*, and *Stieltjes vectors*.

In this paper we shall consider the following more general sets of vectors.

Definition 1. Let (c_n) be a sequence of strictly positive numbers. An element $x \in C^\infty(A)$ belongs to $\mathcal{A}_{(c_n)}(A)$ if

$$\sum_{n=1}^{\infty} \frac{\|A^n x\|}{c_n} t^n < \infty \quad \text{for some } t > 0.$$

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Definition 2. Let (b_n) be a sequence of strictly positive numbers. An element $x \in C^\infty(A)$ belongs to $\mathcal{Q}_{(b_n)}(A)$ if

$$\sum_{n=1}^{\infty} \|A^n x\|^{-1/b_n} = \infty.$$

Note that the same classes $\mathcal{A}_{(c_n)}(A)$ can be obtained by using different sequences (c_n). Indeed,

$$\mathcal{A}_{(c_n)}(A) = \left\{ x \in C^\infty(A) : \sup_{n=1,2,\ldots} \frac{\|A^n\|}{c_n} t^n < \infty \text{ for some } t > 0 \right\},$$

which implies the following

Proposition 1. *Let (c_n) and (c'_n) be two sequences of positive numbers such that*

$$c_n \le c'_n d^n \quad \textit{for some } d > 0.$$

Then

$$\mathcal{A}_{(c_n)}(A) \subset \mathcal{A}_{(c'_n)}(A).$$

Proposition 1 and Stirling's formulae imply that for example $\mathcal{A}_{((n!)^p)}(A) = \mathcal{A}_{(n^{pn})}(A)$.

For the sets $\mathcal{Q}_{(b_n)}(A)$ we have the following

Proposition 2. *If $0 < b_n \le b'_n < \infty$ for every $n = 1, 2, \ldots$, then*

$$\mathcal{Q}_{(b_n)}(A) \subset \mathcal{Q}_{(b'_n)}(A).$$

Proof. Indeed, if $\|A^n x\| \le 1$ for infinitely many n's, then $x \in \mathcal{Q}_{(b_n)}(A) \cap \mathcal{Q}_{(b'_n)}(A)$. Furthermore, if $\|A^n x\| > 1$, then $\|A^n x\|^{-1/b_n} \le \|A^n x\|^{-1/b'_n}$, so that if $\|A^n x\| \le 1$ for finitely many n's only and $x \in \mathcal{Q}_{(b_n)}(A)$, then $x \in \mathcal{Q}_{(b_n)}(A)$. ■

For unbounded symmetric operators A in a Hilbert space H the denseness of $\operatorname{lin} \mathcal{A}_{(c_n)}(A)$ or $\operatorname{lin} \mathcal{Q}_{(b_n)}(A)$ in H may imply the essential self-adjointness of A.

If $c_n = n!$ (or n^n) then $\mathcal{A}_{(c_n)}(A)$ coincides with the set of analytic vectors introduced by Nelson [4], who proved that a symmetric operator with a linearly dense set of analytic vectors is essentially self-adjoint.

In the case of $b_n = n$ we obtain the quasi-analytic vectors introduced by Nussbaum [5], who showed a more general result stating that a symmetric operator with a linearly dense set of quasi-analytic vectors is essentially self-adjoint.

If $c_n = (2n)!$, then $\mathcal{A}_{(c_n)}(A)$ coincides with the semi-analytic vectors introduced by Simon [10], who proved that a symmetric semi-bounded operator with a linearly dense set of semi-analytic vectors is essentially self-adjoint.

If $b_n = 2n$, then $\mathcal{Q}_{(b_n)}(A)$ is equal to the set of Stieltjes vectors introduced by Nussbaum [6], who showed that a symmetric semi-bounded operator with a linearly dense set of Stieltjes vectors is essentially self-adjoint.

The following diagram displays the relationships between various classes of vectors:

$$\begin{array}{ccc} \text{analytic} & \subset & \text{quasi-analytic} \\ \cap & & \cap \\ \text{semi-analytic} & \subset & \text{Stieltjes vectors} \end{array}$$

The sets $\mathcal{A}_{(c_n)}(A)$ also play an important role for unbounded operators A in Banach spaces. Let p be a positive real number. If $c_n = n^{pn}$ or $c_n = (n!)^p$, then we obtain the space called in [1] the *abstract Gevrey space of order* p *associated with* A; this space is denoted by $G(p)$ and called the space of p-*analytic vectors* in [3] and [8].

In [1] only closed operators were considered and it was proved that under some assumptions on the resolvent, a closed operator A generates a strongly continuous semigroup in $G(p)$ equipped with some locally convex topology.

If $b_n = pn$, then we obtain the p-quasi-analytic vectors considered in [3] and [8]. With this terminology, analytic vectors are simply 1-analytic, semi-analytic ones are 2-analytic, quasi-analytic ones are 1-quasi-analytic, and Stieltjes vectors are 2-quasi-analytic.

When E is the space of bounded continuous functions on an interval in $\mathbb{R}$ and $A = d/dx$, special cases of spaces $\mathcal{A}_{(c_n)}(A)$ are considered in [7], namely such that from $f \in \mathcal{A}_{(c_n)}(A)$ and $(A^n f)(x_0) = 0$ for $n = 0, 1, 2, \ldots$ it follows that $f(x) \equiv 0$. Such classes are called *quasi-analytic*.

If $c_n = (n!)^p$ and E, A are as above and $p \in (1, \infty)$, then $\mathcal{A}_{(c_n)}(A)$ is the classical space of Gevrey functions of order p (see [2]), and if $p \in (0, 1]$, then $\mathcal{A}_{(c_n)}(A)$ is a quasi-analytic class.

$\mathcal{A}_{(c_n)}(A)$ **vectors and** $\mathcal{Q}_{(b_n)}(A)$ **vectors.** We shall show some connections between the two classes of vectors defined above. We start with the following result.

THEOREM 1. *Let* (c_n) *and* (b_n) *be sequences of positive numbers such that for some* $a > 0$,

$$b_n \geq \max\left(an, \frac{\ln c_n}{\ln n}\right) \qquad (n \in \mathbb{N}).$$

Then, *for any operator* A,

$$\mathcal{A}_{(c_n)}(A) \subset \mathcal{Q}_{(b_n)}(A).$$

Proof. Let $x \in \mathcal{A}_{(c_n)}(A)$. Then

$$\sum_{n=1}^{\infty} \frac{\|A^n x\|}{c_n} t^n$$

has a positive radius of convergence equal to $1/r$, where

$$r = \limsup_{n\to\infty} \sqrt[n]{\|A^n x\|/c_n} < \infty.$$

Let $M > \max(1, r)$. Then there exists n_0 such that for $n > n_0$,

$$\sqrt[n]{\|A^n x\|/c_n} < M.$$

Hence

$$\|A^n x\|^{-1/b_n} > \frac{1}{M^{n/b_n}(c_n)^{1/b_n}}.$$

Since $b_n \ge an$, we see that $M^{n/b_n} \le M^{1/a}$, and since $b_n \ge \ln c_n / \ln n$, we have $(c_n)^{1/b_n} \le n$. Finally,

$$\|A^n x\|^{-1/b_n} > 1/M^{1/a} n,$$

which implies $\sum_{n=1}^{\infty} \|A^n x\|^{-1/b_n} = \infty$. ∎

As a simple corollary, we establish the horizontal inclusions in the diagram above. Indeed, it suffices to let $c_n = n^n$ and $b_n = n$ to obtain the upper inclusion, and $c_n = n^{2n}$ and $b_n = 2n$ to obtain the lower inclusion.

Remark. The condition $b_n \ge an$ is a very natural one if the denseness of $\mathcal{Q}_{(b_n)}(A)$ for a bounded A is to be ensured. Suppose that each (b_n) is slightly less than an, for example $b_n = n^{1-\varepsilon}$. Let A be the scalar operator $Ax = 2x$ and let x be an arbitrary non-zero vector from E. Then $\|A^n x\| = 2^n \|x\|$, whence $\|A^n x\|^{-1/b_n} = 2^{-(n^{\varepsilon})} \|x\|^{1-\varepsilon}$. Since $2^{n^{\varepsilon}} > n^2$ for large n, we see that the series $\sum_{n=1}^{\infty} \|A^n x\|^{-1/b^n}$ is convergent, and consequently that $x \notin \mathcal{Q}_{(b_n)}(A)$.

If we let $c_n = n^n$ and $b_n = n$ and consider analytic vectors and quasi-analytic vectors, then the expected growth of $(\|A^n x\|)$ is similar. The main difference is that in the latter case this growth can be much more irregular. Therefore results concerning vectors in $\mathcal{Q}_{(b_n)}(A)$ are in general stronger than those for $\mathcal{A}_{(c_n)}(A)$. This phenomenon is demonstrated by the following theorem which is the main result of this paper.

Theorem 2. *Let (c_n) and (b_n) be sequences of strictly positive numbers with $b_n \ge an$ for some $a > 0$. There exists a symmetric operator A acting in a Hilbert space H such that the set $\mathcal{Q}_{(b_n)}(A)$ is dense in H and the set $\mathcal{A}_{(c_n)}(A)$ comprises only the zero vector.*

Proof. Without loss of generality we may assume that $c_n \ge 1$ for $n = 1, 2, \ldots$ Let $H = l^2$ be the Hilbert space of all square-summable complex sequences. Let $e_k = (0,\dots,0,1,0,\dots)$ (1 at the kth place) be the standard basis in H. Let m_0 be the linear subspace of H spanned by $e_1, e_2, \dots$ Of course, m_0 is dense in H.

For a sequence (a_k) $(k = 1,2,\dots)$, let additionally $a_0 = 0$ and consider the operator A defined on m_0 as follows:

$$Ae_k = a_{k-1}e_{k-1} + a_k e_{k+1} \quad \text{for } k = 1,2,\dots$$

In matrix form, A is given by the Jacobi matrix

$$(*) \qquad \begin{bmatrix} 0 & a_1 & 0 & 0 & 0 & \dots \\ a_1 & 0 & a_2 & 0 & 0 & \dots \\ 0 & a_2 & 0 & a_3 & 0 & \dots \\ 0 & 0 & a_3 & 0 & a_4 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

If the a_k are real numbers, then A is symmetric.

First we shall prove the following

LEMMA 1. *If $a_j > 0$ for each $j = 1,2,\dots$, then*

$$A^n e_k = \sum_{i=1}^{n+k} \alpha_i^{n,k} e_i,$$

and

(1) $\alpha_i^{n,k} \geq 0$,

(2) $\alpha_{n+k}^{n,k} = a_k a_{k+1} \dots a_{n+k-1}$,

(3) *for any $q \leq n+k$, $\alpha_q^{n,k} \leq 2^n M^n$, where $M = \sup\{a_i : 1 \leq i \leq n+k\}$.*

Proof. We proceed by induction on n. For $n = 1$ conditions (1)–(3) result from the definition of A. Suppose that (1)–(3) hold for some n, all k and all $q \leq k+n$. We have

$$(**) \qquad A^{n+1}e_k = AA^n e_k = A\Big(\sum_{i=1}^{n+k} \alpha_i^{n,k} e_i\Big) = \sum_{i=1}^{n+k} \alpha_i^{n,k}(a_{i-1}e_{i-1} + a_i e_{i+1}).$$

This establishes condition (1) for $n+1$.

The vector e_{n+k+1} occurs in $(**)$ only once, for $i = k+1$, with coefficient $\alpha_{n+k}^{n,k} \cdot a_{n+k}$. Hence we get (2) for $n+1$ and (3) for $q = n+k+1$.

Now let $q < n+k+1$. The vector e_q occurs in $(**)$ twice: for $i = q-1$ and $i = q+1$. The coefficient $\alpha_q^{n+1,k}$ is equal to $a_{q-1}\alpha_{q-1}^{n,k} + a_q \alpha_{q+1}^{n,k}$. Hence, by the inductive hypothesis, $\alpha_q^{n+1,k} \leq 2M2^n M^n = 2^{n+1}M^{n+1}$. ∎

Let now (c_n) and (b_n) be sequences of positive numbers with $b_n \geq an$ $(a > 0)$. We inductively define a Jacobi matrix of the form as in $(*)$, some increasing sequences (k_n) and (k'_n) of natural numbers and also a sequence (Q_{2n+1}) of positive numbers.

First let $k_1 = k'_1 = 1$, $k_2 = 3$, $k'_2 = 2$ and let $a_1 = a_2 = a_3 = Q_1 = Q_3 = Q_5 = 1$. Suppose that we have defined $k_1 < k_2 < \ldots < k_{2n-1} < k_{2n}$; $k'_2 < k'_4 < \ldots < k'_{2n}$; $Q_1, Q_3, \ldots, Q_{2n-1}$ and $a_1, a_2, \ldots, a_{k_{2n}}$ such that for $i = 0, 1, \ldots, n-1$ the following three conditions are satisfied:

(a) $a_p = c_{p+1}(p+1)^{p+1}$ for $p = k_{2i+1}$, $i = 0, 1, \ldots, n-1$.
(b) $a_p = 1$ for $p \neq k_{2i+1}$, $i = 0, 1, \ldots, n-1$.
(c) $k'_{2i+2} - k_{2i+1} > (2Q_{2i+1})^{1/a}$.

Let $k_{2n+1} = k_{2n} + 1$ and let $a_{k_{2n+1}} = c_p p^p$ with $p = k_{2n+1} + 1$. Define $Q_{2n+1} = \sup\{a_i : 1 \le i \le k_{2n+1}\}$, choose k'_{2n+2} large enough so that $k'_{2n+2} - k_{2n+1} > (2Q_{2n+1})^{1/a}$, and let $k_{2n+2} = k'_{2n+2} + n$. Finally, for $i = k_{2n} + 2, k_{2n} + 3, \ldots, k_{2n+2}$, let $a_i = 1$. Directly from the construction it follows that conditions (a), (b), (c) hold for every $i = 0, 1, \ldots$

We now prove two lemmas.

LEMMA 2. *Each $x \in m_0$ belongs to $\mathcal{Q}_{(b_n)}(A)$.*

Proof. Let

$$x = \sum_{k=1}^{K} d_k e_k, \qquad L = \sup\{|d_k| : k = 1, \ldots, K\}.$$

Let $n > K$ and p be such that $k_{2n+1} < p \le k'_{2n+2}$. Then

$$A^p x = \sum_{k=1}^{K} d_k A^p e_k.$$

By (3) and Lemma 1,

$$\|A^p x\| \le KL2^p M^p, \quad \text{where} \quad M = \sup\{a_i : 1 \le i \le p + K\}.$$

Since $p + K \le k'_{2n+2} + K < k'_{2n+2} + n < k_{2n+2}$, we see that $a_i = 1$ for $i > k_{2n+1}$, and $a_i \le Q_{2n+1}$ for $i \le k_{2n+1}$. Hence, for $k_{2n+1} < p \le k'_{2n+2}$,

$$\|A^p x\|^{1/b_p} \le (KL2^p Q_{2n+1}^p)^{1/b_p} \le (KL2^p Q_{2n+1}^p)^{1/ap} = (KL)^{1/ap}(2Q_{2n+1})^{1/a}.$$

Obviously, $k_{2n+1} > n$ for each n. Thus, for $n > K$,

$$\sum_{p=k_{2n+1}+1}^{k'_{2n+2}} \|A^p x\|^{-1/b_p} \ge \frac{k'_{2n+2} - k_{2n+1}}{(KL)^{1/an}(2Q_{2n+1})^{1/a}} > \frac{1}{(KL)^{1/an}} = \frac{1}{\sqrt[n]{(KL)^{1/a}}}.$$

Since $\sqrt[n]{(KL)^{1/a}}$ tends to 1 as $n \to \infty$, there exists n_0 such that for $n > n_0$

the above sum is greater than 1/2. Therefore

$$\sum_{p=1}^{\infty}\|A^p x\|^{-1/b_p} \geq \sum_{p=k_{2n_0}}^{\infty}\|A^p x\|^{-1/b_p}$$

$$\geq \sum_{n=n_0}^{\infty}\sum_{p=k_{2n+1}+1}^{k'_{2n+2}}\|A^p x\|^{-1/b_p} > \sum_{n=n_0}^{\infty} 1/2 = \infty. \blacksquare$$

LEMMA 3. *No nonzero $x \in m_0$ belongs to $\mathcal{A}_{(c_n)}(A)$.*

Proof. Let $x = \sum_{k=1}^{K} d_k e_k$, and $|d_K| > 0$. We shall estimate $\|A^p x\|$ for $p = k_{2n+1}$ with $p > K$. Since

$$A^p x = \sum_{k=1}^{K} d_k A^p e_k,$$

it follows from Lemma 1 that

$$A^p x = \sum_{k=1}^{K} d_k \sum_{i=1}^{p+k} \alpha_i^{p,k} e_i.$$

The vector e_{p+K} occurs in this sum only once: when $k = K$ and $i = p + K$. Thus by condition (2) of Lemma 1, the corresponding coefficient is equal to $\alpha_{p+K}^{p,K} = a_K a_{K+1} \ldots a_{p+K-1}$. As $K < p \leq p + K - 1$, one of the factors of this product is $a_p = c_p p^p$. Since $c_n \geq 1$ for every n, the remaining factors are not less than 1, and it follows that

$$\|A^p x\| \geq |d_K| \cdot c_p p^p.$$

Thus

$$\sqrt[p]{\|A^p x\|/c_p} \geq \sqrt[p]{|d_K|}\, p,$$

and so

$$\limsup_{p\to\infty} \sqrt[p]{\|A^p x\|/c_p} = \infty. \blacksquare$$

The theorem results immediately from the last two lemmas. ■

As a corollary, we obtain

COROLLARY 1. *For any strictly positive sequence (c_n) there exists an essentiallly self-adjoint operator A for which $\mathcal{A}_{(c_n)}(A)$ consists only of the zero vector.*

Proof. An application of Theorem 2 with $b_n = n$ for each $n \in \mathbb{N}$ yields a symmetric operator A with a linearly dense set of quasi-analytic vectors and with $\mathcal{A}_{(c_n)}(A) = \{0\}$. This operator is essentially self-adjoint by Nussbaum's theorem.

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