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PROBABILITY THEORY AND STOCHASTIC PROCESSES

On Multivalued Amarts

by

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Summary. In recent years, convergence results for multivalued functions have been developed and used in several areas of applied mathematics: mathematical economics, optimal control, mechanics, etc. The aim of this note is to give a criterion of almost sure convergence for multivalued asymptotic martingales (amarts). For every separable Banach space B the fact that every L^1-bounded B-valued martingale converges a.s. in norm to an integrable B-valued random variable (r.v.) is equivalent to the Radon–Nikodym property [6]. In this paper we solve the problem of a.s. convergence of multivalued amarts by giving a topological characterization.

1. Preliminaries. Let $\mathcal{X}_S$ be the set of all random elements (r.e.) defined on a probability space $(\Omega, \mathcal{A}, P)$ with values in a Polish (separable, complete metric) space (S, ϱ), i.e. $\mathcal{X}_S = \{X : \Omega \to S;\ X^{-1}(\mathcal{B}) \subset \mathcal{A}\}$, where $\mathcal{B} = \mathcal{B}_S$ stands for the σ-field generated by the open subsets of S.

Let $\mathcal{P}(S)$ denote the set of all probability measures defined on $(S, \mathcal{B})$. The *Lévy–Prokhorov* metric on $\mathcal{P}(S)$ is defined as follows:

$$L(X,Y) = L(P_X, P_Y) = \inf\{\varepsilon > 0 : P_X(A) < P_Y(A^\varepsilon) + \varepsilon,\quad P_Y(A) < P_X(A^\varepsilon) + \varepsilon \text{ for all } A \in \mathcal{B}\},$$

where P_X is the probability distribution of the r.e. X, $A^\varepsilon = \{x : d(x,A) = \inf_{y\in A} \varrho(x,y) < \varepsilon\}$, and $P_X(B) = P[X \in B]$ for $B \in \mathcal{B}$.

It is known [1] that the convergence of a sequence of probability measures in the Lévy–Prokhorov metric and the weak convergence of this sequence coincide.

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A collection $\{P_j : j \in J\}$ of probability measures is *tight* if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset S$ such that

$$P_j(K_\varepsilon) > 1 - \varepsilon \quad \text{for all } j \in J.$$

We shall say that a sequence $\{X_n : n \geq 1\}$ of r.e. is tight if the sequence of their distributions is tight.

By the Prokhorov theorem [1] if a sequence $\{X_n\}$ of r.e. is convergent in law to a r.e. X, then the sequence is tight.

We denote by T the collection of all bounded stopping times relative to the sequence $\{\sigma(X_1, \ldots, X_n) : n \geq 1\}$, where $\sigma(X_1, \ldots, X_n)$ denotes the smallest σ-algebra with respect to which $X_1, \ldots, X_n$ are measurable.

Now we recall some further notation and definitions. A sequence $\{X_n\}$ of random elements is *randomly convergent in law* to a random element X ($X_\tau \xrightarrow{D} X$) if for each $\varepsilon > 0$ there exists $\tau_0 \in T$ such that $L(X_\tau, X) < \varepsilon$ for every $\tau \in T$, $\tau \geq \tau_0$ a.s. (see [9]).

A sequence $\{X_n\}$ is *essentially convergent in law* to a random element X ($X_n \xrightarrow{E.D.} X$) if for every P_X-continuity set A, i.e. $P_X(\partial A) = 0$, where ∂A denotes the boundary of A, we have

$$P(\limsup_{n\to\infty}[X_n \in A]) = P(\liminf_{n\to\infty}[X_n \in A]) = P([X \in A]).$$

A sequence $\{X_n\}$ of random elements is said to *converge almost surely* (with probability 1) to a random element X ($X_n \xrightarrow{a.s.} X$) if

$$P([\lim_{n\to\infty} \varrho(X_n, X) = 0]) = 1.$$

It was proved in [9] that $X_\tau \xrightarrow{D} X$ iff $X_n \xrightarrow{E.D.} X$ iff there exists a r.e. X' such that $X_n \xrightarrow{a.s.} X'$ and $P_X = P_{X'}$.

Definition 1. We say that a sequence $\{X_n\}$ of r.e. is *strongly tight* if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset S$ such that

$$P\Big(\bigcap_{n=1}^{\infty}[\omega : X_n(\omega) \in K_\varepsilon]\Big) > 1 - \varepsilon.$$

Uniform boundedness of $E\|X_n\|$ is a necessary condition for almost sure convergence of real-valued amarts. However this condition is not sufficient in Banach spaces. It turns out that strong tightness is a necessary and sufficient condition for almost sure convergence of L^1 bounded Banach space valued amarts [6].

Theorem 1 (see [7]). *If $X_n \xrightarrow{a.s.} X$, then the sequence $\{X_n\}$ is strongly tight.*

It is easy to see that this theorem fails in the case of convergence in probability.

2. Hausdorff metric. Let (S,ϱ) be a Polish space. By $(\mathcal{B}(S),\widehat{\varrho})$ we denote the metric space of non-empty closed bounded subsets of S equipped with the *Hausdorff metric* defined as follows:

$$\widehat{\varrho}(F,F')=\max\{\sup_{x\in F} d(x,F'),\ \sup_{x'\in F'} d(x',F)\},\qquad F,F'\in\mathcal{B}(S).$$

Let $\mathcal{C}(S)$ denote the class of compact subsets of S.

The following theorem is known (see [8, Proposition 1.2.5]):

THEOREM 2. *Let (S,ϱ) be a metric space with the property that every closed bounded subset of S is compact, and let $\{F_n\}$ be a sequence in $\mathcal{B}(S)$ such that $\bigcup_{n=1}^{\infty} F_n$ is bounded. Suppose that there exists a dense set $D\subset S$ such that for every $x\in D$ the limit $\lim_{n\to\infty} d(x,F_n)$ exists and is finite. Then $\{F_n\}$ converges in $(\mathcal{B}(S),\widehat{\varrho})$.*

THEOREM 3. *Let $\mathcal{L}$ denote a compact subset of $(\mathcal{C}(S),\widehat{\varrho})$. Then $\bigcup_{A\in\mathcal{L}} A$ is a compact subset of S.*

Proof. Let $\{x_i\}$ be a sequence in $\bigcup_{A\in\mathcal{L}} A$. Without loss of generality, we can assume that $x_i\in A_i$ for $i=1,2,\ldots$ We can extract from $\{A_n\}$ a subsequence $\{A_{n_k}\}$ converging to an $A\in\mathcal{L}$. If $\lim_{k\to\infty}\widehat{\varrho}(A_{i_k},A)=0$ then there exists a sequence $\{x'_{i_k}\}$ such that $x'_{i_k}\in A$ and $\lim_{k\to\infty}\varrho(x'_{i_k},x_{i_k})=0$. Since A is compact we can extract from $\{x'_{i_k}\}$ a subsequence convergent to an $x\in A\in\mathcal{L}$. This implies that every sequence in $\bigcup_{A\in\mathcal{L}} A$ contains a convergent subsequence, which means that $\bigcup_{A\in\mathcal{L}} A$ is compact.

3. Multivalued functions. Expectations. In what follows, B will denote a real separable Banach space with a norm $\|\ \|$, and B^* its dual.

For $1\le p<\infty$, $L^p(\Omega;B)$ denotes the Banach space of measurable functions $f:\Omega\to B$ such that $\|f\|_p=\{\int_\Omega \|f(\omega)\|^p\,dP\}^{1/p}<\infty$. A multivalued function $F:\Omega\to 2^B$ with nonempty closed values is called *measurable* if for each Borel set $D\subset B$, $F^{-1}(D)=\{\omega\in\Omega : F(\omega)\cap D\ne\emptyset\}\in\mathcal{A}$. Denote by $M(\Omega;B)$ the family of such multivalued functions. By $M_{\mathrm{c}}(\Omega;B)\subset M(\Omega;B)$ we denote the subset of measurable multifunctions with nonempty compact values. A measurable function $f:\Omega\to B$ is called a *measurable selection* of F if $f(\omega)\in F(\omega)$ for all $\omega\in\Omega$. For $F\in M(\Omega;B)$ we define

$$S_F^p=\{f\in L^p(\Omega;B): f(\omega)\in F(\omega)\ \text{a.s.}\}.$$

The following basic theorem concerning the measurability of multivalued functions is given in [4].

THEOREM 4. *Let $(\Omega,\mathcal{A},P)$ be a probability space and (S,ϱ) a Polish metric space. Let $F:\Omega\to 2^S$ be a multifunction with closed values. The following conditions are equivalent*:

(i) *F is measurable.*

(ii) *For each closed set $C \subset B$, $F^{-1}(C) \in \mathcal{A}$.*

(iii) *For each open set $O \subset B$, $F^{-1}(O) \in \mathcal{A}$.*

(iv) *$D(F) = \{\omega : F(\omega) \neq \emptyset\} \in \mathcal{A}$, and $d(x, F(\omega))$ is a measurable function of $\omega \in D(F)$ for each $x \in S$.*

(v) *$D(F) \in \mathcal{A}$, and there exists a sequence $\{f_n\}$ of measurable functions $f_n : D(F) \to S$ such that $F(\omega) = \mathrm{cl}\{f_n(\omega) : n \geq 1\}$ for all $\omega \in D(F)$.*

(vi) *$G(F) = \{(\omega, x) \in \Omega \times B : x \in F(\omega)\}$ is $\mathcal{A} \otimes \mathcal{B}_S$-measurable.*

By [2, Theorem III.2], a multivalued function $F : \Omega \to 2^S$ with compact values is measurable iff it is measurable as a function from Ω to $(\mathcal{C}(S), \widehat{\varrho})$.

By Theorem 3 we have the following

Corollary. *Let $\{F_n\}$ be a strongly tight (in $(\mathcal{C}(S), \widehat{\varrho})$) sequence of multifunctions defined on Ω with values in $\mathcal{C}(S)$. Then every sequence $\{f_n\}$ of selectors $\{f_n \in F_n\}$ is strongly tight (in (S, ϱ)).*

Proof. For every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset S$ such that

$$P\Big(\bigcap_{n=1}^{\infty}[\omega : F_n(\omega) \in K_\varepsilon]\Big) > 1 - \varepsilon.$$

If we put $A = \bigcap_{n=1}^{\infty}[\omega : F_n(\omega) \in K_\varepsilon]$ then $K = \bigcup_{\omega \in A}\bigcup_{n=1}^{\infty} F_n(\omega)$ is a compact subset of S, and $P(\bigcap_{n=1}^{\infty}[\omega : f_n(\omega) \in K]) > 1 - \varepsilon$.

Theorem 5. *Let F be a multifunction such that $F(\omega)$ is the set of all cluster points of a sequence $\{f_n : \Omega \to S\}$ of r.e. almost surely. If the sequence $\{f_n\}$ is strongly tight then F is a measurable multifunction.*

Proof. By strong tightness of $\{f_n\}$ we have $F(\omega) \neq \emptyset$ and $F(\omega)$ is closed almost surely. For every $x \in S$ we have $d(x, F(\omega)) = \liminf_{n\to\infty} \varrho(x, f_n(\omega))$, and in view of Theorem 4 we see that F is a measurable multifunction.

Theorem 6. *Let $\{f_n\}$ be a strongly tight sequence of r.e. with values in a Polish space S. If the sequence is not a.s. convergent then there exist two r.e. h_1 and h_2 such that $P([h_1 = h_2]) < 1$, and $h_1(\omega)$ and $h_2(\omega)$ are cluster points of the sequence $\{f_n(\omega)\}$ with probability 1.*

Proof. The multifunction F defined in Theorem 5 is measurable and by Theorem 4 there exists a sequence $\{g_n\}$ such that $F(\omega) = \mathrm{cl}\{g_n(\omega) : n \geq 1\}$ a.s. From the sequence $\{g_n(\omega)\}$ we can choose two functions h_1 and h_2 which are measurable selections of F and $P([h_1 = h_2]) < 1$.

Theorem 7. *Let $\{f_n(\omega)\}$ be a sequence of r.e. adapted to an increasing sequence $\{\mathcal{A}_n\}$ of sub-σ-fields of $\mathcal{A}$, and f be a r.e. such that $f(\omega)$ is a cluster point of the sequence $\{f_n(\omega)\}$ with probability 1. Then there exists a sequence $\{T \ni \tau_n \geq n\}$ of stoping times such that $f_{\tau_n} \xrightarrow{a.s.} f$ as $n \to \infty$.*

Proof. The proof is a simple modification of the proof of [7, Lemma 3.1(a)].

4. Convergence. The basic theory of integrals, conditional expectations and martingales of multifunctions is due to Hiai and Umegaki [5].

We shall deal with multivalued functions in $M(\Omega;B)$.

A multifunction F is L^1 *integrably bounded* ($F \in L^1(\Omega;\mathcal{B}(B))$) if

$$E\widehat{\varrho}(0,F) < \infty.$$

We denote by $\mathcal{K}(B)$ the family of all nonempty closed convex bounded subsets of B, and by $\mathcal{KC}(B)$ the family of all nonempty compact convex subsets of B.

We define two subspaces of $L^1(\Omega;\mathcal{B}(B)$ as follows:

$$L^1_{\mathcal{K}}(\Omega;B) = \{F \in L^1(\Omega;\mathcal{B}(B)) : F(\omega) \in \mathcal{K}(B) \text{ a.s.}\},$$
$$L^1_{\mathcal{KC}}(\Omega;B) = \{F \in L^1(\Omega;\mathcal{B}(B)) : F(\omega) \in \mathcal{KC}(B) \text{ a.s.}\}.$$

DEFINITION 2 ([5]). Let $F \in L^1(\Omega;\mathcal{B}(B))$. We call $E(F\,|\,\mathcal{G}) \in L^1(\Omega,\mathcal{G},\mathcal{B}(B))$ satisfying $S^1_{E(F|\mathcal{G})}(\mathcal{G}) = \mathrm{cl}\{E(f\,|\,\mathcal{G}) : f \in S^1_F\}$ the *multivalued conditional expectation* of F relative to $\mathcal{G}$. If $\mathcal{G}$ is trivial ($\mathcal{G} = \{\emptyset,\Omega\}$), then $E(F\,|\,\mathcal{G})$ is the *expectation* of F.

DEFINITION 3. Let $\{F_n\}$ be an integrable family of multifunctions which is adapted to $\{\mathcal{F}_n\}$. We call $\{F_n,\mathcal{F}_n\}$ an *amart* if the net $\{EF_\tau, \tau \in T\}$ is convergent to some set H,

$$EF_\tau \to H, \qquad \tau \in T.$$

An amart $\{F_n,\mathcal{F}_n\}$ is L^1 *integrable* ($F_n \in L^1$) if

$$\sup_{n\ge 1} E\widehat{\varrho}(0,F_n) < \infty.$$

We will need

THEOREM 8 ([4, Theorem 2.7]). *Suppose that B is reflexive and $E\sup_{n\ge1}\widehat{\varrho}(0,F_n) < \infty$. If $F, F_n \in L^1_{\mathcal{KC}}$ and the sequence $\{F_n\}$ converges almost surely to F then $EF_n \to EF$.*

THEOREM 9. *Let B^* be a separable space and $F_1, F_2 \in L^1_{\mathcal{K}}$. If $P[F_1 = F_2] < 1$ then there exist two functions τ_1 and τ_2, which are measurable with respect to the σ-field $\sigma(F_1,F_2)$,*

$$\tau_1(\omega) = \begin{cases} 1 & \text{for } \omega \in A,\\ 2 & \text{for } \omega \notin A,\end{cases} \qquad \tau_2(\omega) = \begin{cases} 1 & \text{for } \omega \in B,\\ 2 & \text{for } \omega \notin B,\end{cases} \qquad A,B \in \sigma(F_1,F_2),$$

and $\widehat{\varrho}(EF_{\tau_1}, EF_{\tau_2}) > 0$.

Proof. By the assumption $P[F_1 = F_2] < 1$ there exists a selection $f \in F_1$ such that $P[\widehat{\varrho}(f, F_2) > 0] > 0$. By [5, Lemma 4.4] there exists a set $A \in \sigma(F_1, F_2)$ such that $\int_A f\,dP \notin \mathrm{cl}\int_A F_2\,dP$ and $\widehat{\varrho}(\int_A f\,dP, \int_A F_2\,dP) > 0$. For

$$\tau_1(\omega) = \begin{cases} 1 & \text{for } \omega \in A, \\ 2 & \text{for } \omega \notin A, \end{cases} \qquad \tau_2 \equiv 2,$$

we have

$$\widehat{\varrho}(EF_{\tau_1}, EF_{\tau_2}) \geq \widehat{\varrho}\Big(\int_A f\,dP, \int_A F_2\,dP\Big) > 0.$$

THEOREM 10. *Let B be a reflexive separable Banach space. Every strongly tight amart $\{L^1_{\mathcal{KC}}(\Omega; B) \ni F_n, \mathcal{F}_n\}$ such that* $\sup_{n\geq 1} E\widehat{\varrho}(0, F_n) < \infty$ *converges a.s.*

Proof. Assume that this is false. There exist two multifunctions F_1' and F_2' such that for every ω, $F_1'(\omega)$ and $F_2'(\omega)$ are cluster points of the sequence $\{F_n, \mathcal{F}_n\}$ and $P([\widehat{\varrho}(F_1', F_2') = 0]) < 1$. In view of Theorem 9 there exist multifunctions F_1^*, F_2^* such that for every ω, $F_1^*(\omega)$ and $F_2^*(\omega)$ are cluster points of $\{F_n, \mathcal{F}_n\}$, $P[\widehat{\varrho}(F_1^*, F_2^*) = 0] < 1$ and $\widehat{\varrho}(EF_1^*, EF_2^*) > 0$. Then there exist two sequences $\{\tau_n \in T\}$ and $\{\sigma_n \in T\}$ such that $F_{\tau_n} \xrightarrow{a.s.} F_1^*$ and $F_{\sigma_n} \xrightarrow{a.s.} F_2^*$, and hence by the definition of amart it follows that $EF_{\tau_n} \to H$ and $EF_{\sigma_n} \to H$, which together with Theorem 8 yields $EF_1^* = EF_2^* = H$. This contradiction ends the proof.

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