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THE V_a-DEFORMATION OF THE CLASSICAL CONVOLUTION

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Abstract. We study deformations of the classical convolution. For every invertible transformation $T : \mu \mapsto T\mu$, we are able to define a new associative convolution of measures by

$$\mu *_T \nu = T^{-1}(T\mu * T\nu).$$

We deal with the V_a-deformation of the classical convolution. We prove the analogue of the classical Lévy–Khintchine formula. We also show the central limit measure, which turns out to be the standard Gaussian measure. Moreover, we calculate the Poisson measure in the V_a-deformed classical convolution and give the orthogonal polynomials associated to the limiting measure.

1. Introduction. For every invertible transformation of probability measures on the real line $T : \mu \mapsto T\mu$, we are able to define a new associative convolution of measures by the requirement

$$\mu *_T \nu = T^{-1}(T\mu * T\nu).$$

In this paper we consider the V_a-deformation, which was defined and considered in the paper [KW] in the case of deformations of the free and conditionally free convolutions.

In Section 2 we remind the notion of the classical convolution and the Fourier transform. We also recall the moment-cumulant formula for the classical convolution and the classical Lévy–Khintchine theorem for the infinitely divisible distributions.

In Section 3 we recall a definition of the V_a-deformation and show its basic properties.

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In Section 4 we define the V_a-deformed classical convolution and show that this convolution is not a generalized convolution defined by Urbanik. We also prove an analogue of the classical Lévy–Khintchine formula for the V_a-deformed classical convolution. Namely, we show that a probability measure μ on $\mathbb{R}$ is $*_a$-infinitely divisible if and only if there exist a real number α, a positive number σ^2 and a measure ν with $\nu(\{0\}) = 0$ and $\int \frac{x^2}{1+x^2}\,\mathrm{d}\nu(x) < \infty$, such that the Fourier transform of the deformation of the measure μ is of the following form

$$\mathcal{F}[V_a\mu](t) = \exp\left(it\alpha - \frac{t^2\sigma^2}{2} + \int_{-\infty}^{\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\mathrm{d}\nu(x)\right).$$

In Section 5 we find the central limit measure, which turns out to be the standard Gaussian measure. In Section 6 we calculate the Poisson measure in the V_a-deformed classical convolution and give the orthogonal polynomials associated to the limiting measure.

2. Classical convolution. Let us denote by $\mathcal{F}[\mu](z)$ the Fourier transform of a probability measure μ,

$$\mathcal{F}[\mu](t) = \int_{\mathbb{R}} e^{itx}\,\mathrm{d}\mu(x), \quad t \in \mathbb{R}.$$

It is well known that for the classical convolution of measures

$$\mu * \nu(E) = \int_{-\infty}^{\infty} \mu(E - x)\,\mathrm{d}\nu(x), \quad E \text{ a Borel set},$$

we have

$$\mathcal{F}[\mu * \nu](t) = \mathcal{F}[\mu](t) \cdot \mathcal{F}[\nu](t),$$

which means that for the logarithm we have

$$\ln \mathcal{F}[\mu * \nu](t) = \ln \mathcal{F}[\mu](t) + \ln \mathcal{F}[\nu](t),$$

that is, the logarithm of the Fourier transform is a linearizing transform for the classical convolution. This is well defined for some neighbourhood of 0.

Assume that a probability measure μ has all moments. Then

$$\ln \mathcal{F}[\mu](t) = \sum_{n=0}^{\infty} \frac{F^*_\mu(n)}{n!}(it)^n$$

for t in some neighbourhood of 0. We will call the coefficients $F^*_\mu(n)$ the *classical cumulants* or *semiinvariants*.

We have the following relation between the Fourier transform and moments, see [GK] and [N]:

$$m_\mu(n) = \int x^n\,\mathrm{d}\mu(x) = \sum_{\pi \in \mathrm{P}(n)} F^*_\mu(\pi) = \sum_{\substack{\pi \in \mathrm{P}(n) \\ \pi = \{B_1,\ldots,B_k\}}} \prod_{j=1}^{k} F^*_\mu(|B_j|), \tag{2.1}$$

where $\mathrm{P}(n)$ is the set of all partitions of the set $\{1, \ldots, n\}$. Solving the above equation for $F^*_\mu(n)$ we obtain

$$F^*_\mu(n) = m_\mu(n) - \sum_{\substack{\pi\in \mathrm{P}(n)\\ \pi\neq(1,\ldots,n)}} F^*_\mu(\pi) = m_\mu(n) - \sum_{\substack{\pi\in \mathrm{P}(n)\\ \pi\neq(1,\ldots,n)}} \prod_{i=1}^{k} F^*_\mu(|B_i|), \tag{2.2}$$

where $(1, \ldots, n)$ denotes the partition with one block containing all the points.

Remark 1. Let us note that

$$F^*_\mu(1) = m_\mu(1) = \int_{-\infty}^{\infty} x \,\mathrm{d}\mu(x)$$

is the mean of the measure μ and

$$F^*_\mu(2) = \sigma^2_\mu = \int_{-\infty}^{\infty} (x - m_\mu(1))^2 \,\mathrm{d}\mu(x)$$

is equal to the variance of μ.

Let us recall that a probability measure μ is infinitely divisible with respect to the classical convolution $*$ if for every $N \in \mathbb{N}$ there exists a probability measure μ_N such that $\mu = \mu_N^{*N}$.

The well-known Lévy–Khintchine theorem, see for instance [D], characterizes infinitely divisible measures:

Theorem 1. *A probability measure μ on $\mathbb{R}$ is infinitely divisible if and only if there exist a real number α, a positive number σ^2 and a measure ν with $\nu(\{0\}) = 0$ and $\int \frac{x^2}{1+x^2}\,\mathrm{d}\nu(x) < \infty$, such that the Fourier transform of the measure μ is of the following form*

$$\mathcal{F}[\mu](t) = \exp\left(it\alpha - \frac{t^2\sigma^2}{2} + \int_{-\infty}^{\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\mathrm{d}\nu(x)\right).$$

Some deformations of the classical convolutions were considered by Urbanik [U1]–[U5] and called the *generalized convolutions*. We will recall his definition

Definition 1. A commutative and associative operation $\diamond$ defined on $\mathrm{Prob}\,(\mathbb{R}_+)$, the probability measures on the Borel subsets of $[0, \infty)$, is called a *generalized convolution* if it satisfies the following conditions:

(i) The measure δ_0 is a unit element, that is, for all $\mu \in \mathrm{Prob}\,(\mathbb{R}_+)$

$$\delta_0 \diamond \mu = \mu.$$

(ii) For all $\mu, \nu, \rho \in \mathrm{Prob}\,(\mathbb{R}_+)$ and $0 \le p \le 1$

$$(p\mu + (1-p)\nu) \diamond \rho = p(\mu \diamond \rho) + (1-p)(\nu \diamond \rho)$$

(convex linearity).

(iii) For all $\mu, \nu \in \mathrm{Prob}\,(\mathbb{R}_+)$ and $a > 0$

$$(\mathbb{D}_a\mu) \diamond (\mathbb{D}_a\nu) = \mathbb{D}_a(\mu \diamond \nu)$$

(homogeneity).

(iv) If $\mu_n \to \mu$ weakly, then for all $\nu \in \mathrm{Prob}(\mathbb{R}_+)$ and $a > 0$

$$\mu_n \diamond \nu \to \mu \diamond \nu$$

(continuity).

(v) There exists a sequence $c_1, c_2 \dots$ of positive numbers such that the sequence

$$\mathbb{D}_{c_n} \delta_1^{\diamond n}$$

weakly converges to a measure different from δ_0 (the law of large numbers for measure concentrated at a single point), where

$$\delta_a^{\diamond 1} = \delta_a, \quad \delta_a^{\diamond n+1} = \delta_a^{\diamond n} \diamond \delta_a.$$

3. V_a**-deformation.** Let us recall the definition and basic properties of the V_a-deformation, which can be found in [KW].

DEFINITION 2. For a measure μ with finite second moment and $a \in \mathbb{R}$ let us define its deformation $V_a\mu$ by letting

$$\frac{1}{G_{V_a\mu}(z)} = \frac{1}{G_\mu(z)} + a\sigma_\mu^2, \tag{3.1}$$

where $G_\mu(z)$ is the Cauchy transform of the measure μ, for $z \in \mathbb{C}^+$, defined as follows:

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{d\mu(x)}{z - x}.$$

From the Nevanlinna theorem on the reciprocals of Cauchy transforms of measures, [A], we get that $V_a\mu$ is again a probability measure.

REMARK 2. The V_a-transformation of a compactly supported measure can be illustrated with the use of the continued fraction representation of the Cauchy transform. If

$$G_\mu(z) = \cfrac{1}{z - \alpha_0 - \cfrac{\lambda_0}{z - \alpha_1 - \cfrac{\lambda_1}{z - \alpha_2 - \cfrac{\lambda_2}{z - \alpha_3 - \ddots}}}},$$

then

$$G_{V_a\mu}(z) = \cfrac{1}{z - \alpha_0 + a\lambda_0 - \cfrac{\lambda_0}{z - \alpha_1 - \cfrac{\lambda_1}{z - \alpha_2 - \cfrac{\lambda_3}{z - \alpha_3 - \ddots}}}}.$$

This means that the V_a-transformation adds the amount $a\lambda_0$ to the coefficient α_0.

We also have the following

Proposition 1. *The V_a-transformation leaves the λ_1 coefficient (and thus the variance) unchanged:*

$$\sigma^2_{V_a\mu} = \sigma^2_\mu.$$

This follows from the definition of V_a.

Example 1. We compute the V_a-transformation of a probability measure which is supported in two points. Let

$$\mu = p\delta_x + q\delta_y, \quad x < y, \quad p, q \geq 0, \quad p + q = 1.$$

then $V_a\mu$ is again a two point measure

$$V_a\mu = P\delta_A + Q\delta_B,$$

where

$$\begin{aligned} A = {} & \frac{x + y - apq(x-y)^2}{2} \\ & - \frac{\sqrt{(x + y - apq(x-y)^2)^2 - 4(xy - apq(x-y)^2(qx + py))}}{2}, \\ B = {} & \frac{x + y - apq(x-y)^2}{2} \\ & + \frac{\sqrt{(x + y - apq(x-y)^2)^2 - 4(xy - apq(x-y)^2(qx + py))}}{2}, \end{aligned}$$

and

$$\begin{aligned} P &= \frac{1}{2} - \frac{(q-p)(x-y) + apq(x-y)^2}{2\sqrt{(x + y - apq(x-y)^2)^2 - 4(xy - apq(x-y)^2(qx + py))}}, \\ Q &= \frac{1}{2} + \frac{(q-p)(x-y) + apq(x-y)^2}{2\sqrt{(x + y - apq(x-y)^2)^2 - 4(xy - apq(x-y)^2(qx + py))}}. \end{aligned}$$

Proposition 2. *We have*

$$\frac{1}{G_{V_a\mathbb{D}_\lambda\mu}(z)} = \frac{1}{G_{\mathbb{D}_\lambda V_a\mu}(z)} + a(\lambda^2 - \lambda)\sigma^2_\mu,$$

which means that nontrivial dilations of measures different from δ_b, $b \in \mathbb{R}$ do not commute with V_a for $a \neq 0$:

$$\mathbb{D}_\lambda V_a\mu \neq V_a\mathbb{D}_\lambda\mu.$$

A proof of the proposition above can be found in [KW].

4. V_a-deformation of the classical convolution. For every invertible transformation $T : \mu \mapsto T\mu$, we are able to define a new associative convolution of measures by

$$\mu *_T \nu = T^{-1}(T\mu * T\nu).$$

Because the V_a-deformation is invertible, let us define

Definition 3. The *V_a-deformed classical convolution* $*_a$ is defined by

$$\mu *_a \nu = V_{-a}(V_a\mu * V_a\nu), \tag{4.1}$$

where μ and ν have finite second moments.

We can also define the deformed Fourier transform and deformed classical cumulants. Let

$$\mathcal{F}^a[\mu](t) = \mathcal{F}[V_a\mu](t).$$

Then we have

$$\ln \mathcal{F}^a[\mu](t) = \sum_{n=0}^{\infty} \frac{F_\mu^{*_a}(n)}{n!}(it)^n \tag{4.2}$$

where for any positive integer n

$$F_\mu^{*_a}(n) := F_{V_a\mu}^{*}(n).$$

Thus by the above formulae

$$\mathcal{F}^a[\mu *_a \nu](t) = \mathcal{F}^a[\mu](t)\cdot \mathcal{F}^a[\nu](t),$$
$$\ln \mathcal{F}^a[\mu *_a \nu](t) = \ln \mathcal{F}^a[\mu](t) + \ln \mathcal{F}^a[\nu](t).$$

Let us start by recalling that from [KW] and Example 1 it follows that the V_a-deformation has the following properties:

$$V_a\delta_x = \delta_x,$$
$$V_a(p\delta_x + (1-p)\delta_y) = P\delta_X + (1-P)\delta_Y, \quad p \neq P, x \neq X, y \neq Y.$$

In particular, we have

$$V_a\left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) = \left(\frac{1}{2} - \frac{a}{8\sqrt{1+\frac{a}{4}}}\right)\delta_{\frac{1-\sqrt{1+\frac{a}{4}}-\frac{a}{4}}{2}} + \left(\frac{1}{2} + \frac{a}{8\sqrt{1+\frac{a}{4}}}\right)\delta_{\frac{1+\sqrt{1+\frac{a}{4}}-\frac{a}{4}}{2}}.$$

Since

$$\delta_x * \delta_y = \delta_{x+y},$$

we obtain that

$$\begin{aligned}\left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) *_a \delta_z &= V_{-a}\left(V_a\left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) * V_a\delta_z\right)\\ &= V_{-a}\left(V_a\left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) * \delta_z\right)\\ &= V_{-a}\left(\left(\frac{1}{2} - \frac{a}{8\sqrt{1+\frac{a}{4}}}\right)\delta_{\frac{1-\sqrt{1+\frac{a}{4}}-\frac{a}{4}}{2}+z}\right.\\ &\qquad \left. + \left(\frac{1}{2} + \frac{a}{8\sqrt{1+\frac{a}{4}}}\right)\delta_{\frac{1+\sqrt{1+\frac{a}{4}}-\frac{a}{4}}{2}+z}\right).\end{aligned}$$

On the other hand

$$\frac{1}{2}\delta_0 *_a \delta_z + \frac{1}{2}\delta_1 *_a \delta_z = \frac{1}{2}\delta_0 * \delta_z + \frac{1}{2}\delta_1 * \delta_z = \frac{1}{2}\delta_z + \frac{1}{2}\delta_{1+z}.$$

Because

$$\begin{aligned}V_a\left(\frac{1}{2}\delta_z + \frac{1}{2}\delta_{1+z}\right) &= \left(\frac{1}{2} - \frac{a}{4\sqrt{4+a-8z+4az-16z^2}}\right)\delta_{\frac{4-a+8z-2\sqrt{4+a-8z+4az-16z^2}}{8}}\\ &\quad + \left(\frac{1}{2} + \frac{a}{4\sqrt{4+a-8z+4az-16z^2}}\right)\delta_{\frac{4-a+8z+2\sqrt{4+a-8z+4az-16z^2}}{8}},\end{aligned}$$

it follows that

$$P\delta_{X+z} + (1-P)\delta_{Y+z} \neq V_a(p\delta_{x+z} + (1-p)\delta_{y+z}),$$

or equivalently

$$V_{-a}(P\delta_{X+z} + (1-P)\delta_{Y+z}) \neq p\delta_{x+z} + (1-p)\delta_{y+z},$$

which implies that this convolution is not a generalized convolution of Urbanik, because it does not satisfy the linearity condition.

Let us start with recalling a general definition:

DEFINITION 4. We say that a probability measure μ is *infinitely divisible with respect to a convolution* $\oplus$ if for every $N \in \mathbb{N}$ there exists a probability measure μ_N such that $\mu = \mu_N^{\oplus N}$.

This can be rewritten equivalently in terms of respective linearizing $R^{\oplus}$-transforms: a probability measure μ is infinitely divisible with respect to a convolution $\oplus$ if for every $N \in \mathbb{N}$ there exists a measure μ_N such that $R_{\mu}^{\oplus}(z) = N \cdot R_{\mu_N}^{\oplus}(z)$.

In the case of the deformed classical convolution, V_a-convolution, we will use the deformed Fourier transforms.

For the V_a-deformed classical convolution we can prove an analogue of the classical Lévy–Khintchine formula.

THEOREM 2. *A probability measure μ on $\mathbb{R}$ with finite second moment is $*_a$-infinitely divisible if and only if there exist a real number α, a positive number σ^2 and a measure ν with $\nu(\{0\}) = 0$ and $\int \frac{x^2}{1+x^2}\,\mathrm{d}\nu(x) < \infty$ such that the deformed Fourier transform of the measure μ is of the following form:*

$$\mathcal{F}^a[\mu](t) = \exp\left(it\alpha - \frac{t^2\sigma^2}{2} + \int_{-\infty}^{\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\mathrm{d}\nu(x)\right).$$

Proof. A measure μ is $*_a$-infinitely divisible if for every $N \in \mathbb{N}$ there exists a measure μ_N such that on some domain

$$\mathcal{F}^a[\mu](t) = (\mathcal{F}^a[\mu_N](t))^N.$$

By a double application of the definition of the V_a-deformed Fourier transform to the left and right hand side of the above equation we get

$$\mathcal{F}[V_a\mu](t) = \mathcal{F}^a[\mu](t) = (\mathcal{F}^a[\mu_N](t))^N = \Big(\mathcal{F}[V_a\mu_N](t)\Big)^N,$$

hence, the measure μ is $*_a$-infinitely divisible if and only if

$$\mathcal{F}[V_a\mu](t) = \Big(\mathcal{F}[V_a\mu_N](t)\Big)^N,$$

which is equivalent to classical infinite divisibility of the measure $V_a\mu$.

By the Lévy–Khintchine formula (see [D], [GK]) we know that the measure $V_a\mu$ is infinitely divisible if and only if there exist a real number α, a positive number σ^2 and a measure ν with $\nu(\{0\}) = 0$ and $\int \frac{x^2}{1+x^2}\,\mathrm{d}\nu(x) < \infty$, such that the Fourier transform of the measure $V_a\mu$ is of the following form:

$$\mathcal{F}[V_a\mu](t) = \exp\left(it\alpha - \frac{t^2\sigma^2}{2} + \int_{-\infty}^{\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\mathrm{d}\nu(x)\right).$$

Hence

$$\mathcal{F}^a[\mu](t) = \mathcal{F}[V_a\mu](t) = \exp\left(it\alpha - \frac{t^2\sigma^2}{2} + \int_{-\infty}^{\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\mathrm{d}\nu(x)\right). \ \blacksquare$$

5. Central limit theorem. We now prove a central limit theorem for the $*_a$ convolution. First we prove a technical lemma. By [KW] we have

LEMMA 1. *Let μ be a compactly supported probability measure on the real line. Then*

$$m_{V_a\mathbb{D}_\lambda\mu}(n) = \lambda^n m_\mu(n) + o(\lambda^n). \tag{5.1}$$

Using the relation between moments and the Fourier transform, we will obtain:

LEMMA 2. *Let μ be a compactly supported probability measure on the real line with mean zero and variance equal to 1. Then*

$$F^*_{V_a\mathbb{D}_\lambda\mu}(1) = -a\lambda^2, \qquad F^*_{V_a\mathbb{D}_\lambda\mu}(2) = \lambda^2, \qquad F^*_{V_a\mathbb{D}_\lambda\mu}(k) = o(\lambda^2) \quad \textit{for } k \geq 3.$$

Proof. Let us note that the V_a-transformation of a dilation of measure μ with mean zero and variance σ^2_μ equal to 1 has

$$m_{V_a\mathbb{D}_\lambda\mu}(1) = -a\lambda^2, \qquad \sigma^2_{V_a\mathbb{D}_\lambda\mu} = \lambda^2.$$

Because of the moment-cumulant formulae for the classical convolution (2.1), for the V_a-deformation of dilation of measures we have

$$F^*_{V_a\mathbb{D}_\lambda\mu}(1) = -a\lambda^2, \qquad F^*_{V_a\mathbb{D}_\lambda\mu}(2) = \sigma^2_{V_a\mathbb{D}_\lambda\mu} = \lambda^2.$$

Moreover, for $k = 3$ by the moment-cumulant formula (2.1)

$$F^*_{V_a\mathbb{D}_\lambda\mu}(3) = m_{V_a\mathbb{D}_\lambda\mu}(3) - 3F^*_{V_a\mathbb{D}_\lambda\mu}(2)F^*_{V_a\mathbb{D}_\lambda\mu}(1) - F^*_{V_a\mathbb{D}_\lambda\mu}(1)^3,$$

by Lemma 1, we have

$$F^*_{V_a\mathbb{D}_\lambda\mu}(3) = \lambda^3 m_\mu(3) + o(\lambda^3) + 3a\lambda^4 + a^3\lambda^6 = o(\lambda^2),$$

and by induction: if in π there exists B such that $|B| \geq 3$, then $F^*_{V_a\mathbb{D}_\lambda\mu}(\pi) = o(\lambda^2)$. If not, there must be at least two blocks, $|B_1| + |B_2| \geq 2$, hence

$$\prod_{B_i\in\pi} F^*_{V_a\mathbb{D}_\lambda\mu}(B_i) = o(\lambda^3). \ \blacksquare$$

THEOREM 3 (Central limit theorem). *Let μ be a compactly supported probability measure on the real line with mean zero and variance equal to 1. Then the sequence*

$$\mathbb{D}_{1/\sqrt{N}}\mu *_a \ldots *_a \mathbb{D}_{1/\sqrt{N}}\mu$$

is $*$-weakly convergent to the normal distribution $N_{(0,1)}$ with density

$$\mathrm{d}N_{(0,1)}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x.$$

Proof. Let us denote by $\mathcal{F}[\mu](z)$ the Fourier transform of the measure μ on $\mathbb{R}$ and by $\mathcal{F}^a[\mathbb{D}_\lambda\mu](z)$ the deformed Fourier transform of the dilation of the measure μ, defined in (4.2). This means that we have

$$\mathcal{F}^a[\mathbb{D}_\lambda\mu](z) = \mathcal{F}[V_a(\mathbb{D}_\lambda\mu)](z).$$

The sequence of N-fold V_a-convolution of the measure μ is of the form

$$\mu_N = \mathbb{D}_{1/\sqrt{N}}\mu *_a \ldots *_a \mathbb{D}_{1/\sqrt{N}}\mu = V_{-a}(V_a\mathbb{D}_{1/\sqrt{N}}\mu * \ldots * V_a\mathbb{D}_{1/\sqrt{N}}\mu).$$

Denote for simplicity

$$\nu_N = \underbrace{V_a\mathbb{D}_{1/\sqrt{N}}\mu * \cdots * V_a\mathbb{D}_{1/\sqrt{N}}\mu}_{N \text{ times}}.$$

Then $\mu_N = V_{-a}\nu_N$ and we have

$$\mathcal{F}[\nu_N](z) = (\mathcal{F}[V_a\mathbb{D}_{1/\sqrt{N}}\mu](z))^N = (\mathcal{F}^a[\mathbb{D}_{1/\sqrt{N}}\mu](z))^N$$

and by Lemma 2

$$\begin{aligned}
F^*_{\nu_N}(1) &= -N \cdot \frac{a}{N} = -a,\\
F^*_{\nu_N}(2) &= N \cdot \frac{1}{N} = 1,\\
F^*_{\nu_N}(k) &= N \cdot o\left(\frac{1}{N}\right) \overset{N\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where the cumulants $F^{*_a}_{\nu_N}(n)$ and the Fourier transform $\mathcal{F}[\nu_N](t)$ are related by

$$\ln \mathcal{F}[\nu_N](t) = \sum_{n=0}^{\infty} \frac{F^{*_a}_{\nu_N}(n)}{n!}(it)^n.$$

Therefore

$$\ln \mathcal{F}[\nu_N](z) \overset{N\to\infty}{\longrightarrow} \ln \mathcal{F}[N_{(-a,1)}](z) = -az + z^2,$$

and $N_{(-a,1)}$ is the normal distribution with the first moment $-a$ and variance 1. We know that

$$G_{N_{(-a,1)}}(z) = \cfrac{1}{z + a - \cfrac{1}{z - \cfrac{2}{z - \cfrac{3}{z - \cfrac{4}{z - \ddots}}}}}$$

Because the measure $N_{(-a,1)}$ is determined by its moments we can apply the following theorem from [ST]:

THEOREM 4. *Let $m_\mu(n)$ be the sequence of moments of a probability measure μ. A necessary and sufficient condition that the moment problem $m_\mu(n)$ be determined is that the* *continued fraction form of the Cauchy transform of the measure μ converges completely for all complex z.*

Hence the above continued fraction is convergent to the respective Cauchy transform. Thus we have

$$G_{V_{-a}N_{(-a,1)}}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{2}{z - \cfrac{3}{z - \cfrac{4}{z - \ddots}}}}} = G_{N_{(0,1)}}(z).$$

Therefore

$$\mathcal{F}[\mu_N](z) \overset{N\to\infty}{\longrightarrow} \mathcal{F}[N_{(0,1)}](z)$$

and

$$\mathbb{D}_{1/\sqrt{N}}\mu *_a \ldots *_a \mathbb{D}_{1/\sqrt{N}}\mu = V_{-a}(V_a\mathbb{D}_{1/\sqrt{N}}\mu * \ldots * V_a\mathbb{D}_{1/\sqrt{N}}\mu) \overset{N\to\infty}{\longrightarrow} N_{(0,1)}. \blacksquare$$

REMARK 3. Because the V_a-deformation does not commute with dilation of measures, we cannot use the usual central limit theorem, see for instance [B]. The central measure, however, is the normal distribution.

6. Poisson type limit theorem. In this section we will study a Poisson limit theorem for the $*_a$-convolution.

Recall that by Example 1, we know that the V_a-transformation of the Bernoulli measure $\mu_N = \big(1 - \frac{\lambda}{N}\big)\delta_0 + \frac{\lambda}{N}\delta_1$ is also a two-point measure

$$V_a\mu = P_N\delta_{A_N} + Q_N\delta_{B_N},$$

with

$$\begin{aligned}
A_N &= \frac{1 - a\big(1-\frac{\lambda}{N}\big)\frac{\lambda}{N} - \sqrt{4a\big(1-\frac{\lambda}{N}\big)^2\frac{\lambda}{N} + \big(1 - a\big(1-\frac{\lambda}{N}\big)\frac{\lambda}{N}\big)^2}}{2},\\
B_N &= \frac{1 - a\big(1-\frac{\lambda}{N}\big)\frac{\lambda}{N} + \sqrt{4a\big(1-\frac{\lambda}{N}\big)^2\frac{\lambda}{N} + \big(1 - a\big(1-\frac{\lambda}{N}\big)\frac{\lambda}{N}\big)^2}}{2},\\
P_N &= \frac{1}{2} - \frac{\big(1-2\frac{\lambda}{N}\big) + a\big(1-\frac{\lambda}{N}\big)\frac{\lambda}{N}}{2\sqrt{\big(1 - a\big(1-\frac{\lambda}{N}\big)\frac{\lambda}{N}\big)^2 + 4a\big(1-\frac{\lambda}{N}\big)^2\frac{\lambda}{N}}},\\
&= \frac{B_N - \big(1 - a\big(1-\frac{\lambda}{N}\big)\big)\frac{\lambda}{N}}{\beta_N}\\
Q_N &= \frac{1}{2} + \frac{\big(1-2\frac{\lambda}{N}\big) + a\big(1-\frac{\lambda}{N}\big)\frac{\lambda}{N}}{2\sqrt{\big(1 - a\big(1-\frac{\lambda}{N}\big)\frac{\lambda}{N}\big)^2 + 4a\big(1-\frac{\lambda}{N}\big)^2\frac{\lambda}{N}}}\\
&= \frac{\big(1 - a\big(1-\frac{\lambda}{N}\big)\big)\frac{\lambda}{N} - A_N}{\beta_N}.
\end{aligned}$$

with

$$\beta_N = \sqrt{\left(1 - a\left(1 - \frac{\lambda}{N}\right)\frac{\lambda}{N}\right)^2 + 4a\left(1 - \frac{\lambda}{N}\right)^2 \frac{\lambda}{N}}.$$

We will use the Fourier transform. Let

$$\mu^N := \mu_N *_a \ldots *_a \mu_N.$$

It is clear that

$$\mathcal{F}^a[\mu^N](x) = \mathcal{F}[V_a \mu^N](x) = (\mathcal{F}[V_a \mu_N](x))^N = (\mathcal{F}^a[\mu_N](x))^N.$$

Because the deformed Fourier transform of the measure μ_N is of the form

$$\mathcal{F}^a[\mu_N](x) = P_N e^{ixA_N} + Q_N e^{ixB_N},$$

its Taylor expansion is the following

$$\mathcal{F}^a[\mu_N](x) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}(P_N A_N^n + Q_N B_N^n).$$

By the fact that

$$A_N B_N = -a\left(1 - \frac{\lambda}{N}\right)^2 \frac{\lambda}{N} = -a\frac{\lambda}{N} + 2a\left(\frac{\lambda}{N}\right)^2 - a\left(\frac{\lambda}{N}\right)^3 = -a\frac{\lambda}{N} + O\left(\frac{1}{N^2}\right) \tag{6.1}$$

we get the following

Lemma 3. *For each positive integer $n \in \mathbb{N}$*

$$A_N^n + B_N^n = 1 + O\left(\frac{1}{N}\right).$$

Proof. For $n = 1$ we have

$$A_N + B_N = 1 - a\left(1 - \frac{\lambda}{N}\right)\frac{\lambda}{N} = 1 - a\frac{\lambda}{N} + a\left(\frac{\lambda}{N}\right)^2 = 1 + O\left(\frac{1}{N}\right)$$

and for $n \geq 2$

$$\begin{aligned} 1 + O\left(\frac{1}{N}\right) &= (A_N + B_N)^n = A_N^n + B_N^n + A_N B_N \sum_{k=1}^{n-1} \binom{n}{k} A_N^k B_N^{n-k} \\ &= A_N^n + B_N^n + O\left(\frac{1}{N}\right). \ \blacksquare \end{aligned}$$

Lemma 4. *For each positive integer $n \in \mathbb{N}$ we have*

$$P_N A_N^n + Q_N B_N^n = \begin{cases} (1-a)\frac{\lambda}{N} + O\left(\frac{1}{N^2}\right) & \textit{if } n = 1, \\ \frac{\lambda}{N} + O\left(\frac{1}{N^2}\right) & \textit{if } n \geq 2. \end{cases}$$

Proof. Because $\beta_N = B_N - A_N$, for $n = 1$ we have

$$\begin{aligned}
P_N A_N + Q_N B_N &= \frac{B_N - \big(1 - a\big(1 - \frac{\lambda}{N}\big)\big)\frac{\lambda}{N}}{\beta_N} A_N + \frac{\big(1 - a\big(1 - \frac{\lambda}{N}\big)\big)\frac{\lambda}{N} - A_N}{\beta_N} B_N \\
&= \frac{\lambda}{N}\left(1 - a\left(1 - \frac{\lambda}{N}\right)\right)\frac{B_N - A_N}{\beta_N} \\
&= (1-a)\frac{\lambda}{N} + O\left(\frac{1}{N^2}\right).
\end{aligned}$$

For $n = 2$ one easily computes

$$\begin{aligned}
P_N A_N^2 + Q_N B_N^2 &= \frac{B_N - \big(1 - a\big(1 - \frac{\lambda}{N}\big)\big)\frac{\lambda}{N}}{\beta_N} A_N^2 + \frac{\big(1 - a\big(1 - \frac{\lambda}{N}\big)\big)\frac{\lambda}{N} - A_N}{\beta_N} B_N^2 \\
&= \frac{\lambda}{N}\left(1 - a\left(1 - \frac{\lambda}{N}\right)\right)\frac{B_N^2 - A_N^2}{\beta_N} - \frac{A_N B_N (B_N - A_N)}{\beta_N} \\
&= \frac{\lambda}{N}\left(1 - a\left(1 - \frac{\lambda}{N}\right)\right)(B_N + A_N) - A_N B_N \\
&= \left((1-a)\frac{\lambda}{N} + O\left(\frac{1}{N^2}\right)\right)(B_N + A_N) - A_N B_N.
\end{aligned}$$

Thus by the equation (6.1) and Lemma 3 we have

$$\begin{aligned}
P_N A_N^2 + Q_N B_N^2 &= \left((1-a)\frac{\lambda}{N} + O\left(\frac{1}{N^2}\right)\right)\left(1 + O\left(\frac{1}{N}\right)\right) + a\frac{\lambda}{N} + O\left(\frac{1}{N^2}\right) \\
&= \frac{\lambda}{N} + O\left(\frac{1}{N^2}\right).
\end{aligned}$$

We may write for $n \geq 3$

$$\begin{aligned}
P_N A_N^n + Q_N B_N^n &= \frac{B_N - \big(1 - a\big(1 - \frac{\lambda}{N}\big)\big)\frac{\lambda}{N}}{\beta_N} A_N^n + \frac{\big(1 - a\big(1 - \frac{\lambda}{N}\big)\big)\frac{\lambda}{N} - A_N}{\beta_N} B_N^n \\
&= \frac{\lambda}{N}\left(1 - a\left(1 - \frac{\lambda}{N}\right)\right)\frac{B_N^n - A_N^n}{\beta_N} - \frac{A_N B_N (B_N^{n-1} - A_N^{n-1})}{\beta_N} \\
&= \frac{\lambda}{N}\left(1 - a\left(1 - \frac{\lambda}{N}\right)\right)\frac{B_N^n - A_N^n}{B_N - A_N} - \frac{A_N B_N (B_N^{n-1} - A_N^{n-1})}{B_N - A_N} \\
&= \left((1-a)\frac{\lambda}{N} + O\left(\frac{1}{N^2}\right)\right)(B_N^{n-1} + B_N^{n-2}A_N + \ldots + A_N^{n-1}) \\
&\quad - A_N B_N (B_N^{n-2} + B_N^{n-3}A_N + \ldots + B_N A_N^{n-3} + A_N^{n-2}).
\end{aligned}$$

Because

$$\begin{aligned}
&B_N^k + B_N^{k-1}A_N + \ldots + B_N A_N^{k-1} + A_N^k \\
&= B_N^k + A_N^k + A_N B_N (B_N^{k-2} + B_N^{k-3}A_N + \ldots + A_N^{k-2}) \\
&= B_N^k + A_N^k + O\left(\frac{1}{N}\right)(B_N^{k-2} + B_N^{k-3}A_N + \ldots + A_N^{k-2}) \\
&= B_N^k + A_N^k + O\left(\frac{1}{N^2}\right)
\end{aligned}$$

and by (6.1) and Lemma 3 we have

$$\begin{aligned}P_N A_N^n + Q_N B_N^n &= \left((1-a)\frac{\lambda}{N} + O\left(\frac{1}{N^2}\right)\right)(B_N^{n-1} + A_N^{n-1}) \\ &\quad + a\frac{\lambda}{N}(B_N^{n-2} + A_N^{n-2}) + O\left(\frac{1}{N^2}\right) \\ &= \frac{\lambda}{N} + O\left(\frac{1}{N^2}\right). \quad \blacksquare\end{aligned}$$

THEOREM 5. *The Fourier transform of the V_a-transformation of the limiting measure is given by the formula*

$$\mathcal{F}^a[p_\lambda](x) = \mathcal{F}[V_a(p_\lambda)](x) = \exp\left(\lambda(e^{ix} - 1 - ixa)\right) = e^{\lambda(e^{ix}-1)}e^{-\lambda ixa}.$$

Proof. It follows from the above lemmas that

$$\begin{aligned}\mathcal{F}^a[\mu_N](x) &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}(P_N A_N^n + Q_N B_N^n) \\ &= 1 + ix(P_N A_N + Q_N B_N) + \sum_{n=2}^{\infty} \frac{(ix)^n}{n!}(P_N A_N^n + Q_N B_N^n) \\ &= 1 + ix\left((1-a)\frac{\lambda}{N} + O\left(\frac{1}{N^2}\right)\right) + \sum_{n=2}^{\infty} \frac{(ix)^n}{n!}\left(\frac{\lambda}{N} + O\left(\frac{1}{N^2}\right)\right) \\ &= 1 + ix(1-a)\frac{\lambda}{N} + \frac{\lambda}{N}\sum_{n=2}^{\infty} \frac{(ix)^n}{n!} + O\left(\frac{1}{N^2}\right)\end{aligned}$$

Hence

$$\begin{aligned}\lim_{N\to\infty} \mathcal{F}^a[\mu^N](x) &= \lim_{N\to\infty} (\mathcal{F}^a[\mu_N](x))^N \\ &= \lim_{N\to\infty} \left(1 + ix(1-a)\frac{\lambda}{N} + \frac{\lambda}{N}\sum_{n=2}^{\infty} \frac{(ix)^n}{n!} + O\left(\frac{1}{N^2}\right)\right)^N \\ &= \lim_{N\to\infty} \left(1 + ix(1-a)\frac{\lambda}{N} + \frac{\lambda}{N}\sum_{n=2}^{\infty} \frac{(ix)^n}{n!}\right)^N \\ &= \lim_{N\to\infty} \left(1 + ix(1-a)\frac{\lambda}{N} + \frac{\lambda}{N}(e^{ix} - 1 - ix)\right)^N \\ &= \lim_{N\to\infty} \left(1 + (e^{ix} - ixa - 1)\frac{\lambda}{N}\right)^N \\ &= \exp \lambda(e^{ix} - 1 - ixa) \\ &= e^{\lambda(e^{ix}-1)}e^{-\lambda ixa}. \quad \blacksquare\end{aligned}$$

COROLLARY 1. *The V_a-transformation of the Poisson measure p_λ for the V_a-deformed classical convolution is given by*

$$\mathcal{F}^a[p_\lambda](x) = \mathcal{F}[V_a p_\lambda](x) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta_{k-\lambda a}.$$

We would like to find the orthogonal polynomials for the probability measure p_λ. Recall the Charlier polynomials, which belong to the classical Poisson measure

$$e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta_k.$$

The Charlier monic polynomials satisfy the recurrence relation

$$\begin{aligned} &P_{-1}(x,\lambda) = 0, \quad P_0(x,\lambda) = 1, \\ &(x-\lambda-n)P_n(x,\lambda) = P_{n+1}(x,\lambda) + n\lambda P_{n-1}(x,\lambda), \quad \text{for } n \geq 0. \end{aligned}$$

Because the measure $V_a(p_\lambda)$ can be obtained as the right shift by $-a\lambda$ of the classical Poisson measure of parameter λ, so the monic orthogonal polynomials $\{\widetilde{P}_n(x)\}$ for the measure $V_a(p_\lambda)$ are given by

$$\widetilde{P}_n(x) = P_n(x+a\lambda, \lambda),$$

and have the following recurrence relation

$$\begin{aligned} &\widetilde{P}_0(x) = 1, \quad \widetilde{P}_1(x) = x - \lambda, \\ &\widetilde{P}_{n+1}(x) = (x + a\lambda - n)\widetilde{P}_n(x) - n\lambda \widetilde{P}_{n-1}(x), \quad \text{for } n \geq 1. \end{aligned}$$

That means that the Jacobi parameters for $n \geq 1$ are given by

$$\alpha_n = -a\lambda + n - 1, \qquad \lambda_n = n\lambda.$$

Hence we can obtain the Cauchy transform of the probability measure $V_a p_\lambda$ in the continued fraction (Stieltjes expansion) form

$$G_{V_a p_\lambda}(z) = \cfrac{1}{z + a\lambda - \cfrac{\lambda}{z + a\lambda - 1 - \cfrac{2\lambda}{z + a\lambda - 2 - \cfrac{3\lambda}{z + a\lambda - 3 - \cfrac{4\lambda}{\ddots}}}}}.$$

Thus by definition the Poisson measure for V_a-transformation of a classical convolution is equal to

$$G_{p_\lambda}(z) = \cfrac{1}{z - \cfrac{\lambda}{z + a\lambda - 1 - \cfrac{2\lambda}{z + a\lambda - 2 - \cfrac{3\lambda}{z + a\lambda - 3 - \cfrac{4\lambda}{\ddots}}}}}.$$

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