PERSPECTIVES IN OPERATOR THEORY
BANACH CENTER PUBLICATIONS, VOLUME 75
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2007

SOME SIMPLE PROOFS IN HOLOMORPHIC SPECTRAL THEORY

GRAHAM R. ALLAN

Department of Pure Mathematics and Mathematical Statistics
Centre for Mathematical Sciences, University of Cambridge
Wilberforce Road, Cambridge CB3 0WB, U.K.
E-mail: g.r.allan@dpmms.cam.ac.uk

Abstract. This paper gives some very elementary proofs of results of Aupetit, Ransford and others on the variation of the spectral radius of a holomorphic family of elements in a Banach algebra. There is also some brief discussion of a notorious unsolved problem in automatic continuity theory.

1. Introduction. Let A be a complex, unital Banach algebra, let U be an open subset of $\mathbb{C}$ and let $f : U \to A$ be an A-valued holomorphic function. There is a considerable literature concerned with the way in which the spectrum $\mathrm{Sp}(f(z))$ (and, in particular, the spectral radius $\rho(f(z))$) depends on z (see, e.g. [2], [3], [7], [8]).

In this note, we will give elementary proofs of various results on the variation of the spectral radius of $f(z)$. In all that follows, A will be a complex, unital Banach algebra (which is not necessarily commutative). To avoid confusion with other conventions, note that, for every $a \in A$, we *define* its *spectral radius* $\rho(a) \equiv \rho_A(a)$ to be

$$\rho(a) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}\, a\},$$

and it is recalled that

$$\rho(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_{n\geq 1} \|a^n\|^{1/n}.$$

From this last 'inf' formula, it is immediate that the function $\rho : A \to \mathbb{R}^+$ is *upper-semicontinuous*; it is very important to realize that ρ is not in general continuous (see e.g. [2], Chap. 1, §5).

It is also worth remarking that if, following [4] (Chapter 1, §4, Corollary 2), we define $\mathrm{Eun}(A)$ to be the set of all unital, submultiplicative norms on A that are equivalent to

---

2000 *Mathematics Subject Classification*: Primary 46H40; Secondary 31A05, 46H10.

The paper is in final form and no version of it will be published elsewhere.

the initial norm $\|.\|$, then it is simple to show that

$$\rho(a) = \inf\{p(a) : p \in \mathrm{Eun}(A)\}.$$

(This remark gives another proof that the spectral radius is upper-semicontinuous.)

For holomorphic $f : U \to A$, we will generally write $\rho f : U \to \mathbb{R}^+$ for the function $z \mapsto \rho(f(z))$ $(z \in U)$. In [8], Vesentini proved that ρf is *subharmonic* on U. This property was famously exploited by Aupetit [1] in his remarkable proof of Johnson's uniqueness-of-norm theorem. For later reference it is recalled that the Johnson theorem may be stated as follows:

THEOREM (B. E. Johnson). *Let A, B be Banach algebras, with B semisimple, and let $T : A \to B$ be an algebra homomrphism such that $T(A) = B$. Then T is continuous.*

In [6], Ransford gave a still more elementary proof of the Johnson theorem, in which subharmonic functions do not appear explicitly. The aim of this note is to show that a good deal of the holomorphic theory of the spectral radius may be obtained, in a very elementary way, by using a device similar to that of Ransford.

The essential step in Ransford's proof was provided by the deduction of a special case of the classical Hadamard three-circles theorem, as follows:

LEMMA 1 (Ransford). *Let A be a unital Banach algebra, let $R > 1$ and let f be a holomorphic A-valued function on a neighbourhood of the annulus $\{z \in \mathbb{C} : R^{-1} \le |z| \le R\}$. Then*

$$(\rho f)(1)^2 \le \sup_{|z|=R^{-1}} (\rho f)(z) \sup_{|z|=R} (\rho f)(z).$$

REMARK. Ransford stated his lemma for polynomials with coefficients from A; the form of the lemma for holomorphic A-valued f is no harder to prove, and fits better with what follows.

The present note is greatly influenced by Ransford's proof of Lemma 1; but it is useful to give a slightly more general version of the result (Theorem 4).

2. A maximum principle and its consequences. We have recalled that, for every $a \in A$, $\rho(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$. It is a very useful triviality that, if we let n run through powers of 2, say let $g_k(a) = \|a^{2^k}\|^{2^{-k}}$, then g_k is a *monotone decreasing* sequence with, of course, $g_k(a) \to \rho(a)$ as $k \to \infty$. This fact will be combined with the following simple variant of Dini's theorem:

LEMMA 2 (Dini lemma). *Let T be a compact Hausdorff space, and let $(h_n)_{n\ge1}$ be a monotone decreasing sequence of continuous functions $h_n : T \to \mathbb{R}^+$; define $h(t) = \lim_{n\to\infty} h_n(t)$ $(t \in T)$. Then $\sup_T h_n \to \sup_T h$ as $t \to \infty$.*

Note that, in this lemma, h is *not* required to be continuous and, indeed the convergence of h_n to h need not be uniform. (This point is important since, otherwise, we would have a false deduction of continuity of the spectral radius.)

In fact Lemma 2 had also been proved by Zemánek in his doctoral thesis [9], as a step in the proof of his well known characterization of the radical of a Banach algebra. But, in the published version in [10], the lemma does not appear, since a different method of proof was used. The use of the Dini lemma in [9] was essentially the same as in the proof of the following Lemma 3. We remark that (very elementary) classical function theory enters the present account entirely via that lemma. As throughout the paper, A is a complex, unital Banach algebra.

LEMMA 3 (Weak maximum principle for the spectral radius). *Let open $U \subseteq \mathbb{C}$ and let $f : U \to A$ be holomorphic. Then for every compact $K \subset U$ and every $z_0 \in K$,*

$$(\rho f)(z_0) \leq \sup_{z \in \partial K} (\rho f)(z).$$

Proof. For every holomorphic $g : U \to A$, we may use the Hahn-Banach theorem to find $\chi \in A^*$ with $\|\chi\| = 1$ and $\chi(g(z_0)) = \|g(z_0)\|$.

Then $\chi \circ g : U \to \mathbb{C}$ is holomorphic so that, by the classical maximum principle,

$$\|g(z_0)\| = |(\chi g)(z_0)| \leq \sup_{z \in \partial K} |(\chi g)(z)| \leq \sup_{z \in \partial K} \|g(z)\|.$$

Apply the outer inequality to $g = f^{2^k}$ $(k = 1, 2, \dots)$, take 2^kth roots and then let $k \to \infty$. Using the Dini lemma we deduce that

$$(\rho f)(z_0) \leq \sup_{z \in \partial K} (\rho f)(z).$$

The key step for all the other results in this paper, is the following form of three-circles theorem (Lemma 1 being essentially the case $|z|^2 = R_1 R_2$). The proof adapts one of the standard proofs of the Hadamard three-circles theorem.

THEOREM 4 (Spectral three-circles theorem). *Let $0 < R_1 < R_2$, let U be an open neighbourhood of the annulus $\Delta(R_1, R_2) \equiv \{z \in \mathbb{C} : R_1 \leq |z| \leq R_2\}$ and let $f : U \to A$ be holomorphic. For $j = 1, 2$ let $M_j = \sup_{|z|=R_j} (\rho f)(z)$. Then, for every z with $R_1 < |z| < R_2$,*

$$(\rho f)(z) \leq M_1^t M_2^{1-t},$$

where $t \in (0, 1)$ is the unique number with $|z| = R_1^t R_2^{1-t}$.

Proof. Write $r = |z|$. For integers p, q with $q \geq 1$, we apply Lemma 3 to the function $|z|^p \rho(f(z))^q = \rho(z^p f(z)^q)$ on $\Delta(R_1, R_2)$ and then take qth roots, obtaining

$$r^{p/q} (\rho f)(z) \leq \max(R_1^{p/q} M_1, R_2^{p/q} M_2). \tag{$*$}$$

Let α be the real number such that $R_1^\alpha M_1 = R_2^\alpha M_2$, and then apply $(*)$ to a sequence of integer pairs (p_n, q_n) with $p_n / q_n \to \alpha$. Then we deduce

$$r^\alpha (\rho f)(z) \leq R_1^\alpha M_1 = R_2^\alpha M_2. \tag{$**$}$$

But $\log r = t \log R_1 + (1 - t) \log R_2$ and $\alpha \log R_1 + \log M_1 = \alpha \log R_2 + \log M_2$, so we easily deduce from $(**)$ that $\log(\rho f)(z) \leq t \log M_1 + (1 - t) \log M_2$, which is equivalent to the stated result.

The first application is a Liouville theorem: we write the proof to show that, for this result, Lemma 1 is sufficient.

COROLLARY 5. *Let A be a complex unital Banach algebra, and let $f : \mathbb{C} \to A$ be an A-valued entire function. Suppose that ρf is bounded on $\mathbb{C}$; then ρf is constant.*

Proof. Let $M = \sup_{z\in\mathbb{C}}(\rho f)(z)$; we may clearly suppose that $M > 0$. Suppose, towards a contradiction, that ρf is not constant; then $m \equiv \inf_{z\in\mathbb{C}}(\rho f)(z) < M$ and we may choose $\epsilon > 0$ with $m+\epsilon < M$. Choose $z_0 \in \mathbb{C}$ with $(\rho f)(z_0) < m+\epsilon$; by considering $f(z+z_0)$ if necessary, we may assume that $z_0 = 0$. By the upper-semicontinuity of ρf at 0, there is some $\delta > 0$ such that $(\rho f)(z) < m+\epsilon$ whenever $|z| \le \delta$.

Let $0 \ne a \in \mathbb{C}$ and choose $R > 1$ for which $|a|/R < \min(|a|,\delta) \le |a| < R/|a|$. Then, by applying Theorem 4 (or just Lemma 1) to the annulus centered at 0, with radii $|a|/R$ and $R/|a|$ we have

$$(\rho f)(a)^2 \le \sup_{|z|=|a|/R}(\rho f)(z) \sup_{|z|=R/|a|}(\rho f)(z) \le (m+\epsilon)M.$$

Thus $(m+\epsilon)M$ is an upper bound for $(\rho f)^2$ on $\mathbb{C}$ (trivially at $a = 0$), so that $m+\epsilon > M$, which is a contradiction. This proves the theorem.

We next show that the three-circles theorem may be used to give a stronger maximum principle than that in Lemma 3.

THEOREM 6 (Maximum principle for the spectral radius). *Let D be a connected open subset of $\mathbb{C}$ and let $f : D \to A$ be holomorphic. Suppose that there is some $z_0 \in D$ with $(\rho f)(z) \le (\rho f)(z_0)$ for all $z \in D$. Then ρf is constant on D.*

Proof. We first show that ρf is constant on a neighbourhood of z_0.

Let $R > 0$ be such that $\{z : |z-z_0| \le R\} \subset D$. Let $m = \inf\{(\rho f)(z) : |z-z_0| < R/2\}$, let $\epsilon > 0$ and choose z_1 with $|z_1-z_0| < R/2$ and such that $(\rho f)(z_1) < m+\epsilon$. Then use the upper-semicontinuity of ρ to choose δ, with $0 < \delta < |z_1 - z_0|$, such that $(\rho f)(z) < m+\epsilon$ for all z with $|z - z_1| \le \delta$.

Write $M = (\rho f)(z_0)$, observe that $(\rho f)(z) \le M$ for all z with $|z - z_1| \le R/2$ and apply Theorem 4 to an annulus centered at z_1 with radii δ and $R/2$. Then, for a certain $t \in (0, 1)$,

$$M = (\rho f)(z_0) \le (m+\epsilon)^t M^{1-t},$$

whence $M \le m+\epsilon$ for all $\epsilon > 0$. So $m = M$ and thus $(\rho f)(z)$ is constant for $|z-z_0| < R/2$.

Now define

$$E = \{z \in D : (\rho f)(z) = (\rho f)(z_0)\} = \{z \in D : (\rho f)(z) \ge (\rho f)(z_0)\}.$$

By the upper-semicontinuity of ρ, E is relatively closed in D and, by the argument of the last paragraph, E is open. Since $z_0 \in E$, then $E \ne \varnothing$ and so, by the connectedness of D, $E = D$ and the theorem is proved.

We now again use the three-circles theorem, this time to prove a lemma of Aupetit ([3], Theorem 5.5.1) which is a key step in his proof of Johnson's theorem.

Recall that, if X, Y are Banach spaces and if $T : X \to Y$ is a linear mapping (not necessarily continuous), then the *separating subspace*, $\Sigma(T)$, of T is the set of all $y \in Y$ such that $Tx_n \to y$ for some sequence (x_n) in X for which $x_n \to 0$ as $n \to \infty$. It is elementary that $\Sigma(T)$ is a closed subspace of Y, and that T is continuous if and only if $\Sigma(T) = \{0\}$ (which is a formulation of the closed graph theorem).

If A, B are unital Banach algebras and $T : A \to B$ is a unital homomorphism with $\overline{T(A)} = B$, then $\Sigma(T)$ is a proper closed ideal of B.

Remark also that, if $T : A \to B$ is a unital homomorphism of Banach algebras and if $a \in A$, then certainly $\mathrm{Sp}_B(Ta) \subseteq \mathrm{Sp}_A(a)$, so that $\rho_B(Ta) \leq \rho_A(a)$. Following Aupetit, we give the following lemma for any linear mapping $T : A \to B$ for which $\rho_B(Ta) \leq \rho_A(a)$ $(a \in A)$; by the remark just made, this includes the case in which T is a homomorphism.

Theorem 7 (Aupetit). *Let A, B be unital Banach algebras and let $T : A \to B$ be a linear mapping such that $\rho_B(Ta) \leq \rho_A(a)$ for every $a \in A$. Then, for every $a \in A$ and every $b \in \Sigma(T)$,*

$$\rho_B(Ta) \leq \rho_B(T(a) + b).$$

Remark. It appears that the more special three-circles lemma of Ransford (Lemma 1) leads to the weaker inequality: $\rho_B(Ta)^2 \leq \rho_A(a)\rho_B(Ta + b)$; but it should be remarked that this inequality is still sufficient for the deduction of Corollary 8. (That is the version of the result given in [5], Theorem 5.1.9.)

Proof of Theorem 7. Let $0 < r < 1 < R$ and let $f : \mathbb{C} \to B$ be holomorphic. By Theorem 4,

$$(\rho_B f)(1) \leq \sup_{|z|=r} (\rho_B f)(z)^t \sup_{|z|=R} (\rho_B f)(z)^{1-t}, \tag{$*$}$$

where $t = \log R/(\log R - \log r)$.

Let $b \in \Sigma(T)$, let $a \in A$ and let $\epsilon > 0$. By an elementary result (referred to in §1) we may, by choice of equivalent norm on B, suppose that $\|Ta + b\| < \rho_B(Ta + b) + \epsilon$. Choose a sequence (a_n) in A with $a_n \to 0$ in A and $Ta_n \to b$ in B.

For each $n = 1, 2, \ldots$, apply $(*)$ to $f_n(z) = T(a + a_n) - zTa_n$. Then, since $\rho_B(Tx) \leq \rho_A(x)$ for all $x \in A$,

$$\begin{aligned}\rho_B(Ta) = (\rho_B f_n)(1) &\leq \sup_{|z|=r} \rho_B(T(a + a_n) - zTa_n)^t \sup_{|z|=R} \rho_A(a + a_n - za_n)^{1-t}\\ &\leq (\|T(a + a_n)\| + r\|Ta_n\|)^t(\|a + a_n\| + R\|a_n\|)^{1-t}.\end{aligned}$$

First let $n \to \infty$ and deduce that

$$\rho_B(Ta) \leq (\|Ta + b\| + r\|b\|)^t\|a\|^{1-t}.$$

Now, for each fixed $r > 0$, let $R \to \infty$ and note that then $t = \log R/(\log R - \log r) \to 1$, so that

$$\rho_B(Ta) \leq \|Ta + b\| + r\|b\|,$$

for every $r > 0$. Thus $\rho_B(Ta) \leq \|Ta + b\| < \rho_B(Ta + b) + \epsilon$. This holds for every $\epsilon > 0$ so that the result follows.

Corollary 8 (Generalized Johnson theorem). *Let A, B be Banach algebras and let $T : A \to B$ be a linear mapping with $\rho_B(Ta) \leq \rho_A(a)$ for all $a \in A$. If $T(A) = B$ then $\Sigma(T) \subseteq J(B)$ (the Jacobson radical of B); in particular, if B is semisimple then T is continuous.*

Proof. Let $b \in \Sigma(T)$ and let c be any quasi-nilpotent element of B. Since $T(A) = B$, there is an $a \in A$ with $Ta = c - b$. From Theorem 7, $\rho_B(c - b) \leq \rho_B(c) = 0$. By a well known result of Zemánek [10], $b \in J(B)$.

Corollary 9. *Let A, B be unital Banach algebras and let $T : A \to B$ be a linear mapping such that $\rho_B(Ta) \leq \rho_A(a)$ for every $a \in A$. Let $b \in \Sigma(T)$; then for every sequence (b_n)*

in $T(A)$ such that $b_n \to b$ (or even such that, merely, $\rho_B(b_n - b) \to 0$), it follows that $\rho_B(b_n) \to 0$.

Finally, we recall the notorious unsolved problem in this area:

Let A, B be Banach algebras, with B semisimple, and let $T : A \to B$ be a homomorphism with $\overline{T(A)} = B$. Is it true that T is necessarily continuous?

REMARKS

1. It is equivalent to drop the requirement that B be semisimple, and then to ask whether $\Sigma(T) \subseteq J(B)$.

2. An equivalent formulation is to ask whether $\mathrm{Sp}(b) = \{0\}$ for every $b \in \Sigma(T)$. It is known (and quite easy to prove) that $\mathrm{Sp}(b)$ is connected, and, of course, $0 \in \mathrm{Sp}(b)$.

3. We may divide the problem into two parts:

(i) show that $\ker T$ is closed;

(ii) with the additional hypothesis that $\ker T$ is closed, prove that T is continuous.

Then each of these two sub-problems appears to be open.

4. If we just assume (with B semisimple) that $T : A \to B$ is linear with $\rho_B(Ta) \le \rho_A(a)$ for every $a \in A$ and that $\overline{T(A)} = B$, then it is *not* the case that T need be continuous (see e.g. [5], following Theorem 5.1.9; it is easy to see that, in this example, $\ker T$ is not even closed). But it appears that the question of whether every element of $\Sigma(T)$ need be quasi-nilpotent may still be open in this more general case.

We conclude with a simple result that, just possibly, might have relevance to the unsolved problem under discussion (see Remark 2 above).

COROLLARY 10. *Let A, B be Banach algebras and let $T : A \to B$ be a linear mapping with $\rho_B(Ta) \le \rho_A(a)$ for all $a \in A$ and $\overline{T(A)} = B$. Let b be an element of the separating subspace $\Sigma(T)$. Suppose that, for some open neighbourhood D of 0 in $\mathbb{C}$, there is a holomorphic function $f : D \to B$ with $f(0) = b$ and such that $f(z) \in T(A)$ for all $z \in U \setminus \{0\}$. Then b is quasi-nilpotent.*

Proof. Since $f(z) \to f(0) = b$ as $z \to 0$ in D, it follows from Corollary 9 that for every $\epsilon > 0$ there exists $\delta > 0$ such that $\rho_B(f(z)) < \epsilon$ for all $z \in U$ with $0 < |z| \le \delta$. It then follows from the weak maximum principle (Lemma 3) that $\rho_B(b) = 0$.

References

[1] B. Aupetit, *The uniqueness of the complete norm topology in Banach algebras and Banach Jordan algebras*, J. Functional Analysis 47 (1982), 1–6.

[2] B. Aupetit, *Propriétés Spectrales des Algèbres de Banach*, Lecture Notes in Math. 735, Springer-Verlag, Berlin, 1979.

[3] B. Aupetit, *A Primer on Spectral Theory*, Springer-Verlag, Berlin, 1991.

[4] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, Berlin, 1973.

[5] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Mathematical Society Monographs 24, Clarendon Press, Oxford, 2000.

[6] T. J. Ransford, *A short proof of Johnson's uniqueness-of-norm theorem*, Bull. London Math. Soc. 21 (1989), 487–488.

[7] Z. Słodkowski, *On subharmonicity of the capacity of the spectrum*, Proc. Amer. Math. Soc. 81 (1981), 243–249.

[8] E. Vesentini, *On the subharmonicity of the spectral radius*, Boll. Un. Mat. Ital. 4 (1968), 427–249.

[9] J. Zemánek, *Properties of the spectral radius in Banach algebras*, Rozprawa doktorska, Warszawa, 1977.

[10] J. Zemánek, *A note on the radical of a Banach algebra*, Manuscripta Math. 20 (1977), 191–196.