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# FREDHOLM DETERMINANTS AND THE EVANS FUNCTION FOR DIFFERENCE EQUATIONS

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**Abstract.** We develop a difference equations analogue of recent results by F. Gesztesy, K. A. Makarov, and the second author relating the Evans function and Fredholm determinants of operators with semi-separable kernels.

**1. Introduction.** The purpose of this paper is to provide a difference equations version of some of the most recent results in [GM, GML, GML1] relating the Evans function and Fredholm determinants of operators with semi-separable kernels. Although our general strategy is close to that in [GM, GML] for the differential equations case, and for simplicity we consider less general assumptions than in [GML], the arguments and the results in the difference equations setting have some important differences. For a related work we cite [GGK, GKvS] and [BCK, KK]. For a detailed historical account and the bibliography we refer to [GML].

We consider an unperturbed difference equation $x_{j+1} = A_j x_j$ and its perturbation in the form $x_{j+1} = A_j^{\times} x_j$, $j \in \mathbb{Z}$, where $A_j^{\times} = A_j + B_j C_j$. Here and below, $A_j, B_j$, and $C_j$ are $(\mathrm{d} \times \mathrm{d})$ matrices with complex entries, and $\mathbf{x} = (x_j)_{j\in\mathbb{Z}}$ is a sequence of vectors $x_j \in \mathbb{C}^{\mathrm{d}}$. Throughout, we assume that the matrices $A_j$ and $A_j^{\times}$ are invertible and that the unperturbed equation has an exponential dichotomy over $\mathbb{Z}$ with the (unstable) dichotomy projection $P$. Let $\mathbf{U} = (U_j)_{j\in\mathbb{Z}}$ denote the fundamental matrix solution of the unperturbed equation normalized by $U_0 = I$, the identity matrix.

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In the first part of the paper, following [GM], we give formulas for the (modified) Fredholm determinants of the difference operator $\mathcal{T} = (T_{jk})_{j,k\in\mathbb{Z}}$ on $\ell^2(\mathbb{Z};\mathbb{C}^{\rm d})$ whose kernel is given by the formulas

$$T_{jk} = C_jU_j(I-P)U_{k+1}^{-1}B_k \text{ for } j>k \text{ and } T_{jk} = -C_jU_jPU_{k+1}^{-1}B_k \text{ for } j\le k. \tag{1.1}$$

Note that the kernel of every difference operator with a semi-separable kernel admits a representation (1.1), see formulas (3.9) - (3.10) below. The choice of kernel (1.1) is related to the following elementary observation. Given the matrix sequence $\mathbf{A} = (A_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z};\mathbb{C}^{\rm d\times d})$, define on $\ell^2(\mathbb{Z};\mathbb{C}^{\rm d})$ an operator, $G_{\mathbf{A}}$, by $(G_{\mathbf{A}}\mathbf{x})_j = x_{j+1} - A_jx_j$ so that the inhomogeneous equations $x_{j+1} = A_jx_j + y_j, j\in\mathbb{Z}$, becomes $G_{\mathbf{A}}\mathbf{x} = \mathbf{y}$. Due to the exponential dichotomy [CL], the operator $G_{\mathbf{A}}$ is invertible in $\ell^2(\mathbb{Z},\mathbb{C}^{\rm d})$, and by a direct computation, its inverse is a difference operator, $\mathcal{K} = (K_{jk})_{j,k\in\mathbb{Z}}$, with kernel defined by

$$K_{jk} = U_j(I-P)U_{k+1}^{-1} \text{ for } j>k \text{ and } K_{jk} = -U_jPU_{k+1}^{-1} \text{ for } j\le k. \tag{1.2}$$

If $\mathbf{A}^\times = (A_j^\times)_{j\in\mathbb{Z}}$ then the operator $G_{\mathbf{A}^\times} = G_{\mathbf{A}} - \operatorname{diag}(B_jC_j)_{j\in\mathbb{Z}}$ can be represented as $G_{\mathbf{A}^\times} = G_{\mathbf{A}}(I - \mathcal{K}\operatorname{diag}(B_j)_{j\in\mathbb{Z}}\operatorname{diag}(C_j)_{j\in\mathbb{Z}})$, and $G_{\mathbf{A}^\times}$ is invertible if and only if the operator $I-\mathcal{T}$ is invertible; the kernel of $\mathcal{T} = \operatorname{diag}(C_j)_{j\in\mathbb{Z}}\mathcal{K}\operatorname{diag}(B_j)_{j\in\mathbb{Z}}$ is given by (1.1).

In the second part of the paper, following [GML, GML1], we construct appropriate matrix solutions of the perturbed difference equation whose determinant, $\mathcal{E}$, is called the *Evans determinant.* If the sequence $\mathbf{A}^\times = \mathbf{A}^\times(z)$ depends on a spectral parameter $z\in\mathbb{C}$, then the corresponding function $\mathcal{E} = \mathcal{E}(z)$ becomes the *Evans function*, a Wronskian type object widely used to detect unstable modes for operators obtained by linearizing nonlinear equations along special particular solutions such as travelling waves, see [AGJ] and recent reviews [JK, S] and the bibliographies therein. We stress that the Evans determinant, as defined in the current paper, is uniquely determined by the sequences $\mathbf{A}^\times$ and $\mathbf{A}$. Moreover (and this is the central result of this paper), we derive a formula relating $\mathcal{E}$ and the Fredholm determinant of $I-\mathcal{T}$ (for results in this spirit in the case of the Schrödinger differential operator see [KS, p. 861] and [KS1]). Finally, for the discrete Schrödinger operator, we show that the Evans function coincides with the Jost function, the classical object familiar from scattering theory, see e.g. [CS, Chap. XVII], [FT, Sec. III.2], [GH, Sec. 6], [T, Chap. 10], and [To, Chap. 3].

**2. Notation and preliminaries.** The set of $(\rm d\times d)$ matrices with complex entries is denoted by $\mathbb{C}^{\rm d\times d}$. Where possible, we abbreviate $\ell^2 = \ell^2(\mathbb{Z};\mathbb{C}^{\rm d})$ or $\ell^2 = \ell^2(\mathbb{Z};\mathbb{C}^{\rm d\times d})$. We use boldface to denote sequences of vectors or matrices, e.g. $\mathbf{x} = (x_j)_{j\in\mathbb{Z}}$, $x_j\in\mathbb{C}^{\rm d}$, or $\mathbf{a} = (a_j)_{j\in\mathbb{Z}}$, $a_j\in\mathbb{C}^{\rm d\times d}$. We denote by $\sigma(\cdot)$ the spectrum of an operator, and by $I$ (or sometimes $I_{\rm d\times d}$) the identity operator. For a projection $P$ on $\mathbb{C}^{\rm d}$ with $\dim\operatorname{Im}P = {\rm d}_1$ we often identify $I_{{\rm d}_1\times{\rm d}_1}$ and $P$ on $\operatorname{Im}P$. The restriction of an operator $A$ on a subspace $(\cdot)$ is denoted by $A|_{(\cdot)}$. If $A$ satisfies $A = AP$ then we denote $\|A\|_\bullet = \inf\{\|Ax\| : x = Px, \|x\| = 1\}$.

The sets of trace-class and Hilbert-Schmidt operators on a Hilbert space $(\cdot)$ are denoted, respectively, by $\mathcal{B}_1 = \mathcal{B}_1(\cdot)$ and $\mathcal{B}_2 = \mathcal{B}_2(\cdot)$. Recall that $\ell^1\subset\ell^2\subset\ell^\infty$ and $\mathcal{B}_1(\cdot)\subset\mathcal{B}_2(\cdot)$. We will use the following properties of the (modified) Fredholm determinants, see, e.g. [GGK, Si] for more information:

$$\det(I-A)=\prod_{\lambda\in\sigma(A)}(1-\lambda),\quad A\in\mathcal{B}_1, \tag{2.1}$$

$$\det{}_2(I-A)=\det\left[(I-A)e^{A}\right]=\prod_{\lambda\in\sigma(A)}(1-\lambda)e^{\lambda},\quad A\in\mathcal{B}_2, \tag{2.2}$$

$$\det{}_2(I-A)=\det(I-A)e^{\mathrm{tr}A},\quad A\in\mathcal{B}_1, \tag{2.3}$$

$$\det{}_2\left[(I-A)(I-B)\right]=\det{}_2(I-A)\det{}_2(I-B)e^{-\mathrm{tr}(AB)},\quad A,B\in\mathcal{B}_2. \tag{2.4}$$

Given matrix sequences $\mathbf{a}_j=(a_j)_{j\in\mathbb{Z}}$, $\mathbf{b}_j=(b_j)_{j\in\mathbb{Z}}$, and $\mathbf{d}_j=(d_j)_{j\in\mathbb{Z}}$, we define the upper triangular operator $V^+_{\mathbf{a},\mathbf{b}}$, the lower triangular operator $V^-_{\mathbf{a},\mathbf{b}}$, the diagonal operator $D_{\mathbf{d}}$, and the operator $V_{\mathbf{a},\mathbf{b}}$ on $\ell^2(\mathbb{Z};\mathbb{C}^{\mathrm{d}})$ as follows:

$$(V^+_{\mathbf{a},\mathbf{b}}\mathbf{x})_j=\sum_{k=j+1}^{\infty}a_jb_kx_k,\quad (V^-_{\mathbf{a},\mathbf{b}}\mathbf{x})_j=\sum_{k=-\infty}^{j-1}a_jb_kx_k, \tag{2.5}$$

$$(D_{\mathbf{d}}\mathbf{x})_j=d_jx_j,\quad (V_{\mathbf{a},\mathbf{b}}\mathbf{x})_j=\sum_{k=-\infty}^{\infty}a_jb_kx_k,\quad j\in\mathbb{Z}. \tag{2.6}$$

We summarize properties of these operators in the following elementary lemmas.

Lemma 2.1. *Assume* $\mathbf{a},\mathbf{b},\mathbf{d}\in\ell^2(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}})$. *Then:*

$$\sigma(D_{\mathbf{d}})=\{0\}\cup(\cup_{j\in\mathbb{Z}}\sigma(d_j)); \tag{2.7}$$

$$V_{\mathbf{a},\mathbf{b}},V^{\pm}_{\mathbf{a},\mathbf{b}},D_{\mathbf{d}}\in\mathcal{B}_2(\ell^2(\mathbb{Z};\mathbb{C}^{\mathrm{d}})); \tag{2.8}$$

$$\sigma(V^+_{\mathbf{a},\mathbf{b}})=\sigma(V^-_{\mathbf{a},\mathbf{b}})=\{0\}; \tag{2.9}$$

$$\det{}_2(I-V^+_{\mathbf{a},\mathbf{b}})=\det{}_2(I-V^-_{\mathbf{a},\mathbf{b}})=1; \tag{2.10}$$

$$\det{}_2(I-D_{\mathbf{d}})=\prod_{j\in\mathbb{Z}}\det(I_{\mathrm{d}\times\mathrm{d}}-d_j)e^{\mathrm{tr}d_j}. \tag{2.11}$$

Lemma 2.2. *Assume* $\mathbf{a},\mathbf{b},\mathbf{d}\in\ell^1(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}})$. *Then:*

$$V^{\pm}_{\mathbf{a},\mathbf{b}}\ \textit{are compact operators on}\ \ell^\infty(\mathbb{Z};\mathbb{C}^{\mathrm{d}})\ \textit{and}\ \sigma(V^{\pm}_{\mathbf{a},\mathbf{b}})=\{0\}; \tag{2.12}$$

$$V_{\mathbf{a},\mathbf{b}},V^{\pm}_{\mathbf{a},\mathbf{b}},D_{\mathbf{d}}\in\mathcal{B}_1(\ell^2(\mathbb{Z};\mathbb{C}^{\mathrm{d}})); \tag{2.13}$$

$$\det(I-V^+_{\mathbf{a},\mathbf{b}})=\det(I-V^-_{\mathbf{a},\mathbf{b}})=1; \tag{2.14}$$

$$\det(I-D_{\mathbf{d}})=\prod_{j\in\mathbb{Z}}\det(I_{\mathrm{d}\times\mathrm{d}}-d_j). \tag{2.15}$$

*Proof.* If $\mathbf{d}\in\ell^2$ then $\|d_j\|\to0$ as $|j|\to\infty$ and then, for any $r>0$, there are only finitely many $d_j$'s having $\lambda\in\sigma(d_j)$ with $|\lambda|\geq r$. The formula $\sigma(D_{\mathbf{d}})=\mathrm{closure}(\cup_{j\in\mathbb{Z}}\sigma(d_j))$ now implies (2.7).

Note that $V_{\mathbf{a},\mathbf{b}}=D_{\mathbf{d}}+V^+_{\mathbf{a},\mathbf{b}}+V^-_{\mathbf{a},\mathbf{b}}$ with $\mathbf{d}=(a_jb_j)_{j\in\mathbb{Z}}$ and $V^-_{\mathbf{a},\mathbf{b}}=(V^+_{\mathbf{b}^*,\mathbf{a}^*})^*$ with $\mathbf{a}^*=(a_j^*)_{j\in\mathbb{Z}}$ and $\mathbf{b}^*=(b_j^*)_{j\in\mathbb{Z}}$. Since $\mathbf{a}\in\ell^\infty$ and thus $(a_jb_j)_{j\in\mathbb{Z}}$ is in $\ell^2$, resp. $\ell^1$, it is enough to prove (2.9) and (2.8), resp. (2.13), only for $V^+_{\mathbf{a},\mathbf{b}}$ and $D_{\mathbf{d}}$. Using (2.7), we infer:

$$\|D_{\mathbf{d}}\|_{\mathcal{B}_1(\ell^2)}=\mathrm{tr}[(D^*_{\mathbf{d}}D_{\mathbf{d}})^{\frac12}]=\mathrm{tr}[D_{(\mathbf{d}^*\mathbf{d})^{\frac12}}]=\textstyle\sum_{j\in\mathbb{Z}}\mathrm{tr}(d_j^*d_j)^{\frac12}\leq\mathrm{d}\sum_{j\in\mathbb{Z}}\|d_j\|=\mathrm{d}\|\mathbf{d}\|_{\ell^1},$$

$$\|D_{\mathbf{d}}\|_{\mathcal{B}_2(\ell^2)}=\mathrm{tr}(D_{\mathbf{d}^*\mathbf{d}})=\sum_{j\in\mathbb{Z}}\mathrm{tr}(d_j^*d_j)\leq\mathrm{d}\sum_{j\in\mathbb{Z}}\|d_j\|^2=\mathrm{d}\|\mathbf{d}\|_{\ell^2}. \tag{2.16}$$

Assuming $\mathbf{a},\mathbf{b}\in\ell^1$ and writing $V^+_{\mathbf{a},\mathbf{b}}=\sum_{k=1}^\infty \mathcal{S}^{-k}D_{(a_{j-k}b_j)_{j\in\mathbb{Z}}}$, where $(\mathcal{S}\mathbf{x})_j=x_{j-1}$ is the shift operator, we have:

$$\begin{aligned}\|V^+_{\mathbf{a},\mathbf{b}}\|_{\mathcal{B}_1(\ell^2)}&\le\sum_{k=1}^\infty\|D_{(a_{j-k}b_j)_{j\in\mathbb{Z}}}\|_{\mathcal{B}_1(\ell^2)}=\sum_{k=1}^\infty\sum_{j\in\mathbb{Z}}\|a_{j-k}b_j\|_{\mathcal{B}_1(\mathbb{C}^{\mathrm{d}})}\\&\le \mathrm{d}\sum_{j\in\mathbb{Z}}\sum_{k=1}^\infty\|b_j\|\|a_{j-k}\|\le \mathrm{d}\|\mathbf{a}\|_{\ell^1}\|\mathbf{b}\|_{\ell^1}.\end{aligned}$$

Assuming $\mathbf{a},\mathbf{b}\in\ell^2$ and considering the basis $\mathbf{y}_{n,i}=(y_{n,i}(j))_{j\in\mathbb{Z}}$, $i=1,\dots,\mathrm{d}$, $n\in\mathbb{Z}$, in $\ell^2(\mathbb{Z};\mathbb{C}^{\mathrm{d}})$ given by $y_{n,i}(j)=0$ for $j\ne n$ and $y_{n,i}(n)=\mathbf{e}_i$, the standard ort in $\mathbb{C}^{\mathrm{d}}$, we have:

$$\begin{aligned}\|V^+_{\mathbf{a},\mathbf{b}}\|^2_{\mathcal{B}_2(\ell^2)}&=\operatorname{tr}[(V^+_{\mathbf{a},\mathbf{b}})^*V^+_{\mathbf{a},\mathbf{b}}]=\sum_{n\in\mathbb{Z}}\sum_{i=1}^{\mathrm{d}}\|V^+_{\mathbf{a},\mathbf{b}}\mathbf{y}_{n,i}\|^2_{\ell^2}\\&=\sum_{n\in\mathbb{Z}}\sum_{i=1}^{\mathrm{d}}\sum_{j\in\mathbb{Z}}\Big\|a_j\sum_{k=j+1}^\infty b_ky_{n,i}(k)\Big\|^2=\sum_{n\in\mathbb{Z}}\sum_{i=1}^{\mathrm{d}}\sum_{j=n-1}^\infty\|a_jb_n\mathrm{e}_i\|^2\\&\le \mathrm{d}\|\mathbf{a}\|^2_{\ell^2}\|\mathbf{b}\|^2_{\ell^2}.\end{aligned}$$

This proves (2.8) and (2.13). To prove (2.9), observe that $\sigma(V^+_{\mathbf{a},\mathbf{b}})$ consists of eigenvalues since the operator $V^+_{\mathbf{a},\mathbf{b}}$ is compact by (2.8). Suppose there are $\lambda\ne0$ and $\mathbf{0}\ne\mathbf{x}\in\ell^2(\mathbb{Z};\mathbb{C}^{\mathrm{d}})$, so that $a_j\sum_{k=j+1}^\infty b_kx_k=\lambda x_j$. Then

$$\sum_{j=n+1}^\infty c_jy_j=\lambda y_n,\quad n\in\mathbb{Z},\tag{2.17}$$

for $y_n=\sum_{j=n+1}^\infty b_jx_j$ and $c_j=b_ja_j$. Using Cauchy-Schwarz, we have $y_k\to0$ as $k\to\infty$. Using (2.17), we have

$$y_n=\prod_{j=1}^k\left(I+c_{n+k}/\lambda\right)y_{n+k},\quad n\in\mathbb{Z},\quad k=1,2,\dots.\tag{2.18}$$

Since $(c_j)\in\ell^1$, the product $\prod_{j=1}^\infty(1+\|c_{n+j}\|/\lambda)$ converges for each $n$. Using this in (2.18) and letting $k\to\infty$, we conclude that for each $n\in\mathbb{Z}$ one has $0=y_n=\sum_{j=n+1}^\infty b_jx_j=b_{n+1}x_{n+1}+y_{n+1}=b_{n+1}x_{n+1}$. Since $b_nx_n=0$, we conclude that $\lambda x_j=0$ for all $j$, a contradiction. Formulas (2.10), (2.11), (2.14), and (2.15) now follow from (2.2) and (2.7). To prove (2.12), represent $V^+_{\mathbf{a},\mathbf{b}}=D_{\mathbf{a}}V^+_{\mathbf{1},\mathbf{b}}$ where $\mathbf{1}=(I)_{j\in\mathbb{Z}}$. Since $\|a_j\|\to0$ as $|j|\to\infty$ and $\|V^+_{\mathbf{1},\mathbf{b}}\|\le\|\mathbf{b}\|_{\ell^1}$, the operator $V^+_{\mathbf{a},\mathbf{b}}$ is compact on $\ell^\infty(\mathbb{Z};\mathbb{C}^{\mathrm{d}})$. The existence of nonzero $\mathbf{x}\in\ell^\infty$ and $\lambda$ so that $V^+_{\mathbf{a},\mathbf{b}}\mathbf{x}=\lambda\mathbf{x}$ leads to a contradiction, as in (2.17) - (2.18) above, since $\|y_k\|\le\|\mathbf{x}\|_{\ell^\infty}\sum_{j=k+1}^\infty\|b_k\|\to0$ as $k\to\infty$ due to $\mathbf{b}\in\ell^1$, and $\mathbf{c}=(a_jb_j)_{j\in\mathbb{Z}}\in\ell^1$. ∎

Let $E$ denote the $(\mathrm{d}\times\mathrm{d})$ matrix having 1's on the diagonal above the main diagonal, and with zero remaining entries.

Lemma 2.3. *If $A=\lambda I+E$, $\lambda\in\mathbb{C}$, is a $(\mathrm{d}\times\mathrm{d})$ Jordan block, then $\|A^j\|\le c|j|^{\mathrm{d}}|\lambda|^j$ for all $j\in\mathbb{Z}$ and some positive constant $c=c(\mathrm{d},\lambda)$.*

*Proof.* Since $E^{\mathrm{d}}=0$, we have $A^j=\lambda^j(I+\sum_{k=1}^{\mathrm{d}-1}\binom{j}{k}(E/\lambda)^k)$ for $j>0$. The polynomial growth with $j$ of the binomial coefficients $\binom{j}{k}$ gives the result. For $j>0$, we estimate the norm of $A^{-j} = \lambda^{-j}\sum_{k_1=0}^{\mathrm{d}-1}\sum_{k_2=k_1}^{\mathrm{d}-1}\cdots\sum_{k_j=k_{j-1}}^{\mathrm{d}-1}(-E/\lambda)^{k_j}$ by $c|\lambda|^{-j}S(0,j)$, where we denote for $k\in[0,\mathrm{d}-1]$:

$$S(k,j)=\sum_{k_1=k}^{\mathrm{d}-1}\sum_{k_2=k_1}^{\mathrm{d}-1}\cdots\sum_{k_j=k_{j-1}}^{\mathrm{d}-1}1=\sum_{k_1=k}^{\mathrm{d}-1}S(k_1,j-1).$$

Using the formula $\sum_{k_1=1}^{\mathrm{d}-k}k_1(k_1+1)\dots(k_1+j-1)=(\mathrm{d}-k)\dots(\mathrm{d}-k+j)/(j+1)$ and induction, one obtains $S(k,j)=(\mathrm{d}-k)(\mathrm{d}-k+1)\dots(\mathrm{d}-k+j-1)/j!$ and the lemma follows. ∎

**3. Fredholm determinants.** Below, we will make use of the following *assumptions*:

(3.1) $\quad A_j$ is invertible for each $j\in\mathbb{Z}$, and $(A_j)_{j\in\mathbb{Z}},\ (A_j^{-1})_{j\in\mathbb{Z}}\in\ell^\infty(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}})$,

(3.2) $\quad$ equation $x_{j+1}=A_jx_j$, $j\in\mathbb{Z}$, has an exponential dichotomy $P$ on $\mathbb{Z}$,

(3.3) $\quad (C_jU_j)_{j\in\mathbb{Z}},\quad (U_{j+1}^{-1}B_j)_{j\in\mathbb{Z}}\in\ell^2(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}})$,

(3.4) $\quad A_j^\times=A_j+B_jC_j$ is invertible for each $j\in\mathbb{Z}$.

Assume (3.1). Then the fundamental matrix solution $\mathbf{U}=(U_j)_{j\in\mathbb{Z}}$ satisfying $U_{j+1}=A_jU_j$, $j\in\mathbb{Z}$, and $U_0=I$ is given by the formulas

$$U_j=A_{j-1}\cdot\ldots\cdot A_0,\quad U_{-j}=A_{-j}^{-1}\cdot\ldots\cdot A_{-1}^{-1},\quad j=1,2,\dots. \tag{3.5}$$

Similarly, assume (3.4). Then any matrix solution $(U_j^\times)_{j\in\mathbb{Z}}$ satisfying $U_{j+1}^\times=A_j^\times U_j^\times$, $j\in\mathbb{Z}$, is given by the formulas

$$U_j^\times=A_{j-1}^\times\cdot\ldots\cdot A_0^\times U_0^\times,\quad U_{-j}^\times=(A_{-j}^\times)^{-1}\cdot\ldots\cdot(A_{-1}^\times)^{-1}U_0^\times,\quad j=1,2,\dots. \tag{3.6}$$

Here, $U_0^\times$ is an arbitrary (possibly singular!) matrix. Recall that $\mathbf{U}$ has the exponential dichotomy over $\mathbb{Z}$ with the (unstable) projection $P$ provided the following inequalities hold for some $c\geq 1$ and $\alpha>0$:

$$\|U_j(I-P)U_{k+1}^{-1}\|\leq ce^{-\alpha(j-k)}\text{ for } j>k,\quad \|U_jPU_{k+1}^{-1}\|\leq ce^{-\alpha(k-j)}\text{ for } j\leq k. \tag{3.7}$$

We let

$$\mathrm{d}_1=\dim\operatorname{Im}P\text{ and }\mathrm{d}_2=\dim\operatorname{Im}(I-P)\text{ so that }\mathrm{d}_1+\mathrm{d}_2=\mathrm{d}. \tag{3.8}$$

Under assumptions (3.1) - (3.2) the operator $\mathcal{T}$ in (1.1) is well-defined. Using notation (2.5) - (2.6), it can be written as $\mathcal{T}=V^-_{\mathbf{a}_1,\mathbf{b}_1}+D_{\mathbf{d}}+V^+_{\mathbf{a}_2,\mathbf{b}_2}$, where $\mathbf{a}_1=(C_jU_j(I-P))_{j\in\mathbb{Z}}$, $\mathbf{b}_1=((I-P)U_{j+1}^{-1}B_j)_{j\in\mathbb{Z}}$, $\mathbf{d}=(-C_jU_jPU_{j+1}^{-1}B_j)_{j\in\mathbb{Z}}$, $\mathbf{a}_2=(-C_jU_jP)_{j\in\mathbb{Z}}$, and $\mathbf{b}_2=(PU_{j+1}^{-1}B_j)_{j\in\mathbb{Z}}$. If assumption (3.3) holds then $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{a}_2$, $\mathbf{b}_2\in\ell^2$ and, using Cauchy-Schwarz, $\mathbf{d}\in\ell^1\subset\ell^2$, so that we have $\mathcal{T}\in\mathcal{B}_2(\ell^2)$ by (2.8), and thus $\det_2(I-\mathcal{T})$ is well-defined. Below, we will sometimes assume that $\mathcal{T}\in\mathcal{B}_1(\ell^2)$ so that $\det(I-\mathcal{T})$ is well-defined. Note that $\mathcal{T}\in\mathcal{B}_1(\ell^2)$ provided, say, (3.3) is replaced by the assumption $(C_jU_j)_{j\in\mathbb{Z}}$, $(U_{j+1}^{-1}B_j)_{j\in\mathbb{Z}}\in\ell^1(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}})$. Indeed, under this latter assumption we have $\mathbf{a}_1,\mathbf{b}_1,\mathbf{a}_2,\mathbf{b}_2,\mathbf{d}\in\ell^1$; thus $\mathcal{T}\in\mathcal{B}_1(\ell^2)$ by (2.13).

We remark that every difference operator $\mathcal{T}$ with a semi-separable kernel

$$T_{jk}=\mathbf{a}_2(j)\mathbf{b}_2(k)\text{ for } j>k\text{ and }T_{jk}=\mathbf{a}_1(j)\mathbf{b}_1(k)\text{ for } j\leq k,$$

with $\mathbf{a}_i = (\mathbf{a}_i(j))_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}_i})$ and $\mathbf{b}_i = (\mathbf{b}_i(j))_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z};\mathbb{C}^{\mathrm{d}_i\times\mathrm{d}})$, $i = 1, 2$, and $\mathrm{d}_1 + \mathrm{d}_2 = \mathrm{d}$, can be written in the form (1.1) by setting for a fixed $\alpha > 0$ and all $j \in \mathbb{Z}$:

$$P = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathrm{d}_1\times\mathrm{d}_1} \end{bmatrix}, \quad U_j = \begin{bmatrix} e^{-\alpha j} I_{\mathrm{d}_2\times\mathrm{d}_2} & 0 \\ 0 & e^{\alpha j} I_{\mathrm{d}_1\times\mathrm{d}_1} \end{bmatrix}, \tag{3.9}$$

$$C_j = \begin{bmatrix} e^{\alpha j}\mathbf{a}_2(j) & e^{-\alpha j}\mathbf{a}_1(j) \end{bmatrix}, \quad B_j = \begin{bmatrix} e^{-\alpha(j+1)}\mathbf{b}_2(j) & e^{\alpha(j+1)}\mathbf{b}_1(j) \end{bmatrix}^\top, \tag{3.10}$$

where $\top$ means transposition of the $(1 \times 2)$ block-row $[\cdot \quad \cdot]$.

We will use the following representations of the operator $\mathcal{T}$ defined in (1.1):

$$(\mathcal{T}\mathbf{x})_j = \sum_{k=-\infty}^{j-1} C_j U_j (I-P) U_{k+1}^{-1} B_k x_k - \sum_{k=j}^{\infty} C_j U_j P U_{k+1}^{-1} B_k x_k$$

$$= \sum_{k=-\infty}^{j-1} C_j U_j U_{k+1}^{-1} B_k x_k - \sum_{k=-\infty}^{\infty} C_j U_j P U_{k+1}^{-1} B_k x_k \tag{3.11}$$

$$= -\sum_{k=j}^{\infty} C_j U_j U_{k+1}^{-1} B_k x_k + \sum_{k=-\infty}^{\infty} C_j U_j (I-P) U_{k+1}^{-1} B_k x_k, j \in \mathbb{Z}. \tag{3.12}$$

Accordingly, we define the following operators:

$$(H_-\mathbf{x})_j = \sum_{k=-\infty}^{j-1} C_j U_j U_{k+1}^{-1} B_k x_k, \quad (H_+\mathbf{x})_j = -\sum_{k=j}^{\infty} C_j U_j U_{k+1}^{-1} B_k x_k, \tag{3.13}$$

$$(D\mathbf{x})_j = -C_j U_j U_{j+1}^{-1} B_j x_j, \quad (H_+^0\mathbf{x})_j = -\sum_{k=j+1}^{\infty} C_j U_j U_{k+1}^{-1} B_k x_k, \tag{3.14}$$

$$(Qx)_j = C_j U_j P x, x \in \mathbb{C}^{\mathrm{d}}, \quad R\mathbf{x} = -P \sum_{k=-\infty}^{\infty} U_{k+1}^{-1} B_k x_k \in \mathbb{C}^{\mathrm{d}_1}, \tag{3.15}$$

$$(Sx)_j = C_j U_j (I-P) x, x \in \mathbb{C}^{\mathrm{d}}, \quad W\mathbf{x} = (I-P) \sum_{k=-\infty}^{\infty} U_{k+1}^{-1} B_k x_k \in \mathbb{C}^{\mathrm{d}_2}, \tag{3.16}$$

so that (3.11) - (3.12) in this notation become

$$\mathcal{T} = H_- + QR \tag{3.17}$$

$$= H_+ + SW = D + H_+^0 + SW. \tag{3.18}$$

We stress that the operators $R$ and $W$ have finite ranks $\mathrm{d}_1$ and $\mathrm{d}_2$, respectively. Properties of the operators (3.13) - (3.16) are summarized in the following lemmas.

LEMMA 3.1. *Assume (3.1) - (3.3). Then* $\sigma(H_-) = \{0\}$, $H_- \in \mathcal{B}_2(\ell^2)$, *and* $\det_2(I-H_-) = 1$. *If, in addition to (3.1) - (3.3), we assume that* $\mathcal{T} \in \mathcal{B}_1(\ell^2)$, *then* $H_- \in \mathcal{B}_1(\ell^2)$ *and* $\det(I - H_-) = 1$.

LEMMA 3.2. *Assume (3.1) - (3.4). Then:*

$$D \in \mathcal{B}_1(\ell^2) \text{ and } \det(I-D) = \prod_{j\in\mathbb{Z}} \left(\det A_j^{-1} \det(A_j + B_j C_j)\right), \tag{3.19}$$

$$\text{the operator } (I-D) \text{ is invertible,} \tag{3.20}$$

$$\det_2(I-D)=\prod_{j\in\mathbb{Z}}\big(\det A_j^{-1}\det(A_j+B_jC_j)\big)\exp\sum_{j\in\mathbb{Z}}-\mathrm{tr}(C_jU_jU_{j+1}^{-1}B_j), \tag{3.21}$$

$$\begin{gathered}\sigma((I-D)^{-1}H_+^0)=\{0\},\quad (I-D)^{-1}H_+^0\in\mathcal{B}_2(\ell^2),\ \textit{and}\\ \det_2(I-(I-D)^{-1}H_+^0)=1.\end{gathered} \tag{3.22}$$

*If, in addition to (3.1) - (3.4), we assume that* $\mathcal{T}\in\mathcal{B}_1(\ell^2)$ *then*

$$(I-D)^{-1}H_+^0\in\mathcal{B}_1(\ell^2),\ \textit{and}\ \det(I-(I-D)^{-1}H_+^0)=1. \tag{3.23}$$

LEMMA 3.3. *Assume (3.1) - (3.4). Then* $H_+\in\mathcal{B}_2(\ell^2)$, $\det_2(I-H_+)=\det_2(I-D)$, *and the operator* $I-H_+$ *is invertible. If, in addition to (3.1) - (3.4), we assume that* $\mathcal{T}\in\mathcal{B}_1(\ell^2)$ *then* $H_+\in\mathcal{B}_1(\ell^2)$ *and* $\det(I-H_+)=\det(I-D)$.

*Proof.* Using notation (2.5), we remark that $H_-=V^-_{\mathbf{a},\mathbf{b}}$ with $\mathbf{a}=(C_jU_j)_{j\in\mathbb{Z}}$ and $\mathbf{b}=(U_{j+1}^{-1}B_j)_{j\in\mathbb{Z}}$. Since $\mathbf{a},\mathbf{b}\in\ell^2$ by (3.3), the first three assertions in Lemma 3.1 follow, respectively, from (2.9), (2.8), and (2.10). If $\mathcal{T}\in\mathcal{B}_1(\ell^2)$ then $H_-\in\mathcal{B}_1(\ell^2)$ because $R\in\mathcal{B}_1(\ell^2)$ is of finite rank and (3.17) holds. We already know that $\sigma(H_-)=\{0\}$. Thus, $\det(I-H_-)=1$ follows from (2.1), and Lemma 3.1 is proved.

To prove Lemma 3.2, note that $D=D_{\mathbf{d}}$, see (2.6), where $\mathbf{d}=(-C_jU_jU_{j+1}^{-1}B_j)_{j\in\mathbb{Z}}$. Assumption (3.3) and Cauchy-Schwarz imply $\mathbf{d}\in\ell^1$. Then $D\in\mathcal{B}_1(\ell^2)$ by (2.13). Moreover, using (2.15) and the identity $U_{j+1}=A_jU_j$, $j\in\mathbb{Z}$, we verify (3.19) as follows:

$$\begin{aligned}\det(I-D)&=\prod_{j\in\mathbb{Z}}\det(I_{\mathrm{d}\times\mathrm{d}}+C_jU_jU_{j+1}^{-1}B_j)=\prod_{j\in\mathbb{Z}}\det(I+U_jU_{j+1}^{-1}B_jC_j)\\&=\prod_{j\in\mathbb{Z}}\det(I+A_j^{-1}B_jC_j)=\prod_{j\in\mathbb{Z}}\det A_j^{-1}\det(A_j+B_jC_j).\end{aligned}$$

To prove (3.20), we use assumption (3.4) and the identity $U_{j+1}=A_jU_j$ to infer:

$$\begin{aligned}(I+C_jU_jU_{j+1}^{-1}B_j)^{-1}&=(I+C_jA_j^{-1}B_j)^{-1}=I-C_j(I+A_j^{-1}B_jC_j)^{-1}A_j^{-1}B_j\\&=I-C_j(A_j+B_jC_j)^{-1}B_j=I-C_jU_j(U_{j+1}^{-1}(A_j+B_jC_j)U_j)^{-1}U_{j+1}^{-1}B_j\\&=I-C_jU_j(I+U_{j+1}^{-1}B_jC_jU_j)^{-1}U_{j+1}^{-1}B_j.\end{aligned}$$

Note that $(C_jU_j)_{j\in\mathbb{Z}}$, $(U_{j+1}^{-1}B_j)_{j\in\mathbb{Z}}\in\ell^2\subset\ell^\infty(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}})$ and $\lim_{j\to\infty}\|U_{j+1}^{-1}B_jC_jU_j\|=0$ by assumption (3.3). Thus (3.20) holds and, moreover, $(I-D)^{-1}=D_{\mathbf{d}}$, where

$$\mathbf{d}=((I+C_jU_jU_{j+1}^{-1}B_j)^{-1})_{j\in\mathbb{Z}}\in\ell^\infty(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}}). \tag{3.24}$$

Formula (3.21) follows from (3.19) and (2.3). Using notation (2.5) and (3.24) we remark that $(I-D)^{-1}H_+^0=V^+_{\mathbf{a},\mathbf{b}}$, where $\mathbf{a}=(-(I+C_jU_jU_{j+1}^{-1}B_j)^{-1}C_jU_j)_{j\in\mathbb{Z}}$ and $\mathbf{b}=(U_{j+1}^{-1}B_j)_{j\in\mathbb{Z}}$. By $\mathbf{d}\in\ell^\infty$ in (3.24) and assumption (3.3) we conclude $\mathbf{a},\mathbf{b}\in\ell^2$, and then (3.22) follows from (2.9), (2.8), and (2.10). Finally, if $\mathcal{T}\in\mathcal{B}_1(\ell^2)$ then $H_+=D+H_+^0\in\mathcal{B}_1(\ell^2)$ by (3.18) since $W$ is of finite rank. Since $D=D_{\mathbf{d}}$ with $\mathbf{d}=(-C_jU_jU_{j+1}^{-1}B_j)\in\ell^1$, see assumption (3.3), by (2.13) we have $D\in\mathcal{B}_1(\ell^2)$. Therefore $H_+^0\in\mathcal{B}_1(\ell^2)$ and thus $(I-D)^{-1}H_+^0\in\mathcal{B}_1(\ell^2)$ since $(I-D)^{-1}$ is a bounded operator. The last assertion in (3.23) now follows from $\sigma((I-D)^{-1}H_+^0)=\{0\}$ and (2.1). Assertions in Lemma 3.3 follow from the identity $I-H_+=(I-D)(I-(I-D)^{-1}H_+^0)$. ∎

Our first main result gives a formula for the (modified) Fredholm determinant of the (infinite-dimensional) operator $I - \mathcal{T}$ in terms of finite-dimensional determinants.

THEOREM 3.4. *Assume (3.1), (3.2), (3.3), and (3.4). Then*

$$\det{}_2(I-\mathcal{T}) = \det{}_{\mathbb{C}^{d_1}}(I_{d_1\times d_1} - R(I-H_-)^{-1}Q)\exp\sum_{j\in\mathbb{Z}} -\mathrm{tr}(PU_{j+1}^{-1}B_jC_jU_jP) \tag{3.25}$$

$$= \det{}_{\mathbb{C}^{d_2}}(I_{d_2\times d_2} - W(I-H_+)^{-1}S) \times \prod_{j\in\mathbb{Z}} \det A_j^{-1}\det(A_j+B_jC_j)\exp\sum_{j\in\mathbb{Z}} -\mathrm{tr}(PU_{j+1}^{-1}B_jC_jU_jP). \tag{3.26}$$

*If, in addition to (3.1) - (3.4), we assume that* $\mathcal{T} \in \mathcal{B}_1(\ell^2)$*, then*

$$\det(I-\mathcal{T}) = \det{}_{\mathbb{C}^{d_1}}(I_{d_1\times d_1} - R(I-H_-)^{-1}Q) \tag{3.27}$$

$$= \det{}_{\mathbb{C}^{d_2}}(I_{d_2\times d_2} - W(I-H_+)^{-1}S)\prod_{j\in\mathbb{Z}} \det A_j^{-1}\det(A_j+B_jC_j). \tag{3.28}$$

*Proof.* Using representation (3.17), recalling that $\det_2(I-H_-) = 1$ by Lemma 3.1, noting that $(I-H_-)^{-1}QR \in \mathcal{B}_1 \subset \mathcal{B}_2$ because $R$ is of rank $d_1$, and applying (2.4) and (2.3), we infer:

$$\begin{aligned}
\det{}_2(I-\mathcal{T}) &= \det{}_2[(I-H_-)(I-(I-H_-)^{-1}QR)]\\
&= \det{}_2(I-H_-)\det{}_2(I-(I-H_-)^{-1}QR)\exp(-\mathrm{tr}[H_-(I-H_-)^{-1}QR])\\
&= \det(I-(I-H_-)^{-1}QR)\exp\mathrm{tr}[(I-H_-)^{-1}QR]\exp(-\mathrm{tr}[H_-(I-H_-)^{-1}QR])\\
&= \det(I-R(I-H_-)^{-1}Q)\exp\mathrm{tr}(QR)\\
&= \det{}_{\mathbb{C}^{d_1}}(I_{d_1\times d_1} - R(I-H_-)^{-1}Q)\exp\mathrm{tr}(RQ).
\end{aligned}$$

Using (3.15), we have (3.25). To establish (3.26), we first apply representation (3.18), Lemma 3.3, and (2.4):

$$\begin{aligned}
&\det{}_2(I-\mathcal{T}) = \det{}_2[(I-H_+)(I-(I-H_+)^{-1}SW)]\\
&= \det{}_2(I-H_+)\det{}_2(I-(I-H_+)^{-1}SW)\exp(-\mathrm{tr}[H_+(I-H_+)^{-1}SW])\\
&= \det{}_2(I-D)\det(I-(I-H_+)^{-1}SW)\exp(\mathrm{tr}[(I-H_+)^{-1}SW - H_+(I-H_+)^{-1}SW])\\
&= \det{}_2(I-D)\det(I-W(I-H_+)^{-1}S)\exp\mathrm{tr}(SW)\\
&= \det{}_2(I-D)\det{}_{\mathbb{C}^{d_2}}(I_{d_2\times d_2} - W(I-H_+)^{-1}S)\exp\mathrm{tr}(WS),
\end{aligned}$$

recalling that $W$ is of rank $d_2$, and thus $(I-H_+)^{-1}SW \in \mathcal{B}_1 \subset \mathcal{B}_2$, which allows us to use (2.3). Using (3.21) and (3.16) we therefore have:

$$\begin{aligned}
\det{}_2(I-\mathcal{T}) = \det{}_{\mathbb{C}^{d_2}}(I_{d_2\times d_2} - W(I-H_+)^{-1}S)\prod_{j\in\mathbb{Z}}\det A_j^{-1}\det(A_j+B_jC_j)\\
\times\exp\sum_{j\in\mathbb{Z}}\mathrm{tr}[-C_jU_jU_{j+1}^{-1}B_j + C_jU_j(I-P)U_{j+1}^{-1}B_j],
\end{aligned}$$

which implies (3.26). Formula (3.27) follows from representation (3.17) and Lemma 3.1:

$$\begin{aligned}
\det(I-\mathcal{T}) &= \det[(I-H_-)(I-(I-H_-)^{-1}QR)]\\
&= \det(I-H_-)\det(I-R(I-H_-)^{-1}Q) = \det{}_{\mathbb{C}^{d_1}}(I-R(I-H_-)^{-1}Q).
\end{aligned}$$

Formula (3.28) follows from representation (3.18) and Lemmas 3.2 - 3.3:

$$\det(I-\mathcal{T}) = \det[(I-H_+)(I-(I-H_+)^{-1}SW)]$$
$$= \det(I-H_+)\det(I-(I-H_+)^{-1}SW) = \det(I-D)\det(I-W(I-H_+)^{-1}S),$$

which concludes the proof. ■

Our next objective is to relate the finite-dimensional determinants in the right-hand side of (3.25) - (3.28) to the determinants of a particular matrix solution $(\mathcal{U}_j^\times)_{j\in\mathbb{Z}}$ satisfying $\mathcal{U}_{j+1}^\times = A_j^\times \mathcal{U}_j^\times$. We stress that this solution could be singular. For this, we consider the following matrix difference equations:

$$X_j^+ = C_jU_j(I-P) - \sum_{k=j}^{\infty} C_jU_jU_{k+1}^{-1}B_kX_k^+, \tag{3.29}$$

$$X_j^- = C_jU_jP + \sum_{k=-\infty}^{j-1} C_jU_jU_{k+1}^{-1}B_kX_k^-, \quad j\in\mathbb{Z}. \tag{3.30}$$

Using notation (3.13) - (3.16), equations (3.29) - (3.30) for $\mathbf{X}^\pm = (X_j^\pm)_{j\in\mathbb{Z}}$ can be rewritten as follows:

$$\mathbf{X}^+x = Sx + H_+\mathbf{X}^+x, \quad \mathbf{X}^-x = Qx + H_-\mathbf{X}^-x, \quad x\in\mathbb{C}^{\mathrm{d}}. \tag{3.31}$$

By assumption (3.3), we have $Sx, Qx \in \ell^2(\mathbb{Z};\mathbb{C}^{\mathrm{d}})$. By Lemmas 3.1 and 3.3 operators $I - H_\pm$ are invertible. Thus, (3.29) - (3.30) have a unique pair of solutions $\mathbf{X}^\pm \in \ell^2(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}})$ given by

$$\mathbf{X}^+ = (I-H_+)^{-1}S \text{ and } \mathbf{X}^- = (I-H_-)^{-1}Q. \tag{3.32}$$

Since the solutions $\mathbf{X}^\pm$ of (3.29) - (3.30) are unique, multiplying (3.29) by $(I-P)$ and (3.30) by $P$ from the right, we also have: $X_j^+ = X_j^+(I-P)$ and $X_j^- = X_j^-P$, $j\in\mathbb{Z}$. Thus, we can treat matrices $X_j^\pm$ as operators $X_j^+ : \operatorname{Im}(I-P)\to\mathbb{C}^{\mathrm{d}}$ and $X_j^- : \operatorname{Im}P\to\mathbb{C}^{\mathrm{d}}$. Using (3.32) and notations (3.15) - (3.16) we then have for $X_j^\pm$ from (3.29) - (3.30):

$$\det{}_{\mathbb{C}^{\mathrm{d}_1}}(I_{\mathrm{d}_1\times\mathrm{d}_1} - R(I-H_-)^{-1}Q) = \det{}_{\mathbb{C}^{\mathrm{d}_1}}\Big(I_{\mathrm{d}_1\times\mathrm{d}_1} + \sum_{k=-\infty}^{\infty} PU_{k+1}^{-1}B_kX_k^-P\Big), \tag{3.33}$$

$$\det{}_{\mathbb{C}^{\mathrm{d}_2}}(I_{\mathrm{d}_2\times\mathrm{d}_2} - W(I-H_+)^{-1}S) \tag{3.34}$$
$$= \det{}_{\mathbb{C}^{\mathrm{d}_2}}\Big(I_{\mathrm{d}_2\times\mathrm{d}_2} - \sum_{k=-\infty}^{\infty}(I-P)U_{k+1}^{-1}B_kX_k^+(I-P)\Big).$$

We remark that the series $\sum_{k=-\infty}^{\infty} U_{k+1}^{-1}B_kX_k^\pm$ converge absolutely by assumption (3.3), $\mathbf{X}^\pm \in \ell^2(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}})$, and Cauchy-Schwarz.

Using the direct sum decomposition $\mathbb{C}^{\mathrm{d}} = \operatorname{Im}(I-P)\oplus\operatorname{Im}P$, consider a matrix sequence, $\mathcal{U}^\times = (\mathcal{U}_j^\times)_{j\in\mathbb{Z}}$, defined by $\mathcal{U}_j^\times = U_jV_j$ where $(U_j)_{j\in\mathbb{Z}}$ is the fundamental matrix solution of the unperturbed equation $x_{j+1} = A_jx_j$, $U_0 = I$, and the $(2\times 2)$ block matrix $V_j$ is defined as follows:

$$V_j = \begin{bmatrix} I - P - \sum_{k=j}^{\infty} (I-P)U_{k+1}^{-1}B_kX_k^+(I-P) & \sum_{k=-\infty}^{j-1} (I-P)U_{k+1}^{-1}B_kX_k^-P \\ -\sum_{k=j}^{\infty} PU_{k+1}^{-1}B_kX_k^+(I-P) & P + \sum_{k=-\infty}^{j-1} PU_{k+1}^{-1}B_kX_k^-P \end{bmatrix} \tag{3.35}$$

First, we claim that $\mathcal{U}^\times$ is a solution of the matrix equation $\mathcal{U}_{j+1}^\times = A_j^\times \mathcal{U}_j^\times$, $j \in \mathbb{Z}$. Indeed, (3.35) and $U_{j+1} = A_jU_j$ imply:

$$\begin{aligned} \mathcal{U}_{j+1}^\times &= U_{j+1}V_{j+1} = U_{j+1}V_j + B_j \left[ X_j^+(I-P) \quad X_j^-P \right] \\ &= A_j\mathcal{U}_j^\times + B_j \left[ X_j^+(I-P) \quad X_j^-P \right], \end{aligned}$$

for the $(1 \times 2)$ block row $\left[ X_j^+(I-P) \quad X_j^-P \right] : \operatorname{Im}(I-P) \oplus \operatorname{Im} P \to \mathbb{C}^{\mathrm{d}}$. Using (3.29) - (3.30) and (3.35), we also have $\left[ X_j^+(I-P) \quad X_j^-P \right] = C_jU_jV_j$, and the claim is proved. Second, we observe that there exist limits $V_{\pm\infty} = \lim_{j\to\pm\infty} U_j^{-1}\mathcal{U}_j^\times$, and the operators $V_\infty$ and $V_{-\infty}$ are, respectively, upper- and lower-triangular matrices in the direct-sum decomposition $\mathbb{C}^{\mathrm{d}} = \operatorname{Im}(I-P) \oplus \operatorname{Im} P$. Moreover, $\det_{\mathbb{C}^{\mathrm{d}}} V_\infty$ and $\det_{\mathbb{C}^{\mathrm{d}}} V_{-\infty}$ are equal, respectively, to the right hand sides of (3.33) and (3.34). Finally, since $\mathbf{U}$ and $\mathcal{U}^\times$ are solutions of the equations $U_{j+1} = A_jU_j$ and $\mathcal{U}_{j+1}^\times = A_j^\times\mathcal{U}_j^\times$, we can use formulas (3.5) - (3.6) to recalculate the right hand sides of (3.33) and (3.34) via $\det_{\mathbb{C}^{\mathrm{d}}} \mathcal{U}_0^\times$ (recall that $\det U_0 = 1$ since $U_0 = I$). Using Theorem 3.4, we arrive to the following result.

THEOREM 3.5. *Assume (3.1) - (3.4), let $X_j^\pm$ be the matrix solutions of (3.29) - (3.30), and define $\mathcal{U}^\times = (\mathcal{U}_j^\times)_{j\in\mathbb{Z}}$ as $\mathcal{U}_j^\times = U_jV_j$, where $V_j$ are given by formula (3.35). Then*

$$\det{}_2(I - \mathcal{T}) = (\det{}_{\mathbb{C}^{\mathrm{d}}} \mathcal{U}_0^\times) \prod_{j=0}^{\infty} \frac{\det(A_j + B_jC_j)}{\det A_j} \times \exp \sum_{j\in\mathbb{Z}} -\operatorname{tr}(PU_{j+1}^{-1}B_jC_jU_jP).$$

*If, in addition to (3.1) - (3.4), we assume that $\mathcal{T} \in \mathcal{B}_1(\ell^2)$, then*

$$\det(I - \mathcal{T}) = (\det{}_{\mathbb{C}^{\mathrm{d}}} \mathcal{U}_0^\times) \prod_{j=0}^{\infty} \frac{\det(A_j + B_jC_j)}{\det A_j}. \tag{3.36}$$

**4. The Evans determinant.** In this section we continue to assume that (3.1) - (3.4) hold. Because of (3.1), $\mathbf{U}$ is exponentially bounded: $\sup_{j,k\in\mathbb{Z}} e^{-\alpha(k-j)}\|U_jU_{k+1}^{-1}\| < \infty$ for some $\alpha \in \mathbb{R}$. This allows us to introduce the following notions related to the Bohl (or, in other terminology, general Lyapunov) exponents. For the corresponding theory in differential equations case see [DK]; also, these notions are related to the so-called Sacker-Sell, or dynamical spectrum, see [SS] and the bibliography in [CL].

If $J = \mathbb{Z}$ or $J = \mathbb{Z}^\pm$, and $Q$ is a projection on $\mathbb{C}^{\mathrm{d}}$ so that $\sup_{k\in J} \|U_kQU_k^{-1}\| < \infty$ then the upper and lower Bohl exponents, $\varkappa_g$ and $\varkappa_g'$, are defined as follows:

$$\varkappa_g(Q; J) = \inf\{\alpha \in \mathbb{R} : \sup_{j\geq k\in J} e^{-\alpha(j-k)}\|U_jQU_{k+1}^{-1}\| < \infty\}, \tag{4.1}$$

$$\varkappa_g'(Q; J) = \sup\{\alpha \in \mathbb{R} : \sup_{j\leq k\in J} e^{-\alpha(j-k)}\|U_jQU_{k+1}^{-1}\| < \infty\}. \tag{4.2}$$

For instance, if $P$ is the dichotomy projection, cf. (3.7), then

$$\varkappa_g(I - P; \mathbb{Z}) < 0 < \varkappa_g'(P; \mathbb{Z}). \tag{4.3}$$

A system $\{Q_i\}_{i=1}^{\mathrm{d}_0}$ of disjoint projections on $\mathbb{C}^{\mathrm{d}}$, $1 \le \mathrm{d}_0 \le \mathrm{d}$, is called an *exponential splitting* of order $\mathrm{d}_0$ if the following holds: $Q_1 + \cdots + Q_{\mathrm{d}_0} = I$, $\sup_{k\in J} \|U_k Q_i U_k^{-1}\| < \infty$, and the segments $[\varkappa_g'(Q_i; J), \varkappa_g(Q_i; J)]$ are all disjoint, $i = 1, \ldots, \mathrm{d}_0$. An exponential splitting is called the *finest* if there is no exponential splitting of order $\mathrm{d}_0 + 1$.

Let $\{Q_i\}_{i=1}^{\mathrm{d}_0}$ denote the finest exponential splitting over $\mathbb{Z}$ for $\mathbf{U} = (U_j)_{j\in\mathbb{Z}}$. In what follows we assume that projections $Q_i$ are numbered such that $\varkappa_g(Q_i;\mathbb{Z}) < \varkappa_g'(Q_{i+1};\mathbb{Z})$; we set $Q_0 = 0$, $Q_{\mathrm{d}_0+1} = I$. Since $P$ is the dichotomy projection for $\mathbf{U}$, there exists an $n \in \{1, \ldots, \mathrm{d}_0\}$ such that $I - P = Q_1 + \cdots + Q_n$ and $P = Q_{n+1} + \cdots + Q_{\mathrm{d}_0}$. Thus,

$$\varkappa_g'(Q_i;\mathbb{Z}) \le \varkappa_g(Q_i;\mathbb{Z}) < 0 \quad \text{for } i = 1, \ldots, n, \text{ and} \tag{4.4}$$

$$0 < \varkappa_g'(Q_i;\mathbb{Z}) \le \varkappa_g(Q_i;\mathbb{Z}) \quad \text{for } i = n+1, \ldots, \mathrm{d}_0. \tag{4.5}$$

Clearly, $\{Q_i\}_{i=1}^{\mathrm{d}_0}$ is also an exponential splitting for $\mathbf{U}$ over $\mathbb{Z}_+$ and $\mathbb{Z}_-$. We denote:
(4.6)

$$\varkappa_i' = \varkappa_g'(Q_i;\mathbb{Z}),\ \varkappa_i'^{\pm} = \varkappa_g'(Q_i;\mathbb{Z}_\pm),\ \varkappa_i = \varkappa_g(Q_i;\mathbb{Z}),\ \varkappa_i^{\pm} = \varkappa_g(Q_i;\mathbb{Z}_\pm),\ i = 1, \ldots, \mathrm{d}_0,$$

$$\varepsilon_0 = \frac{1}{2}\min\{-\varkappa_n, \varkappa_{n+1}', \varkappa_i' - \varkappa_{i-1} : i = 1, \ldots, n-1, n+1, \ldots, \mathrm{d}_0\}. \tag{4.7}$$

In this section we will use the following assumptions for the perturbation $(B_jC_j)_{j\in\mathbb{Z}}$: There exists a $\delta \in (0, \varepsilon_0)$ such that

$$\sum_{k=0}^{\infty} e^{(\varkappa_i^+ - \varkappa_i'^+ + \delta)k}\|B_kC_k\| < \infty, \text{ for } i = 1, \ldots, n, \tag{4.8}$$

$$\sum_{k=-\infty}^{0} e^{-(\varkappa_i^- - \varkappa_i'^- + \delta)k}\|B_kC_k\| < \infty, \text{ for } i = n+1, \ldots, \mathrm{d}_0. \tag{4.9}$$

Our next objective is to construct matrix solutions of the perturbed difference equation $x_{j+1} = A_j^\times x_j$ that are asymptotic to the solutions $(U_jQ_i)_{j\in\mathbb{Z}}$ of the unperturbed equation $x_{j+1} = A_jx_j$. For some $N \in \mathbb{N}$ consider the following $(\mathrm{d}\times\mathrm{d})$ matrix difference equations:

$$\begin{aligned} Y_j^{(i)} - U_jQ_i = &-\sum_{k=j}^{\infty} U_j(Q_i + \cdots + Q_{\mathrm{d}_0})U_{k+1}^{-1}B_kC_kY_k^{(i)} \\ &+ \sum_{k=N}^{j-1} U_j(I - (Q_i + \cdots + Q_{\mathrm{d}_0}))U_{k+1}^{-1}B_kC_kY_k^{(i)},\ j > N,\ i = 1, \ldots, n, \end{aligned} \tag{4.10}$$

$$\begin{aligned} Y_j^{(i)} - U_jQ_i = &\sum_{k=-\infty}^{j-1} U_j(Q_1 + \cdots + Q_i)U_{k+1}^{-1}B_kC_kY_k^{(i)} \\ &- \sum_{k=j}^{-N} U_j(I - (Q_1 + \cdots + Q_i))U_{k+1}^{-1}B_kC_kY_k^{(i)},\ j \le -N,\ i = n+1, \ldots, \mathrm{d}_0. \end{aligned} \tag{4.11}$$

REMARK 4.1. If the sequence $\mathbf{Y}^{(i)} = (Y_j^{(i)})$, where $j > N$, resp. $j \le -N$, is a solution of (4.10), resp. (4.11), then this sequence is a solution of the (perturbed) difference equation $Y_{j+1} = A_j^\times Y_j$ for $j > N$, resp. $j \le -N$. If $\mathbf{U}^\times = (U_j^\times)_{j\in\mathbb{Z}}, U_0^\times = I$, denotes the fundamental matrix solution of the equation $U_{j+1}^\times = A_j^\times U_j^\times$ (this solution is non-singular by assumption (3.4)), then the solutions $\mathbf{Y}^{(i)} = (Y_j^{(i)})$, $i = 1, \dots, n$, resp. $i = n+1, \dots, \mathrm{d}_0$, originally given for $j > N$, resp. $j \le -N$, could be extended to $\mathbb{Z}_+ = \{0, 1, \dots\}$, resp. $\mathbb{Z}_- = \{\dots, -1, 0\}$, by setting

$$\begin{aligned} Y_j^{(i)} &= U_j^\times (U_{N+1}^\times)^{-1} Y_{N+1}^{(i)}, \quad j = 0, \dots, N, \\ Y_j^{(i)} &= U_j^\times (U_{-N}^\times)^{-1} Y_{-N}^{(i)}, \quad j = -N+1, \dots, 0. \end{aligned} \tag{4.12}$$

LEMMA 4.2. *Assume (4.8), resp. (4.9). Then, for a sufficiently large $N = N(\delta)$, there exists a solution $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j>N}$, $i = 1, \dots, n$, of (4.10), resp. $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j\le -N}$, $i = n+1, \dots, \mathrm{d}_0$, of (4.11). Moreover, these solutions satisfy $Y_j^{(i)} = Y_j^{(i)} Q_i$ and have the following properties:*

$$\begin{aligned} (a) \qquad & \sup_{j>N} e^{-(\varkappa_i^+ + \frac{\delta}{2})j} \|Y_j^{(i)}\| < \infty, \quad i = 1, \dots, n, \\ & \sup_{j\le -N} e^{-(\varkappa_i'^- - \frac{\delta}{2})j} \|Y_j^{(i)}\| < \infty, \quad i = n+1, \dots, \mathrm{d}_0; \\ (b) \qquad & \inf_{j>N} e^{-(\varkappa_i'^+ - \frac{\delta}{2})j} \|Y_j^{(i)}\|_\bullet > 0, \quad i = 1, \dots, n, \\ & \inf_{j\le -N} e^{-(\varkappa_i^- + \frac{\delta}{2})j} \|Y_j^{(i)}\|_\bullet > 0, \quad i = n+1, \dots, \mathrm{d}_0; \\ (c) \qquad & \sup_{j>N} e^{-(\varkappa_i'^+ - \frac{\delta}{2})j} \|Y_j^{(i)} - U_j Q_i\| < \infty, \quad i = 1, \dots, n, \\ & \sup_{j\le -N} e^{-(\varkappa_i^- + \frac{\delta}{2})j} \|Y_j^{(i)} - U_j Q_i\| < \infty, \quad i = n+1, \dots, \mathrm{d}_0. \end{aligned}$$

*Proof.* For $i = 1, \dots, n$ set $\alpha = \varkappa_i'^+ - \delta/2$ so that $\alpha \in (\varkappa_{i-1}^+, \varkappa_i'^+)$. Using (4.1) - (4.2) with $J = \mathbb{Z}_+$, find constants $c_\alpha$ and $c'_\alpha$ such that

$$\|U_j(I - (Q_i + \dots + Q_{\mathrm{d}_0}))U_{k+1}^{-1}\| \le c_\alpha e^{\alpha(j-k)}, \quad j \ge k \ge 0, \tag{4.13}$$

$$\|U_j(Q_i + \dots + Q_{\mathrm{d}_0})U_{k+1}^{-1}\| \le c'_\alpha e^{\alpha(j-k)}, \quad k \ge j \ge 0. \tag{4.14}$$

Using assumption (4.8), choose $N$ so large that

$$q := \sum_{k=N}^\infty \max\{c_\alpha, c'_\alpha\} e^{(\varkappa_i^+ - \varkappa_i'^+ + \delta)k} \|B_k C_k\| < 1. \tag{4.15}$$

With this $N$, and letting $\beta = \varkappa_i^+ + \delta/2$, we define a Banach space, $\ell^\infty_{+,\beta}$, of $\mathbb{C}^{\mathrm{d}\times\mathrm{d}}$-valued sequences $\mathbf{u} = (u_j)_{j>N}$ as follows: $\ell^\infty_{+,\beta} = \{\mathbf{u} : \|\mathbf{u}\|_{+,\beta} := \sup_{j>N} e^{-\beta j}\|u_j\| < \infty\}$. Let $T$ denote the operator corresponding to the right-hand side of (4.10), so that this equation becomes $\mathbf{Y}^{(i)} - (U_j Q_i)_{j>N} = T\mathbf{Y}^{(i)}$. We claim that $T$ is a contraction in $\ell^\infty_{+,\beta}$. Indeed, using (4.13) - (4.14), and then (4.15), we infer:

$$\begin{aligned} \|T\mathbf{u}\|_{+,\beta} &\le \|\mathbf{u}\|_{+,\beta} \sup_{j>N} e^{-\beta j} \Big( \sum_{k=j}^\infty c'_\alpha e^{\alpha(j-k)} \|B_k C_k\| e^{\beta k} + \sum_{k=N}^{j-1} c_\alpha e^{\alpha(j-k)} \|B_k C_k\| e^{\beta k} \Big) \\ &\le \|\mathbf{u}\|_{+,\beta} \max\{c_\alpha, c'_\alpha\} \sup_{j>N} \sum_{k=N}^\infty e^{(\alpha-\beta)(j-k)} \|B_k C_k\| < q\|\mathbf{u}\|_{+,\beta}, \end{aligned} \tag{4.16}$$

because $(\alpha - \beta)j = -(\varkappa_i^+ - \varkappa_i'^+ + \delta)j < 0$. Since $\sup_{j\ge 0} e^{-\beta j}\|U_j Q_i\| < \infty$ by (4.1) with $J = \mathbb{Z}_+$, $k = -1$, and $\beta > \varkappa_i^+$, we have $(U_j Q_i)_{j>N} \in \ell^\infty_{+,\beta}$. This proves the existence and uniqueness of a solution $\mathbf{Y}^{(i)} \in \ell^\infty_{+,\beta}$ of (4.10), and thus (*a*). Also, multiplying (4.10) by $Q_i$ from the right, and using the uniqueness, we have $Y_j^{(i)} = Y_j^{(i)} Q_i$. Similarly to estimate (4.16), we have

$$\|T\mathbf{u}\|_{+,\alpha} \le \|\mathbf{u}\|_{+,\beta} \max\{c_\alpha, c'_\alpha\} \sum_{k=N}^{\infty} e^{(\beta-\alpha)k} \|B_k C_k\| = q\|\mathbf{u}\|_{+,\beta}.$$

Since $\mathbf{Y}^{(i)} - (U_j Q_i)_{j>N} = T\mathbf{Y}^{(i)}$ and $\mathbf{Y}^{(i)} \in \ell^\infty_{+,\beta}$, we have $\mathbf{Y}^{(i)} - (U_j Q_i)_{j>N} \in \ell^\infty_{+,\alpha}$ and assertion (*c*) in the lemma follows. To prove assertion (*b*), we estimate

$$\begin{aligned} \|Y_j^{(i)}\|_\bullet &= \|Y_j^{(i)} - U_j Q_i + U_j Q_i\|_\bullet \ge \|U_j Q_i\|_\bullet - \|Y_j^{(i)} - U_j Q_i\| \\ &\ge \|U_j Q_i\|_\bullet - c_1 e^{\alpha j} \end{aligned} \tag{4.17}$$

with some positive constant $c_1$ from (*c*). If $\|x\| = 1$ and $Q_i x = x$ then, using (4.2) for $J = \mathbb{Z}_+$, and that $\alpha + \delta/4 = \varkappa_j'^{+} - \delta/4 < \varkappa_j'^{+}$, we infer for any $k > 0$:

$$\begin{aligned} 1 = \|Q_i x\| &\le \|Q_i U_{k+1}^{-1}\| \cdot \|U_{k+1}^{-1} Q_i x\| = (e^{(\alpha+\delta/4)k}\|Q_i U_{k+1}^{-1}\|)(e^{-(\alpha+\delta/4)k}\|U_{k+1}^{-1} Q_i\|) \\ &\le (\sup_{0\le j\le k} e^{-(\alpha+\delta/4)(j-k)}\|U_j Q_i U_{k+1}^{-1}\|)(e^{-(\alpha+\delta/4)k}\|U_{k+1}^{-1} Q_i x\|) \\ &\le c e^{-(\alpha+\delta/4)k}\|U_{k+1}^{-1} Q_i x\|. \end{aligned}$$

This implies that $\|U_j Q_i\|_\bullet \ge c_2 e^{(\alpha+\delta/4)j}$ for all $j \in \mathbb{Z}_+$ and some $c_2 > 0$. Using (4.17), we have $e^{-\alpha j}\|Y_j^{(i)}\|_\bullet \ge c_2 e^{(\delta/4)j} - c_1 \ge 1$ starting from some sufficiently large $j_0 > N$. Since $\mathbf{Y}^{(i)} = \mathbf{Y}^{(i)} Q_i$ is a solution of the equation $Y_{j+1}^{(i)} = A_j^\times Y_j^{(i)}$, the last assertion and assumption (3.4) imply that $\|Y_j^{(i)}\|_\bullet > 0$ for each $j \in [N, j_0]$. This proves (*b*), and the proof of the lemma for $i = 1, \dots, n$ is complete. The proof for $i = n+1, \dots, \mathrm{d}_0$ is similar. ∎

REMARK 4.3. By Lemma 4.2 and Remark 4.1 we thus have matrix solutions $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j\ge 0}$, $i = 1, \dots, n$, resp. $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j\le 0}$, $i = n+1, \dots, \mathrm{d}_0$, of the perturbed equation $x_{j+1} = A_j^\times x_j$ on $\mathbb{Z}_+$, resp. $\mathbb{Z}_-$. Define $\mathbf{Y}^+ = \mathbf{Y}^{(1)} + \cdots + \mathbf{Y}^{(n)}$ and $\mathbf{Y}^- = \mathbf{Y}^{(n+1)} + \cdots + \mathbf{Y}^{(\mathrm{d}_0)}$. Using the property $Y_j^{(i)} Q_i = Y_j^{(i)}$, we may view $Y_j^+$, resp. $Y_j^-$, as operators acting from $\mathrm{Im}(I - P) = \mathrm{Im}\, Q_1 \oplus \cdots \oplus \mathrm{Im}\, Q_n$, resp. $\mathrm{Im}\, P = \mathrm{Im}\, Q_{n+1} \oplus \cdots \oplus \mathrm{Im}\, Q_{\mathrm{d}_0}$, to $\mathbb{C}^{\mathrm{d}}$ and thus write $\mathbf{Y}^+ = [\mathbf{Y}^{(1)} \cdots \mathbf{Y}^{(n)}]$, resp. $\mathbf{Y}^- = [\mathbf{Y}^{(n+1)} \cdots \mathbf{Y}^{(\mathrm{d}_0)}]$ either as $(1 \times n)$-, resp. $(1 \times (\mathrm{d}_0 - n))$ block rows, or, cf. (3.8), as $(\mathrm{d} \times \mathrm{d}_2)$-, resp. $(\mathrm{d} \times \mathrm{d}_1)$-matrices.

Assertions (*a*) and (*b*) of Lemma 4.2 and Remark 4.1 show that the solution $\mathbf{Y}^{(i)}$ of the perturbed equation $x_{j+1} = A_j^\times x_j$, $j \in \mathbb{Z}_+$, resp. $\mathbb{Z}_-$, corresponds to the segment $[\varkappa_i'^{+}, \varkappa_i^{+}]$, resp. $[\varkappa_i'^{-}, \varkappa_i^{-}]$, in the Bohl spectrum for $i = 1, \dots, n$, resp. $i = n+1, \dots, \mathrm{d}_0$. In particular, for $i = 1, \dots, n$ one has $\|Y_j^{(i)}\| \to 0$ as $j \to \infty$, and for $i = n+1, \dots, \mathrm{d}_0$ one has $\|Y_j^{(i)}\| \to 0$ as $j \to -\infty$, see (4.4) - (4.5) and recall that $\varkappa_i' \le \varkappa_i'^{\pm} \le \varkappa_i^{\pm} \le \varkappa_i$ by (4.1) - (4.2). Assertion (*c*) shows that the solutions $\mathbf{Y}^{(i)}$ of the perturbed equation are asymptotic to the solutions $(U_j Q_i)_{j\in\mathbb{Z}}$ of the unperturbed equation $x_{j+1} = A_j x_j$, $j \in \mathbb{Z}$, as $j \to \infty$ for $i = 1, \dots, n$, resp. $j \to -\infty$ for $i = n+1, \dots, \mathrm{d}_0$. This motivates the following definition.

DEFINITION 4.4. If $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j\ge 0}$, $i = 1, \dots, n$ and $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j\le 0}$, $i = n+1, \dots, \mathrm{d}_0$ are the matrix solutions of the perturbation $x_{j+1} = A_j^\times x_j$, $A_j^\times = A_j + B_j C_j$, of the difference equation $x_{j+1} = A_j x_j$, then the *Evans determinant*, $\mathcal{E}$, is defined as follows:

(4.18) $\mathcal{E} = \det[Y_0^+ + Y_0^-]$, where $Y_0^+ = Y_0^{(1)} + \cdots + Y_0^{(n)}$ and $Y_0^- = Y_0^{(n+1)} + \cdots + Y_0^{(\mathrm{d}_0)}$.

The terminology is related to the so-called Evans function, a powerful tool frequently used for detecting isolated eigenvalues of differential (and difference) operators that appear after linearizing nonlinear equations about such special solutions as travelling waves, see [AGJ, JK, S] and [BCK, KK], and the bibliographies therein. The Evans function, $\mathcal{D}(z)$, is usually defined, cf. [S], in the situation when the coefficients of the unperturbed and perturbed equations depend (analytically) on a (spectral) parameter $z$. Thus, in our terminology, the values of the Evans function for fixed $z$'s are called the Evans determinants. In Proposition 4.5 we will show that the definition of the Evans determinant $\mathcal{E}$ given in (4.18) coincides with the definition of the Evans determinant $\mathcal{D}$ standardly accepted in the literature on the Evans function, cf. [JK, S]; moreover, the Evans determinant $\mathcal{E}$ gives a canonical choice among the standard Evans determinants $\mathcal{D}$ whose definition in [JK, S] is not unique.

Out of several available (equivalent) standard definitions of $\mathcal{D}$ we chose the definition using the exponential dichotomies on $\mathbb{Z}_+$ and $\mathbb{Z}_-$, cf. [S, Def. 4.1] for the differential equations case. Recall the definition from [S]. Assume that the perturbed equation $x_{j+1} = A_j^\times x_j$ has exponential dichotomies $P_+$ and $P_-$ on $\mathbb{Z}_+$ and $\mathbb{Z}_-$, respectively, so that for its solution $\mathbf{x} = (x_j)$ one has: $\|x_j\| \to 0$ as $j \to \infty$ if and only if $x_0 \in \operatorname{Im}(I - P_+)$; $\|x_j\| \to \infty$ as $j \to \infty$ if and only if $x_0$ has a nonzero component in $\operatorname{Im} P_+$; $\|x_j\| \to 0$ as $j \to -\infty$ if and only if $x_0 \in \operatorname{Im} P_-$; and $\|x_j\| \to \infty$ as $j \to -\infty$ if and only if $x_0$ has a nonzero component in $\operatorname{Im}(I - P_-)$. In addition, following [S], assume that $\dim \operatorname{Im}(I - P_+) = \dim \operatorname{Im}(I - P_-)$ and denote the common value of these dimensions $\mathrm{d}'$. Choose ordered bases $u_1, \ldots, u_{\mathrm{d}'}$ and $u_{\mathrm{d}'+1}, \ldots, u_{\mathrm{d}}$ of $\operatorname{Im}(I - P_+)$ and $\operatorname{Im} P_-$, respectively, and define

$$\mathcal{D} = \det\nolimits_{\mathbb{C}^{\mathrm{d}}}[u_1 \cdots u_{\mathrm{d}}], \tag{4.19}$$

the Evans determinant. Note that $\mathcal{D}$ depends on the choice of the basis vectors $u_i$; however, if $\widetilde{\mathcal{D}}$ is the determinant corresponding to another choice of $\tilde{u}_i$, then $\mathcal{D} = c\widetilde{\mathcal{D}}$ for a nonzero $c$; if $\mathbf{A}^\times = \mathbf{A}^\times(z)$ and $u_i = u_i(z)$ and $\tilde{u}_i = \tilde{u}_i(z)$ depend on $z$ analytically then one can show [S] that $c = c(z)$ is a nonvanishing analytic multiplier.

Proposition 4.5. *Assume (3.1), (3.2), (3.4), and (4.8) - (4.9). Then the Evans determinant $\mathcal{E}$, as defined in (4.18), coincides with $\mathcal{D}$, as defined in (4.19), where $u_1, \ldots u_{\mathrm{d}'}$ and $u_{\mathrm{d}'+1}, \ldots, u_{\mathrm{d}}$ are the columns of the matrices $Y_0^+$ and $Y_0^-$, respectively.*

*Proof.* First, we claim that

$$\dim \operatorname{Im}(I - P) = \dim \operatorname{Im}(I - P_+) \text{ and } \dim \operatorname{Im} P = \dim \operatorname{Im} P_- \tag{4.20}$$

for the dichotomy projection $P$ over $\mathbb{Z}$ of the unperturbed equation $x_{j+1} = A_j x_j$ and the dichotomy projections $P_\pm$ over $\mathbb{Z}_\pm$ of the perturbed equation $y_{j+1} = A_j^\times y_j$. To prove the first equality in (4.20) (the second is proved similarly), we recall that the perturbation $B_j C_j = A_j^\times - A_j$, $j \in \mathbb{Z}$, is assumed to satisfy $(B_j C_j)_{j \in \mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+; \mathbb{C}^{\mathrm{d} \times \mathrm{d}})$, cf. (4.8). Under this assumption, following the proof of [Co, Prop. 4.3], one can see that the (bounded on $\mathbb{Z}_+$) solutions $\mathbf{x} = (x_j)_{j \in \mathbb{Z}_+}$ of the equation $x_{j+1} = A_j x_j$ and $\mathbf{y} = (y_j)_{j \in \mathbb{Z}}$ of the equation $y_{j+1} = A_j^\times y_j$ are in one-to-one correspondence. Indeed, using (3.7), choose $N$ so large that the operator $T$ defined by

$$(T\mathbf{y})_j = -\sum_{k=j}^{\infty} U_j P U_{k+1}^{-1} B_k C_k y_k + \sum_{k=N}^{j-1} U_j (I-P) U_{k+1}^{-1} B_k C_k y_k, \quad j \in \mathbb{Z}_+,$$

is a contraction on $\ell^\infty(\mathbb{Z}_+; \mathbb{C}^{\mathrm{d}\times\mathrm{d}})$, cf. the proof of Lemma 4.2. Then the above-mentioned correspondence is given by the formula $\mathbf{y} = \mathbf{x} + T\mathbf{y}$. Since the solutions $\mathbf{x}$ and $\mathbf{y}$ are bounded on $\mathbb{Z}_+$ if and only if $x_0 \in \operatorname{Im}(I-P)$, resp. $y_0 \in \operatorname{Im}(I-P_+)$, and using notation (3.8), the equality $\mathrm{d}_2 = \mathrm{d}'$ in (4.20) follows.

Next, we claim that for the solutions $\mathbf{Y}^+ = [\mathbf{Y}^{(1)} \cdots \mathbf{Y}^{(n)}]$ and $\mathbf{Y}^- = [\mathbf{Y}^{(n+1)} \cdots \mathbf{Y}^{(\mathrm{d}_0)}]$, see Remark 4.3, of the equation $y_{j+1} = A_j^\times y_j$, obtained in Lemma 4.2 and Remark 4.1, one has the following equalities:

$$\text{(4.21)} \qquad \operatorname{rank} Y_0^+ = \dim \operatorname{Im}(I-P_+) \text{ and } \operatorname{rank} Y_0^- = \dim \operatorname{Im} P_-.$$

As soon as (4.21) is proved, the equality $\mathcal{E} = \mathcal{D}$ follows. Indeed, recalling (4.4) - (4.5), and the inequality $\delta < \varepsilon_0$ for $\varepsilon_0$ defined in (4.7), we use Lemma 4.2(a) to observe that $\|Y_j^{(i)}\| \to 0$ as $j \to \infty$, resp. $j \to -\infty$, for $i = 1, \dots, n$, resp. $i = n+1 \dots, \mathrm{d}_0$. Therefore, recalling (4.20), the definitions of the exponential dichotomies $P_\pm$, see (3.7), and Remark 4.3, we observe that the columns $u_1, \dots, u_{\mathrm{d}_2}$ and $u_{\mathrm{d}_2+1}, \dots, u_{\mathrm{d}}$ of the matrices $Y_0^+$ and, respectively, $Y_0^-$ belong to the subspace $\operatorname{Im}(I-P_+)$, respectively, to the subspace $\operatorname{Im} P_-$. By (4.21), these columns form bases in the respective subspaces, and thus $\mathcal{E} = \mathcal{D}$.

To prove the first equality in (4.21) (the second is proved similarly), we will use (4.20) and will show that $\operatorname{rank} Y_0^+ = \dim \operatorname{Im}(I-P)$. For this, it is enough to check that $\operatorname{rank} Y_j^+ = \dim \operatorname{Im}(I-P)$ for sufficiently large $j$ because $\operatorname{rank} Y_j^+ = \operatorname{rank} Y_0^+$ using $Y_j^+ = U_j^\times Y_0^+$ for the invertible by (3.4) matrices $U_j^\times$ forming the fundamental matrix solution $\mathbf{U}^\times$ of the equation $y_{j+1} = A_j^\times y_j$. Finally, since matrices $U_j$ are also invertible, and $I - P = Q_1 + \cdots + Q_n$, we have:

$$\dim \operatorname{Im}(I-P) = \operatorname{rank}[Q_1 \cdots Q_n] = \operatorname{rank}[U_j Q_1 \cdots U_j Q_n] = \operatorname{rank}[Y_j^{(1)} \cdots Y_j^{(n)}] = \operatorname{rank} Y_j^+,$$

where the relation $\|Y_j^{(i)} - U_j Q_i\| \to 0$ as $j \to \infty$, $i = 1, \dots, n$, from Lemma 4.2 (c) has been used to justify the third equality. Thus, (4.21) follows, and $\mathcal{E} = \mathcal{D}$ is proved. ∎

One advantage of Definition 4.4 adapted in the current paper is that the Evans determinant given in (4.18) is uniquely defined by $\mathbf{A}$ and $\mathbf{A}^\times$ (indeed, the finest exponential splitting $\{Q_i\}_{i=1}^{\mathrm{d}_0}$ is uniquely defined; although the solutions $\mathbf{Y}^{(i)}$ depend on $N$, see Lemma 4.2, the determinant (4.18) is proved below to be $N$-independent, see (4.22)). In other words, the columns of the matrices $Y_0^+$ and $Y_0^-$ give the canonical choice of the bases in $\operatorname{Im}(I-P_+)$ and $\operatorname{Im} P_-$ needed in (4.19). Also, if $\mathbf{A} = \mathbf{A}(z)$ and $\mathbf{A}^\times = \mathbf{A}^\times(z)$ depend on $z$ analytically, then the analyticity of the Evans function $\mathcal{E} = \mathcal{E}(z)$, where the Evans determinant $\mathcal{E}(z)$ is defined for each fixed $z$ using $\mathbf{A}(z)$ and $\mathbf{A}^\times(z)$ as indicated in (4.18), follows automatically from the analyticity of the corresponding Fredholm determinant whose connection to the Evans determinant is given next.

THEOREM 4.6. *Assume that $(B_j)_{j\in\mathbb{Z}}$, $(C_j)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z}; \mathbb{C}^{\mathrm{d}\times\mathrm{d}})$, and (3.1), (3.2), (3.4), (4.8), (4.9) hold. Then the Fredholm determinant for the operator $\mathcal{T} \in \mathcal{B}_1(\ell^2(\mathbb{Z}; \mathbb{C}^{\mathrm{d}\times\mathrm{d}}))$,* *defined in (1.1), and the Evans determinant (4.18) are related as follows:*

$$\det(I-\mathcal{T}) = \mathcal{E}\prod_{j=0}^{\infty}\frac{\det(A_j+B_jC_j)}{\det A_j}. \tag{4.22}$$

REMARK 4.7. In this paper we study the perturbed equation $x_{j+1}=(A_j+B_jC_j)x_j$ with the perturbation term $R_j=B_jC_j$ having a *predefined* factorization. Given a perturbed equation $x_{j+1}=(A_j+R_j)x_j$, we can *define* the factorization $R_j=B_jC_j$ by using the polar decomposition $R_j=V_{R_j}|R_j|$ and setting $B_j=V_{R_j}|R_j|^{\frac{1}{2}}$ and $C_j=|R_j|^{\frac{1}{2}}$. If (4.8) - (4.9) are assumed for $R_j=B_jC_j$ then $(R_j)_{j\in\mathbb{Z}}\in\ell^1$ and $(B_j)_{j\in\mathbb{Z}}$, $(C_j)_{j\in\mathbb{Z}}\in\ell^2$.

*Proof.* First, we remark that $\mathcal{T}\in\mathcal{B}_1(\ell^2)$ provided $\mathbf{b}=(B_j)_{j\in\mathbb{Z}}$ and $\mathbf{c}=(C_j)_{j\in\mathbb{Z}}$ belong to $\ell^2$. Indeed, let $\mathcal{K}, (\mathcal{K}\mathbf{x})_j=\sum_{k\in\mathbb{Z}}K_{jk}x_k$, denote the (bounded on $\ell^2$) operator with the kernel given by (1.2) and satisfying $\|K_{jk}\|\le ce^{-\alpha|k-j|}$, see (3.7). Then $\mathcal{T}=D_{\mathbf{c}}\mathcal{K}D_{\mathbf{b}}$ using notation (2.6). By (2.8) we have $D_{\mathbf{c}}, D_{\mathbf{b}}\in\mathcal{B}_2(\ell^2)$ and thus $\mathcal{T}\in\mathcal{B}_1$, see, e.g. [GGK, Lem. IV.7.2]. Next, we remark that the product in the right-hand side of (4.22) converges absolutely provided (3.1) and $\mathbf{b},\mathbf{c}\in\ell^2$ hold:

$$\prod_{j=0}^{\infty}\left|\frac{\det(A_j+B_jC_j)}{\det A_j}\right| = \prod_{j=0}^{\infty}|\det(I+A_j^{-1}B_jC_j)| \le \prod_{j=0}^{\infty}(1+\mathrm{d}\|(A_j^{-1})_{j\in\mathbb{Z}}\|_{\ell^\infty}\|B_jC_j\|)<\infty$$

because $(B_jC_j)_{j\in\mathbb{Z}}\in\ell^1$.

Further, we *claim* that it is enough to prove (4.22) for finitely supported $\mathbf{b}$ and $\mathbf{c}$. For $M\in\mathbb{N}$ let $\chi_M$ denote the characteristic function of $[-M,M]\cap\mathbb{Z}$, and set $\mathbf{b}^{(M)}=(\chi_M(j)B_j)_{j\in\mathbb{Z}}$, $\mathbf{c}^{(M)}=(\chi_M(j)C_j)_{j\in\mathbb{Z}}$, $\mathcal{T}^{(M)}=D_{\mathbf{c}^{(M)}}\mathcal{K}D_{\mathbf{b}^{(M)}}$. By (2.16), $D_{\mathbf{c}^{(M)}}\to D_{\mathbf{c}}$ and $D_{\mathbf{b}^{(M)}}\to D_{\mathbf{b}}$ in $\mathcal{B}_2(\ell^2)$-norm as $M\to\infty$. Using e.g. [GGK, Lem. IV.7.2], we conclude that $\mathcal{T}^{(M)}\to\mathcal{T}$ in $\mathcal{B}_1(\ell^2)$-norm and hence $\det(I-\mathcal{T}^{(M)})\to\det(I-\mathcal{T})$ as $M\to\infty$, see e.g. [GGK, (IV.5.14)]. Let $\mathbf{d}_+=(A_j^{-1}B_jC_j)_{j\ge0}$ and $\mathbf{d}_+^{(M)}=(\chi_M(j)A_j^{-1}B_jC_j)_{j\ge0}$. Then the product in the right-hand side of (4.22) is equal to $\det(I+D_{\mathbf{d}_+})=\lim_{M\to\infty}\det(I+D_{\mathbf{d}_+^{(M)}})$ because $D_{\mathbf{d}_+^{(M)}}\to D_{\mathbf{d}_+}$ in $\mathcal{B}_1(\ell^2(\mathbb{Z}_+;\mathbb{C}^{\mathrm{d}}))$-norm, as above. Thus, to prove the claim it remains to show that

$$\mathcal{E}=\lim_{M\to\infty}\mathcal{E}^{(M)} \tag{4.23}$$

where $\mathcal{E}^{(M)}$ is defined as in (4.18) using the solutions $\mathbf{Y}^{(i,M)}$ of equations (4.10) - (4.11) with $B_j$ and $C_j$ replaced by $B_j^{(M)}=\chi_M(j)B_j$ and $C_j^{(M)}=\chi_M(j)C_j$. Note that $q^{(M)}\le q$, cf. (4.15). Thus, $N$ in Lemma 4.2 could be fixed independent of $M$. Note that for $i=1,\dots,n$ (and similarly for $i=n+1,\dots,\mathrm{d}_0$) we have $\mathbf{Y}^{(i)}=(I-T)^{-1}(U_jQ_i)_{j>N}$ and $\mathbf{Y}^{(i,M)}=(I-T^{(M)})^{-1}(U_jQ_i)_{j>N}$ for the operators $T$ and $T^{(M)}$ defined by the right-hand side of (4.10), and that $\|T\|<q<1$ and $\|T^{(M)}\|<q$ uniformly for $M\in\mathbb{N}$, for the operator norm in $l^\infty_{+,\beta}$, cf. (4.16). Also, $U_j^{\times,M}\to U_j^{\times}$ as $M\to\infty$ uniformly for $|j|\le N$, cf. Remark 4.1. Thus, (4.23) is proved as soon as we show that $\|T-T^{(M)}\|\to0$ as $M\to\infty$. But $T-T^{(M)}$ is again given by the same expression as $T$, see (4.10), except the product $B_kC_k$ should be replaced by $B_kC_k-B_k^{(M)}C_k^{(M)}=(1-\chi_M(k))B_kC_k$. We can now use (4.16) to estimate for $i=1,\dots,n$ the operator norm of $T-T^{(M)}$ in $l^\infty_{+,\beta}$:

$$
\begin{aligned}
(4.24)\qquad \|T-T^{(M)}\| &\le \max\{c_\alpha, c'_\alpha\}\sum_{k=N}^{\infty}(1-\chi_M(k))e^{(\varkappa_i^+-\varkappa_i'^{\,+}+\delta)k}\|B_kC_k\| \\
&= \max\{c_\alpha, c'_\alpha\}\sum_{k=M+1}^{\infty} e^{(\varkappa_i^+-\varkappa_i'^{\,+}+\delta)k}\|B_kC_k\| \to 0 \text{ as } M\to\infty,
\end{aligned}
$$

using (4.8). A similar argument works for $i = n+1,\dots,\mathrm{d}_0$. Thus, (4.23) holds, and the claim above is verified.

From now on we therefore assume that $(B_j)_{j\in\mathbb{Z}}$ and $(C_j)_{j\in\mathbb{Z}}$ are finitely supported. Note that then assumption (3.3) holds, and thus (3.36) holds. So, to establish (4.22) we need to prove that $\mathcal{E} = \det\mathcal{U}_0^\times$, where $\mathcal{U}_0^\times = V_0$ is given by (3.35) in terms of the solutions $\mathbf{X}^\pm$ of difference equations (3.29) - (3.30).

Consider the following matrix difference equations:

$$
(4.25)\qquad Z_j^+ - (I-P) = -\sum_{k=j}^{\infty} U_{k+1}^{-1}B_kC_kU_kZ_k^+,
$$

$$
(4.26)\qquad Z_j^- - P = \sum_{k=-\infty}^{j-1} U_{k+1}^{-1}B_kC_kU_kZ_k^-,
$$

and introduce on $\ell^\infty(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}})$ operators $\widetilde{H}_+$ and $\widetilde{H}_-$ corresponding to the right-hand sides of (4.25) - (4.26) so that these equations become $(I-\widetilde{H}_+)\mathbf{Z}^+ = (I-P)_{j\in\mathbb{Z}}$ and $(I-\widetilde{H}_-)\mathbf{Z}^- = (P)_{j\in\mathbb{Z}}$. We claim that the operators $I-\widetilde{H}_\pm$ are invertible on $\ell^\infty(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}})$. Indeed, using notation (2.5) - (2.6) we have $\widetilde{H}_+ = -D_{\mathbf{d}} - V^+_{\mathbf{a},\mathbf{b}}$ and $\widetilde{H}_- = V^-_{\mathbf{a},\mathbf{b}}$ with $\mathbf{a} = (U_{j+1}^{-1}B_j)_{j\in\mathbb{Z}}\in\ell^1$, $\mathbf{b} = (C_jU_j)_{j\in\mathbb{Z}}\in\ell^1$, and $\mathbf{d} = (U_{j+1}^{-1}B_jC_jU_j)_{j\in\mathbb{Z}}\in\ell^1$. The operator $I+D_{\mathbf{d}}$ is invertible in $\ell^\infty(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}})$ since the operator $I+D_{\mathbf{d}'}, \mathbf{d}' = (B_jC_jU_jU_{j+1}^{-1})_{j\in\mathbb{Z}}$ in invertible in $\ell^\infty(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}})$ by assumptions (3.1) and (3.4) and the identities $I+B_jC_jU_jU_{j+1}^{-1} = A_j^\times A_j^{-1}, j\in\mathbb{Z}$ (recall that $A_j^\times = A_j$ for sufficiently large $|j|$ because $(B_j)_{j\in\mathbb{Z}}$ and $(C_j)_{j\in\mathbb{Z}}$ are finitely supported). Moreover, $(I+D_{\mathbf{d}})^{-1} = D_{\mathbf{d}''}$ with some $\mathbf{d}'' = (d_j'')_{j\in\mathbb{Z}}\in\ell^\infty(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}})$. Since

$$
I-\widetilde{H}_+ = (I+D_{\mathbf{d}})(I+(I+D_{\mathbf{d}})^{-1}V^+_{\mathbf{a},\mathbf{b}}) = (I+D_{\mathbf{d}})(I+V^+_{\mathbf{a}',\mathbf{b}})
$$

with $\mathbf{a}' = (d_j''U_{j+1}^{-1}B_j)_{j\in\mathbb{Z}}\in\ell^1$, the operators $I-\widetilde{H}_+$ and $I-\widetilde{H}_-$ are invertible by (2.12), as claimed. Therefore, equations (4.25) - (4.26) have a unique pair of solutions $\mathbf{Z}^\pm\in\ell^\infty(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}})$ given by

$$
(4.27)\qquad \mathbf{Z}^+ = (I-\widetilde{H}_+)^{-1}(I-P)_{j\in\mathbb{Z}} \text{ and } \mathbf{Z}_- = (I-\widetilde{H}_-)^{-1}(P)_{j\in\mathbb{Z}}.
$$

In particular $Z_j^+(I-P) = Z_j^+$ and $Z_j^-P = Z_j^-$, $j\in\mathbb{Z}$. Also, using (4.25) - (4.26), it is easy to see that $(U_jZ_j^\pm)_{j\in\mathbb{Z}}$ are solutions of the equation $x_{j+1} = A_j^\times x_j$, $j\in\mathbb{Z}$. Thus, if $(U_j^\times)_{j\in\mathbb{Z}}$ is the fundamental matrix solution of the equation $x_{j+1} = A_j^\times x_j$, then the following relations hold:

$$
(4.28)\qquad Z_0^+ = (U_{N+1}^\times)^{-1}(U_{N+1}Z_{N+1}^+) \text{ and } Z_0^- = (U_{-N}^\times)^{-1}(U_{-N}Z_{-N}^-)
$$

(recall that $U_0 = U_0^\times = I$, and $N$ here is chosen as in Lemma 4.2). On the other hand, multiplying (4.25) - (4.26) by $C_jU_j$ from the left, letting $X_j^\pm = C_jU_jZ_j^\pm$ and recalling that $(C_jU_j)_{j\in\mathbb{Z}}$ is finitely supported, we conclude that $\mathbf{X}^\pm = (X_j)_{j\in\mathbb{Z}}\in\ell^2(\mathbb{Z};\mathbb{C}^{\mathrm{d}\times\mathrm{d}})$ is the unique pair of solutions of equations (3.29) - (3.30). Using the direct sum decomposition $\mathbb{C}^{\mathrm{d}} = \mathrm{Im}(I-P) \oplus \mathrm{Im}\, P$, and writing $Z_j^+ + Z_j^-$, $j \in \mathbb{Z}$, as a $(2\times 2)$-block matrix, we observe that $Z_j^+ + Z_j^- = V_j$, where $V_j$ is given in (3.35) in terms of $\mathbf{X}^{\pm}$. So, to complete the proof of the theorem we need to show that $\mathcal{E} = \det[Z_0^+ + Z_0^-]$.

Since $(U_{j+1}^{-1}B_j)_{j\in\mathbb{Z}}$ and $(C_jU_j)_{j\in\mathbb{Z}}$ are finitely supported, equations (4.10) - (4.11) can be equivalently rewritten as follows:

$$\begin{aligned} Y_j^{(i)} &= U_jQ_i - \sum_{k=j}^{\infty} U_jU_{k+1}^{-1}B_kC_kY_k^{(i)} \\ &\quad + \sum_{k=N}^{\infty} U_j(I-(Q_i+\cdots+Q_{\mathrm{d}_0}))U_{k+1}^{-1}B_kC_kY_k^{(i)}, \quad j > N, \end{aligned} \tag{4.29}$$

$$\begin{aligned} Y_j^{(i)} &= U_jQ_i + \sum_{k=-\infty}^{j-1} U_jU_{k+1}^{-1}B_kC_kY_k^{(i)} \\ &\quad - \sum_{k=-\infty}^{-N} U_j(I-(Q_1+\cdots+Q_i))U_{k+1}^{-1}B_kC_kY_k^{(i)}, \quad j \le -N, \end{aligned} \tag{4.30}$$

Recall that $Y_j^{(i)}Q_i = Y_j^{(i)}$ by Lemma 4.2, and that $I - P = Q_1 + \cdots + Q_n$, $P = Q_{n+1} + \cdots + Q_{\mathrm{d}_0}$, so that $(I-P)Q_i = Q_i$ for $i = 1, \ldots, n$ and $PQ_i = Q_i$ for $i = n+1, \ldots, \mathrm{d}_0$. Using notation $Y_j^+ = \sum_{i=1}^{n} Y_j^{(i)}$ and $Y_j^- = \sum_{i=n+1}^{\mathrm{d}_0} Y_j^{(i)}$, we derive from (4.29) - (4.30):

$$Y_j^+ = U_j(I-P) - \sum_{k=j}^{\infty} U_jU_{k+1}^{-1}B_kC_kY_k^+ + U_jT_+, \quad j > N, \tag{4.31}$$

$$Y_j^- = U_jP + \sum_{k=-\infty}^{j-1} U_jU_{k+1}^{-1}B_kC_kY_k^- + U_jT_-, \quad j \le -N, \tag{4.32}$$

where we have denoted:

$$T_+ = \sum_{i=1}^{n}(I-(Q_i+\cdots Q_{\mathrm{d}_0}))\eta_iQ_i, \quad \eta_i = \sum_{k=N}^{\infty} U_{k+1}^{-1}B_kC_kY_k^{(i)}, \tag{4.33}$$

$$T_- = \sum_{i=n+1}^{d_0}(I-(Q_1+\cdots Q_i))\eta_iQ_i, \quad \eta_i = \sum_{k=-\infty}^{-N} U_{k+1}^{-1}B_kC_kY_k^{(i)}. \tag{4.34}$$

We remark that, in the direct sum decomposition $\mathbb{C}^{\mathrm{d}} = \mathrm{Im}(I-P) \oplus \mathrm{Im}\, P$,

$$T_+ = (I-P)T_+(I-P), \quad T_- = PT_-P, \quad \text{and} \quad T_+ + T_- = T_+ \oplus T_-. \tag{4.35}$$

Using the operators $\widetilde{H}_{\pm}$, we can also rewrite (4.31) - (4.32) as follows:

$$\begin{aligned} (I-\widetilde{H}_+)(U_j^{-1}Y_j^+)_{j>N} &= ((I-P)(I-P+T_+))_{j>N}, \\ (I-\widetilde{H}_-)(U_j^{-1}Y_j^-)_{j\le -N} &= (P(P+T_-))_{j\le -N}. \end{aligned}$$

Comparison with (4.27) yields:

$$Y_j^+ = U_jZ_j^+(I-P+T_+),\ j > N, \quad \text{and} \quad Y_j^- = U_jZ_j^-(P+T_-),\ j \le -N.$$

Formulas (4.12) and (4.28) with the fundamental matrix solution $(U_j^\times)_{j\in\mathbb{Z}}$ of $x_{j+1} = A_j^\times x_j$ then imply:

$$\begin{aligned} Y_0^+ + Y_0^- &= (U_{N+1}^\times)^{-1} Y_{N+1}^+ + (U_{-N}^\times)^{-1} Y_{-N}^- \\ &= (U_{N+1}^\times)^{-1}(U_{N+1} Z_{N+1}^+ (I - P + T_+)) + (U_{-N}^\times)^{-1}(U_{-N} Z_{-N}^- (P + T_-)) \\ &= Z_0^+(I - P + T_+) + Z_0^-(P + T_-) = (Z_0^+ + Z_0^-)(I + T_+ + T_-). \end{aligned}$$

In the last equality we also used the fact that $Z_0^+ = Z_0^+(I - P)$ and $Z_0^- = Z_0^- P$, and assertions (4.35). Since $\mathcal{E} = \det[Y_0^+ + Y_0^-]$ by (4.18), to finish the proof of the identity $\mathcal{E} = \det[Z_0^+ + Z_0^-]$ (and thus the proof of the theorem), it suffices to prove that

$$\det[I + T_+ + T_-] = 1. \tag{4.36}$$

For this, cf. (4.35), we note that the operators $T_+ = (I-P)T_+(I-P)$ on $\operatorname{Im}(I-P)$ and, resp. $T_- = PT_-P$ on $\operatorname{Im} P$ are represented by lower-, resp. upper-triangular matrices with zero main diagonals in the direct sum decomposition $\operatorname{Im}(I - P) = \operatorname{Im} Q_1 \oplus \cdots \oplus \operatorname{Im} Q_n$, resp. $\operatorname{Im} P = \operatorname{Im} Q_{n+1} \oplus \cdots \oplus \operatorname{Im} Q_{\mathrm{d}_0}$ (cf. (4.33) - (4.34)). Thus, the proof of (4.36) and the theorem is completed. ∎

**5. Constant coefficients.** In this section we specialize to the case where $A_j \equiv A, j \in \mathbb{Z}$, with a given matrix $A \in \mathbb{C}^{\mathrm{d}\times\mathrm{d}}$ satisfying the assumptions

$$0 \notin \sigma(A) \quad \text{and} \quad \sigma(A) \cap \{z \in \mathbb{C} : |z| = 1\} = \emptyset, \tag{5.1}$$

cf. assumptions (3.1) - (3.2). We will continue to assume that $A_j^\times = A + B_j C_j$ is invertible for each $j \in \mathbb{Z}$, cf. (3.4). Note that $U_j = A^j, j \in \mathbb{Z}$. We will show that in this constant coefficients case assumptions (4.8) - (4.9) of the exponential decay of the perturbation could be replaced with a weaker assumption of a polynomial decay. Specifically, let $m$ denote the maximal size of Jordan blocks in the Jordan canonical form of the $(\mathrm{d} \times \mathrm{d})$-matrix $A$. (Then $m = 0$ provided all eigenvalues of $A$ are semi-simple, that is, when all Jordan blocks are diagonal.) The following assumption will replace (4.8) - (4.9):

$$\sum_{k=-\infty}^{\infty} |k|^{2m} \|B_k C_k\| < \infty. \tag{5.2}$$

Let us decompose $\sigma(A) = \cup_{i=1}^{\mathrm{d}_0} \sigma_i$ where $\sigma_i$'s belong to concentric circles whose radii are denoted by $e^{\varkappa_i}$ so that $\sigma_i = \{\lambda \in \sigma(A) : |\lambda| = e^{\varkappa_i}\}$, and enumerate the numbers $\varkappa_i$ so that

$$\varkappa_1 < \cdots < \varkappa_n < 0 < \varkappa_{n+1} < \varkappa_{\mathrm{d}_0} \tag{5.3}$$

for some $n \in \{1, \dots, \mathrm{d}_0\}$. Let $Q_i$ denote the spectral projection for $A$ such that $\sigma(A|_{\operatorname{Im} Q_i}) = \sigma_i$. Note that $\varkappa_i = \varkappa_g(Q_i;\mathbb{Z}) = \varkappa_g'(Q_i;\mathbb{Z}) = \varkappa_i'^{\pm} = \varkappa_i^{\pm}$ for the Bohl exponents, cf. (4.6). Passing to appropriate coordinates, we will assume that $A$ is in the Jordan normal form. Thus, each $A|_{\operatorname{Im} Q_i}$ is a direct sum of (maybe, several) Jordan blocks $\lambda I + E$ with $|\lambda| = e^{\varkappa_i}$. By Lemma 2.3,

$$\|A^j|_{\operatorname{Im} Q_i}\| \le c|j|^m e^{j\varkappa_i}, \quad i = 1, \dots, \mathrm{d}_0, \quad j \in \mathbb{Z}. \tag{5.4}$$

Equations (4.10) - (4.11) now become:

$$\mathbf{Y}^{(i)} - (A^j Q_i)_{j>N} = T_i \mathbf{Y}^{(i)}, \quad i = 1, \dots, n, \tag{5.5}$$

$$\mathbf{Y}^{(i)} - (A^j Q_i)_{j\le -N} = T_i \mathbf{Y}^{(i)}, \quad i = n+1, \dots, \mathrm{d}_0, \tag{5.6}$$

where

$$\begin{aligned}(T_i\mathbf{u})_j &= -\sum_{k=j}^{\infty} A^{j-k-1}(Q_i + \dots + Q_{\mathrm{d}_0})B_k C_k u_k \\ &\quad + \sum_{k=N}^{j-1} A^{j-k-1}(I - (Q_i + \dots + Q_{\mathrm{d}_0}))B_k C_k u_k, \quad \mathbf{u} = (u_j)_{j>N}, i = 1, \dots, n,\\ (T_i\mathbf{u})_j &= \sum_{k=-\infty}^{j-1} A^{j-k-1}(Q_1 + \dots + Q_i)B_k C_k u_k \\ &\quad - \sum_{k=j}^{-N} A^{j-k-1}(I - (Q_1 + \dots + Q_i))B_k C_k u_k, \quad \mathbf{u} = (u_j)_{j\le -N}, i = n+1, \dots, \mathrm{d}_0.\end{aligned} \tag{5.7}$$

LEMMA 5.1. *Assume (5.1) - (5.2). Then for a sufficiently large $N$ there exist solutions $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j>N}$, $i = 1, \dots, n$, resp. $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j\le -N}$, $i = n+1, \dots, \mathrm{d}_0$, of (5.5), resp. (5.6) such that $Y_j^{(i)} = Y_j^{(i)} Q_i$ and the following assertions hold:*

$$\begin{aligned} &(a) \quad \sup_{j>N} |j|^{-m} e^{-j\varkappa_i} \|Y_j^{(i)}\| < \infty, \quad i = 1, \dots, n,\\ &\qquad \sup_{j\le -N} |j|^{-m} e^{-j\varkappa_i} \|Y_j^{(i)}\| < \infty, \quad i = n+1, \dots, \mathrm{d}_0,\\ &(b) \quad e^{-j\varkappa_i}\|Y_j^{(i)} - A^j Q_i\| \to 0 \\ &\qquad \textit{as } j \to \infty \textit{ for } i = 1, \dots, n \textit{ and as } j \to -\infty \textit{ for } i = n+1, \dots, \mathrm{d}_0.\end{aligned} \tag{5.8}$$

*Proof.* We prove the lemma for $i = n+1, \dots, \mathrm{d}_0$; the proof for $i = 1, \dots, n$ is similar. Using (5.3) - (5.4), we infer:

$$\|A^{j-k-1}(Q_1 + \dots + Q_i)\| \le c|j-k|^m e^{(j-k)\varkappa_i}, \quad 0 \ge j \ge k, \tag{5.9}$$

$$\begin{aligned}\|A^{j-k-1}(I - (Q_1 + \dots + Q_i))\| &\le c|j-k|^m e^{(j-k)\varkappa_{i+1}} \\ &\le c'|j-k|^m e^{(j-k)\varkappa_i}, \quad 0 \ge k \ge j.\end{aligned} \tag{5.10}$$

Introduce the Banach space $\ell_-^\infty = \{\mathbf{u} = (u_j)_{j\le -N} : \|u\|_{\ell_-^\infty} := \sup_{j\le -N} |j|^{-m} e^{-j\varkappa_i}\|u_j\| < \infty\}$. Denote $q_N = \sum_{k=-\infty}^{-N} |k|^{2m}\|B_k C_k\|$ and remark that $q_N \to 0$ as $N \to \infty$ by (5.2). Using (5.9) - (5.10) and the fact that $\sup_{k\ge 0} |k|^m e^{-k(\varkappa_{i+1}-\varkappa_i)} < \infty$, we have for $j \le -N$ and the operator $T = T_i(N)$ defined in (5.7):

$$\begin{aligned}|j|^{-m} e^{-j\varkappa_i}\|(T\mathbf{u})_j\| &\le c\|\mathbf{u}\|_{\ell_-^\infty}\Big(\sum_{k=-\infty}^{j-1} |j^{-1}(j-k)k|^m \|B_k C_k\| \\ &\quad + \sum_{k=j}^{-N}(|j-k|^m e^{(j-k)(\varkappa_{i+1}-\varkappa_i)})|j^{-1}k|^m\|B_k C_k\|\Big) \le c' q_N \|\mathbf{u}\|_{\ell_-^\infty}.\end{aligned} \tag{5.11}$$

Note that $|k| \geq |j| \geq N \geq 1$ in the first sum and $|j| \geq |k| \geq N \geq 1$ in the second sum. Thus, $\|T\mathbf{u}\|_{\ell^\infty_-} \leq cq_N\|\mathbf{u}\|_{\ell^\infty_-}$, and $T = T_i(N)$ is a contraction on $\ell^\infty_-$ for a sufficiently large $N$. Since $(A^jQ_i)_{j\leq -N} \in \ell^\infty_-$ by (5.4), assertion (5.8) follows. To show $(b)$ in the lemma, we follow the proof of the celebrated Levinson Theorem in [E, p.14]. Indeed, for $j < -M < -N$ and $\mathbf{u} = (Y_j^{(i)})_{j\leq -N} \in \ell^\infty_-$, similarly to (5.11), we infer:

$$\begin{aligned}
&e^{-j\varkappa_i}\|(T\mathbf{u})_j\| \leq \|e^{-j\varkappa_i}(T_i(M)\mathbf{u})_j\| \\
&\quad + \sum_{k=-M+1}^{-N} \|A^{j-k-1}(I-(Q_1+\cdots+Q_i))B_kC_ku_k\| \\
&\leq c\|\mathbf{u}\|_{\ell^\infty_-}\Big(\sum_{k=-\infty}^{j-1} |(j-k)k|^m\|B_kC_k\| + \sum_{k=j}^{-M} |j-k|^m e^{(j-k)(\varkappa_{i+1}-\varkappa_i)}|k|^m\|B_kC_k\| \\
&\quad + \sum_{k=-M+1}^{-N} (|j-k|^m e^{\frac{1}{2}(j-k)(\varkappa_{i+1}-\varkappa_i)})e^{\frac{1}{2}(j-k)(\varkappa_{i+1}-\varkappa_i)}|k|^m\|B_kC_k\|\Big) \\
&\leq c'\|\mathbf{u}\|_{\ell^\infty_-}(q_M + c(M,N)e^{\frac{1}{2}j(\varkappa_{i+1}-\varkappa_i)}).
\end{aligned}$$

Choosing first a sufficiently large $M$, and then letting $j \to -\infty$, we have $(b)$ in the lemma. ■

As soon as the existence of the solutions of (5.5) - (5.6) is established, we can use formulas (4.12) to obtain the matrix solutions $\mathbf{Y}^+ = \mathbf{Y}^{(1)} + \cdots + \mathbf{Y}^{(n)}$ on $\mathbb{Z}_+$ and $\mathbf{Y}^- = \mathbf{Y}^{(n+1)} + \cdots + \mathbf{Y}^{(\mathrm{d}_0)}$ on $\mathbb{Z}_-$ of the perturbed equation $x_{j+1} = (A + B_jC_j)x_j$, and then define the Evans determinant, $\mathcal{E}$, as indicated in (4.18). Note that the equality $\mathcal{E} = \mathcal{D}$ for the "standard" Evans determinant $\mathcal{D}$ also holds as in Proposition 4.5 since $\mathbf{Y}^\pm$ enjoy the properties listed in Lemma 5.1. The proof of Theorem 4.6 remains unchanged with a natural replacement of the exponential term in (4.24) by $k^{2m}$. Thus, we have the following result.

PROPOSITION 5.2. *Formula (4.22) holds provided $A = A_j$, $j \in \mathbb{Z}$, assumption (3.4) is satisfied, and assumptions (3.1) - (3.2) and (4.8) - (4.9) in Theorem 4.6 are replaced, respectively, by (5.1) and (5.2).*

Next, as an illustration, we will consider a particularly important class of second order difference equations, the discrete Schrödinger equation, and show how to specialize our results for the corresponding $(2\times 2)$ first order system. Given a real-valued potential $\mathbf{v} = (v_j)_{j\in\mathbb{Z}} \in \ell^1(\mathbb{Z};\mathbb{R})$, consider on $\ell^2(\mathbb{Z};\mathbb{C})$ a bounded self-adjoint operator, $L$, defined by $(L\mathbf{y})_j = y_{j+1} + y_{j-1} + v_jy_j$, $\mathbf{y} \in \ell^2(\mathbb{Z};\mathbb{C})$. If $(\mathcal{S}\mathbf{y})_j = y_{j-1}$ denotes the shift operator, and $z \in \mathbb{R}$, then the following identity, cf. [LS, Exmp. 1.6], holds on $(\ell^2(\mathbb{Z};\mathbb{C}))^2 = \ell^2(\mathbb{Z};\mathbb{C}^2)$ for the operator $L = \mathcal{S} + \mathcal{S}^{-1} + D_{\mathbf{v}}$, cf. (2.6):

$$\begin{bmatrix} 0 & I \\ I & \mathcal{S} \end{bmatrix}\begin{bmatrix} L - zI & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 & \mathcal{S} \\ I & -\mathcal{S} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & -I \\ I & D_{\mathbf{v}} - zI \end{bmatrix}\begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{S} \end{bmatrix}. \tag{5.12}$$

Writing $v_j = e^{\mathrm{i}\arg v_j}|v_j|$, and introducing $(2\times 2)$ matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & z \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -e^{\mathrm{i}\arg v_j}|v_j|^{\frac{1}{2}} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & |v_j|^{\frac{1}{2}} \end{bmatrix}, \tag{5.13}$$

and $A_j^\times = A + B_jC_j$, we observe from (5.12) that the operator $L - zI$ has a bounded inverse on $\ell^2(\mathbb{Z};\mathbb{C})$ if and only if the operator $I - D_{\mathbf{A}^\times}\mathcal{S}$, $\mathbf{A}^\times = (A_j^\times)_{j\in\mathbb{Z}}$, has a bounded inverse on $\ell^2(\mathbb{Z};\mathbb{C}^2)$ since the $(2\times 2)$ operator matrices containing $\mathcal{S}$ in the left-hand side of (5.12) are invertible operators on $\ell^2(\mathbb{Z};\mathbb{C}^2)$, and the right-hand side of (5.12) is $I - D_{\mathbf{A}^\times}\mathcal{S}$. Recalling the operators $G_{\mathbf{A}}$ and $G_{\mathbf{A}^\times}$, see the Introduction, corresponding to the difference equations $x_{j+1} = Ax_j$ and $x_{j+1} = A_j^\times x_j$, $x_j \in \mathbb{C}^2$, $j \in \mathbb{Z}$, by the formulas $(G_{\mathbf{A}}\mathbf{x})_j = x_{j+1} - Ax_j$ and $(G_{\mathbf{A}^\times}\mathbf{x})_j = x_{j+1} - A_j^\times x_j$, observe that $(I - D_{\mathbf{A}^\times}\mathcal{S})\mathcal{S}^{-1} = G_{\mathbf{A}^\times}$. Note that $\det A_j^\times = 1$ and thus assumption (3.4) holds. The eigenvalues of $A = A(z)$ will be denoted by $\lambda = \lambda(z)$ and $\lambda^{-1}$; these are the roots of the equation $\lambda^2 - z\lambda + 1 = 0$. If $|z| \le 2$ then $|\lambda| = 1$; if $|z| > 2$ we choose $\lambda$ so that $|\lambda| < 1$ and denote $\Lambda = \lambda^{-1} - \lambda$. Thus, if $|z| > 2$ then $A = A(z)$ satisfies assumptions (5.1). Then $G_{\mathbf{A}}$ is invertible on $\ell(\mathbb{Z};\mathbb{C}^2)$, and the kernel of the operator $\mathcal{K} = G_{\mathbf{A}}^{-1}$ is given by (1.2) where $U_j = A^j$ and $P = P(z)$ is the spectral projection for $A$ so that $\sigma(A|_{\operatorname{Im} P}) = \{\lambda^{-1}\}$, explicitly, $P = \frac{1}{\Lambda}\begin{bmatrix} -\lambda & 1 \\ -1 & \lambda^{-1}\end{bmatrix}$. Using the identity $G_{\mathbf{A}^\times} = G_{\mathbf{A}}(I - G_{\mathbf{A}}^{-1}D_{\mathbf{B}}D_{\mathbf{C}})$ with $\mathbf{B} = (B_j)_{j\in\mathbb{Z}} \in \ell^2$ and $\mathbf{C} = (C_j)_{j\in\mathbb{Z}} \in \ell^2$, recalling that $\mathcal{T} = \mathcal{T}(z)$ from (1.1) satisfies $\mathcal{T} = D_{\mathbf{C}}G_{\mathbf{A}}^{-1}D_{\mathbf{B}}$, and noting that the operators $I - G_{\mathbf{A}}^{-1}D_{\mathbf{B}}D_{\mathbf{C}}$ and $I - \mathcal{T}$ are invertible at the same time, we have the following corollary of Theorem 4.6.

Proposition 5.3. *Assume* $\mathbf{v} = (v_j)_{j\in\mathbb{Z}} \in \ell^1(\mathbb{Z};\mathbb{R})$. *If* $|z| > 2$ *then* $z \in \sigma(L)$ *on* $\ell^2(\mathbb{Z};\mathbb{C})$ *if and only if* $\mathcal{E}(z) = 0$ *for the Evans function* $\mathcal{E}(z) = \det(I - \mathcal{T}(z))$.

We remark that the Evans determinant $\mathcal{E}(z)$ here is defined, cf. (4.18), as $\mathcal{E}(z) = \det[Y_0^+ + Y_0^-]$, where $\mathbf{Y}^\pm = \mathbf{Y}^\pm(z)$ are the $(2\times 2)$ matrix solutions of the following system:

$$\begin{aligned} Y_j^+ - A^j(z)(I - P(z)) &= -\sum_{k=j}^{\infty} A^{j-k-1}(z)B_kC_kY_k^+, \quad j \in \mathbb{Z}, \\ Y_j^- - A^j(z)P(z) &= \sum_{k=-\infty}^{j-1} A^{j-k-1}(z)B_kC_kY_k^-, \quad j \in \mathbb{Z}. \end{aligned} \tag{5.14}$$

Recall that the solutions satisfy $Y_j^+ = Y_j^+(I - P(z))$, $Y_j^- = Y_j^-P(z)$. Also, because (5.14) are Volterra-type equations, passing to $\lambda^{\mp j}Y_j^\pm$ and using (2.12), one can easily see that $\mathbf{Y}^\pm = (Y_j^\pm)$ are indeed defined for all $j \in \mathbb{Z}$.

Next, we will use the special structure of $A$, $B_j$, and $C_j$ to simplify system (5.14). Introduce matrices

$$W = \begin{bmatrix} 1 & 1 \\ \lambda & \lambda^{-1}\end{bmatrix}, \quad W^{-1} = \Lambda^{-1}\begin{bmatrix} \lambda^{-1} & -1 \\ -\lambda & 1\end{bmatrix}, \quad V = \begin{bmatrix} -1 & -1 \\ 1 & 1\end{bmatrix}, \quad \widetilde{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\end{bmatrix},$$

and remark that $\widetilde{A} = W^{-1}AW$ and $\Lambda^{-1}v_jV\widetilde{A} = W^{-1}B_jC_jW$ for the matrices $A$, $B_j$, and $C_j$ introduced in (5.13). The change of variables $\widetilde{x}_j = W^{-1}x_j$ transforms the equation $x_{j+1} = A_j^\times x_j$ to the equation $\widetilde{x}_{j+1} = (\widetilde{A} + \Lambda^{-1}v_jV\widetilde{A})\widetilde{x}_j$ with the diagonal unperturbed coefficient $\widetilde{A}$ whose dichotomy projection $\widetilde{P} = W^{-1}PW$ is given by $\widetilde{P} = \begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}$. Similarly, passing in equations (5.14) to the new unknowns $\widetilde{\mathbf{Y}}^\pm = W^{-1}\mathbf{Y}^\pm W$, we may use the fact that $\mathbf{Y}^+ = \mathbf{Y}^+(I - P)$ and $\mathbf{Y}^- = \mathbf{Y}^-P$ imply $\widetilde{\mathbf{Y}}^+ = \widetilde{\mathbf{Y}}^+(I - \widetilde{P})$ and $\widetilde{\mathbf{Y}}^- = \widetilde{\mathbf{Y}}^-\widetilde{P}$ to conclude that the second column in $\widetilde{\mathbf{Y}}_j^+$ and the first column in $\widetilde{\mathbf{Y}}_j^-$ are equal to zero. Let $u_j^+ \in \mathbb{C}^2$, resp. $u_j^- \in \mathbb{C}^2$ denote the first column of $\widetilde{\mathbf{Y}}_j^+$, resp. the second column of $\widetilde{\mathbf{Y}}_j^-$. Then equations (5.14) are equivalent to the following $(2 \times 1)$ vector equations:

$$\begin{aligned} u_j^+ &= [\lambda^j \quad 0]^\top - \sum_{k=j}^{\infty} \Lambda^{-1} v_k \widetilde{A}^{j-k-1} V \widetilde{A} u_k^+, \quad j \in \mathbb{Z}, \\ u_j^- &= [0 \quad \lambda^{-j}]^\top - \sum_{k=-\infty}^{j-1} \Lambda^{-1} v_k \widetilde{A}^{j-k-1} V \widetilde{A} u_k^-, \quad j \in \mathbb{Z}. \end{aligned} \tag{5.15}$$

Introducing $(2 \times 1)$ vectors $x_j^\pm = W u_j^\pm$ given, in components, by $x_j^\pm = [x_j^\pm(1) \quad x_j^\pm(2)]^\top$ and passing in (5.15) to the new unknowns $x_j^\pm$, we have the following system of equations equivalent to (5.14):

$$\begin{aligned} x_j^+ &= [\lambda^j \quad \lambda^{j+1}]^\top - \sum_{k=j}^{\infty} \Lambda^{-1} v_k [\lambda^{-j+k+1} - \lambda^{j-k-1} \quad \lambda^{-j+k} - \lambda^{j-k}]^\top x_k^+(2), \\ x_j^- &= [\lambda^{-j} \quad \lambda^{-j-1}]^\top + \sum_{k=-\infty}^{j-1} \Lambda^{-1} v_k [\lambda^{-j+k+1} - \lambda^{j-k-1} \quad \lambda^{-j+k} - \lambda^{j-k}]^\top x_k^-(2). \end{aligned} \tag{5.16}$$

Recall that $\mathbf{y} = (y_j)_{j\in\mathbb{Z}}$, $y_j \in \mathbb{C}$, is a solution of the difference equation $L\mathbf{y} = z\mathbf{y}$ if and only if $\mathbf{x} = (x_j)_{j\in\mathbb{Z}}$ with $x_j = [y_{j-1} \quad y_j]^\top \in \mathbb{C}^2$ is a solution of the equation $x_{j+1} = A_j^\times x_j$. Moreover, since $\det A_j^\times = 1$, if $\mathbf{y}^+$ and $\mathbf{y}^-$ are two solutions of the equation $L\mathbf{y} = z\mathbf{y}$ with the corresponding $\mathbf{x}^+$ and $\mathbf{x}^-$, then the Wronskian

$$\mathcal{W}(\mathbf{y}^+, \mathbf{y}^-) := y_{j-1}^+ y_j^- - y_j^+ y_{j-1}^- = \det[x_j^+ \quad x_j^-] \tag{5.17}$$

does not depend on $j \in \mathbb{Z}$.

A direct calculation shows that solutions $\mathbf{x}^\pm = (x_j^\pm)$ of (5.16) have the property $x_{j+1}^\pm(1) = x_j^\pm(2)$. Also, $\mathbf{x}^\pm$ satisfy the equation $x_{j+1}^\pm = A_j^\times x_j^\pm$ because $\mathbf{Y}^\pm$ in (5.14) satisfy this equation. Thus, letting $y_j^\pm = \lambda^{\mp 1} x_j^\pm(2)$ and keeping only the second component of the vectors in (5.16), we have obtained the solutions $\mathbf{y}^\pm = (y_j^\pm)$ of the second order difference equation $L\mathbf{y} = z\mathbf{y}$ that satisfy the following equations:

$$\begin{aligned} y_j^+ &= \lambda^j - \sum_{k=j}^{\infty} \Lambda^{-1} v_k (\lambda^{-j+k} - \lambda^{j-k}) y_k^+, \quad j \in \mathbb{Z}, \\ y_j^- &= \lambda^{-j} + \sum_{k=-\infty}^{j-1} \Lambda^{-1} v_k (\lambda^{-j+k} - \lambda^{j-k}) y_k^-, \quad j \in \mathbb{Z}. \end{aligned} \tag{5.18}$$

Introducing $\varkappa < 0$ so that $\lambda = e^\varkappa$, (5.18) could be rewritten as follows:

$$\begin{aligned} y_j^+ &= e^{j\varkappa} - \sum_{k=j}^{\infty} v_k \frac{\sinh(j-k)\varkappa}{\sinh \varkappa} y_k^+, \quad j \in \mathbb{Z}, \\ y_j^- &= e^{-j\varkappa} + \sum_{k=-\infty}^{j-1} v_k \frac{\sinh(j-k)\varkappa}{\sinh \varkappa} y_k^-, \quad j \in \mathbb{Z}. \end{aligned}$$

The solutions $\mathbf{y}^{\pm} = (y_j^{\pm})_{j\in\mathbb{Z}_{\pm}}$ are asymptotic to $\lambda^{\pm j} = e^{\pm j\varkappa}$ as $j \to \pm\infty$; these are the *Jost solutions* of $L\mathbf{y} = z\mathbf{y}$, cf. [CS, XVII.1.9-10], [GH, Sec. 2], [GM, (4.52)] for the continuous and [C, C1], [FT, Sec. III.2], [GH, (6.6)], [HKS], [T, (10.3)], and [To, (3.3.1)] for the discrete models. Recalling that $\mathbf{y}^{\pm} = \mathbf{y}^{\pm}(z)$ and definition (5.17) of the Wronskian of the solutions, we define the *Jost function*, $\mathcal{J} = \mathcal{J}(z)$, by $\mathcal{J}(z) := \Lambda^{-1}\mathcal{W}(\mathbf{y}^{+}(z), \mathbf{y}^{-}(z))$, cf. [T, Sec. 10.2] and [To, (3.3.19)]. Our last claim is that, in fact, the Evans function for the Schrödinger equation is the same as the Jost function.

PROPOSITION 5.4. *If* $\mathbf{v} = (v_j)_{j\in\mathbb{Z}} \in \ell^1(\mathbb{Z};\mathbb{R})$ *and* $|z| > 2$ *then* $\mathcal{E}(z) = \mathcal{J}(z)$.

*Proof.* Using the choice of transformations converting (5.14) to (5.18), we infer:

$$\begin{aligned}\mathcal{E}(z) &= \det[Y_0^{+} + Y_0^{-}] = \det[W\widetilde{Y}_0^{+}W^{-1} + W\widetilde{Y}_0^{-}W^{-1}] = \det W \det[\widetilde{Y}_0^{+} + \widetilde{Y}_0^{-}] \det W^{-1} \\ &= \det[u_0^{+} \quad u_0^{-}] = \det[W^{-1}x_0^{+} \quad W^{-1}x_0^{-}] = \Lambda^{-1}\det[x_0^{+} \quad x_0^{-}] \\ &= \Lambda^{-1}\mathcal{W}(\mathbf{y}^{+}, \mathbf{y}^{-}) = \mathcal{J}(z),\end{aligned}$$

which concludes the proof. ∎

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