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BANACH SPACES WITH SMALL CALKIN ALGEBRAS

MANUEL GONZÁLEZ

Departamento de Matemáticas, Universidad de Cantabria
E-39071 Santander, Spain
E-mail: gonzalem@unican.es

Abstract. Let X be a Banach space. Let $\mathcal{A}(X)$ be a closed ideal in the algebra $\mathcal{L}(X)$ of the operators acting on X. We say that $\mathcal{L}(X)/\mathcal{A}(X)$ is a *Calkin algebra* whenever the Fredholm operators on X coincide with the operators whose class in $\mathcal{L}(X)/\mathcal{A}(X)$ is invertible. Among other examples, we have the cases in which $\mathcal{A}(X)$ is the ideal of compact, strictly singular, strictly cosingular and inessential operators, and some other ideals introduced as perturbation classes in Fredholm theory.

Our aim is to present some classes of Banach spaces and some concrete examples of Banach spaces for which some of their Calkin algebras are "small" in some sense: finite dimensional, commutative, etc. The first example of such a Banach space was constructed around 1990. However, at this moment there is a great variety of examples of spaces of this kind, which provides interesting examples and counterexamples of operators. Moreover, the methods and techniques of operator theory have been found to be useful in the study of these spaces.

1. Introduction. In [GowersM:93], Gowers and Maurey presented an example of an infinite dimensional Banach space X_{GM} (Example 6.1.1) such that no subspace of X_{GM} can be written as the direct sum of two infinite dimensional closed subspaces. The existence of Banach spaces of this kind (which are called *hereditarily indecomposable*) was quite unexpected. It provided a negative answer to several long-standing open problems in Banach space theory. Moreover, for every operator T acting on the complex version of X_{GM} we can find a complex number λ so that $T = \lambda I + S$, with S a strictly singular operator. Afterwards, many examples of hereditarily indecomposable spaces and some other classes of spaces with unexpected properties were constructed. Now it is common

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to refer to these spaces as "exotic" spaces to distinguish them from the previously known spaces, which are referred to as "classical" spaces. Among other examples, it has been shown that for each p, $1 < p < \infty$, the sequence space ℓ_p can be obtained as a quotient of a separable, reflexive, hereditarily indecomposable space [ArgyrosF:00]. Many other examples have been constructed. See for example, [ArgyrosLT:03], [ArgyrosT:04], [Koszmider:04], [Koszmider:05] and [Koszmider:05a]. We refer to [Maurey:03] for a readable paper describing many interesting examples of spaces of this kind.

For a complex, infinite dimensional, hereditarily indecomposable Banach space X, the quotient algebra $\mathcal{L}(X)/\mathcal{SS}(X)$ is one-dimensional. Here $\mathcal{SS}(X)$ is the ideal of strictly singular operators acting on X. As a consequence, the essential spectrum of each operator $T \in \mathcal{L}(X)$ consists of just one point. This is quite different from what happens for the classical Banach spaces. So these spaces are a good bench-mark for the development of Fredholm theory, as well as a source of nontrivial examples of operators. The hereditarily indecomposable Banach spaces have also found applications in the study of C_0-semigroups of operators. See [RabigerR:96] and [RabigerR:98].

In this paper we deal with closed two-sided ideals $\mathcal{A}(X)$ in $\mathcal{L}(X)$ for which an operator $T \in \mathcal{L}(X)$ is Fredholm if and only if the corresponding class is invertible in $\mathcal{L}(X)/\mathcal{A}(X)$. In this case we say that $\mathcal{L}(X)/\mathcal{A}(X)$ is the *Calkin algebra associated to the ideal* $\mathcal{A}(X)$.

It follows from well-known results that $\mathcal{L}(X)/\mathcal{A}(X)$ is a *Calkin algebra* if and only if $\mathcal{A}(X)$ contains the finite range operators and is contained in the inessential operators. As examples, we mention the compact, the strictly singular, the strictly cosingular and the inessential operators. Also the perturbation classes in Fredholm theory can be taken as examples.

We are interested in the spaces which admit a Calkin algebra which is "small" in some sense: finite dimensional, commutative, etc. For the reader of this paper, it should become clear that there are plenty of spaces of this kind, even among the $C(K)$ spaces of continuous functions defined on a compact space K. In the paper we show the main properties of these spaces, describe some examples and give references for further properties and examples. Our aim is to show that these spaces are useful and not too exotic. So it pays out to have them in mind while working in operator theory.

Let us see a more detailed description of the paper. After a preliminary section, we introduce in Section 3 the concept of *Calkin algebra* $\mathcal{L}(X)/\mathcal{A}(X)$ associated to some ideal $\mathcal{A}(X)$ of $\mathcal{L}(X)$. We give several examples, show the relations among them and give information on the Calkin algebras of some classical sequence spaces and function spaces.

Section 4 deals with the "decomposability" properties of a Banach space as the direct sum of infinite dimensional closed subspaces. We show the relation of these properties with the number of connected components of the essential spectrum of the operators $T \in \mathcal{L}(X)$. We also show that the spaces whose subspaces (quotients) are decomposable admit characterizations in terms of the classes of operators in Fredholm theory.

Section 5 is dedicated to the Calkin algebra associated to the ideal of inessential operators $\mathcal{L}(X)/\mathcal{I}n(X)$, which is the smallest Calkin algebra. In the case $\mathcal{L}(X)/\mathcal{I}n(X)$ is finite dimensional, we obtain some results on the structure of the space X and describe the essential spectrum of the operators $T \in \mathcal{L}(X)$. Also, in this case we give a description of the Calkin algebra $\mathcal{L}(X)/\mathcal{I}n(X)$ as a finite product of algebras of matrices $M_k(\mathcal{C})$. This description of $\mathcal{L}(X)/\mathcal{I}n(X)$ allows us to give examples of spaces X for which $\mathcal{L}(X)/\mathcal{SS}(X)$ or $\mathcal{L}(X)/\mathcal{SC}(X)$ can be represented as special subalgebras of $M_n(\mathcal{C})$; for example, as triangular algebras.

We have chosen to include all the examples in Section 6. In this way we can appreciate directly the variety of Banach spaces with small Calkin algebras. However, we refer to these examples in the sections in which they are relevant. In order the paper be useful, we give a rough description of some of the constructions, and most of the times we give a reference in which the reader can find more details.

Throughout we refer to other results of the paper by giving the number of the result and the section in which it is stated. For example, Theorem 4.3 is Theorem 3 in section 4.

2. Preliminaries. This paper is a blend of Fredholm theory and Banach space theory. So we have to consider definitions and results in both theories. In this section we include definitions and notations for the classes of operators of Fredholm theory. The corresponding definitions of Banach space theory are introduced when needed.

Let X and Y be Banach spaces. For a subspace M of X we denote by J_M the embedding map from M into X, and by Q_M the quotient map onto X/M. We denote by $\mathcal{L}(X,Y)$ the space of continuous operators from X to Y, and $\mathcal{K}(X,Y)$ and $\mathcal{F}(X,Y)$ denote the space of compact and finite dimensional range operators, respectively.

Let $T \in \mathcal{L}(X,Y)$.

- T is *upper semi-Fredholm*, $T \in \Phi_+$, if $R(T)$ is closed and $\dim N(T) < \infty$.
- T is *lower semi-Fredholm*, $T \in \Phi_-$, if $R(T)$ is closed and $\dim Y/R(T) < \infty$.
- T is *Fredholm*, $T \in \Phi$, if T is upper and lower semi-Fredholm.
- For an operator $T \in \mathcal{L}(X)$ the *essential spectrum* is defined by

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi(X)\}.$$

- The *index of a semi-Fredholm operator* $T \in \Phi_+(X,Y) \cup \Phi_-(X,Y)$ is defined by

$$\operatorname{ind}(T) := \dim N(T) - \dim Y/R(T).$$

- T is *strictly singular*, $T \in \mathcal{SS}$, if given a subspace M of X, TJ_M isomorphism (into) implies $\dim M < \infty$.
- T is *strictly cosingular*, $T \in \mathcal{SC}$, if given a subspace N of Y, $Q_N T$ surjective implies $\dim Y/N < \infty$.
- T is *inessential*, $T \in \mathcal{I}n$, if $I - ST \in \Phi(X)$ for all $S \in \mathcal{L}(Y,X)$.

The *perturbation class* $P\mathcal{C}$ of a class of operators $\mathcal{C}$ was defined by Lebow and Schechter [LebowS:71] as follows. Suppose that $\mathcal{C}(X,Y)$ is a non-empty subset of $\mathcal{L}(X,Y)$. Then

$$P\mathcal{C}(X,Y) := \{K \in L(X,Y) : K + T \in \mathcal{C}(X,Y), \text{ for all } T \in \mathcal{C}(X,Y)\}.$$

Note that in the case $X = Y$, $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$ are always non-empty.

Proposition 1 ([LebowS:71]). *The perturbation classes $P\Phi(X)$, $P\Phi_+(X)$ and $P\Phi_-(X)$ are closed, two-sided ideals in $\mathcal{L}(X)$.*

3. Calkin algebras. In this section we introduce the concept of Calkin algebra. We also give several examples of Calkin algebras and describe the relations among them.

DEFINITION 1. Let $\mathcal{A}(X)$ be a closed ideal in $\mathcal{L}(X)$. We say that $\mathcal{L}(X)/\mathcal{A}(X)$ is a *Calkin algebra* whenever the Fredholm operators on X coincide with the operators whose class in $\mathcal{L}(X)/\mathcal{A}(X)$ is invertible.

The following characterization can be easily derived from standard results in Fredholm theory.

PROPOSITION 1. *Let $\mathcal{A}(X)$ be a closed ideal in $\mathcal{L}(X)$. Then $\mathcal{L}(X)/\mathcal{A}(X)$ is a Calkin algebra if and only if $\overline{\mathcal{F}(X)} \subset \mathcal{A}(X) \subset \mathcal{I}n(X)$.*

The existence of spaces X for which $\overline{\mathcal{F}(X)} \neq \mathcal{K}(X)$ is related with the approximation property of Banach spaces. Since this property has no relevance here, $\mathcal{K}(X)$ is the smallest ideal we consider.

REMARK 1. Roughly, we are going to consider two kinds of closed ideals in $\mathcal{L}(X)$:

1. *Intrinsically defined ideals,* for which the fact that the operator belongs to the ideal can be expressed in terms of its action on the spaces in which it is defined, like $\mathcal{K}$, $\mathcal{SS}$ and $\mathcal{SC}$.
2. *Perturbation classes* in which the definition involves the properties of a large class of operators.

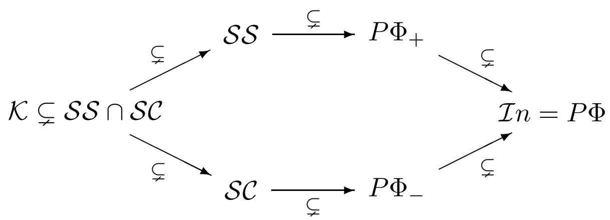
REMARK 2. The spaces $\mathcal{K}(X,Y)$, $\mathcal{SS}(X,Y)$, $\mathcal{SC}(X,Y)$ and $\mathcal{I}n(X,Y)$ are defined for all pairs of Banach spaces X and Y. They form operator ideals in the sense of [Pietsch-Book:80]. However, the perturbation classes $P\Phi(X,Y)$, $P\Phi_+(X,Y)$ and $P\Phi_-(X,Y)$ are defined only when $\Phi(X,Y)$, $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ are non-empty, respectively.

Next we describe the relations of containment between the main ideals that we consider in the paper.

PROPOSITION 2. *The following results hold whenever the perturbation classes are defined.*

- $\mathcal{I}n(X,Y) = P\Phi(X,Y)$ [Kleinecke:63].
- $P\Phi_+(X,Y) \cup P\Phi_-(X,Y) \subset P\Phi(X,Y)$ (Stability of the index).
- $\mathcal{SS}(X,Y) \subset P\Phi_+(X,Y)$ [Kato:58].
- $\mathcal{SC}(X,Y) \subset P\Phi_-(X,Y)$ [Vladimirskii:67].

The following diagram shows the relations between the ideals we have introduced before.



There are natural operators that show that all the inclusions, except $\mathcal{SS} \subsetneq P\Phi_+$ and $\mathcal{SC} \subsetneq P\Phi_-$, are proper. Indeed, let us consider the embedding maps

$$L_\infty[0,1] \xrightarrow{J_2} L_2[0,1] \xrightarrow{J_1} L_1[0,1].$$

Then $J_1J_2 \in (\mathcal{SS} \cap \mathcal{SC}) \setminus \mathcal{K}$, $J_2 \in (\mathcal{SS} \setminus \mathcal{SC})$, $J_1 \in (\mathcal{SC} \setminus \mathcal{SS})$ and $J_1, J_2 \in \mathcal{I}n$. We refer to Proposition 2.6 in [Gonzalez:04] for details.

For the remaining inclusions we need exotic spaces. It was shown in [Gonzalez:03] that there exists a reflexive Banach space X, which can be defined in terms of products of subspaces of X_{GM}, such that $\mathcal{SS}(X) \neq P\Phi_+(X)$ and $\mathcal{SC}(X^*) \neq P\Phi_-(X^*)$. See [Gonzalez:03] for details.

Remark 3. It follows from the inequalities $\mathcal{SS}(X) \neq P\Phi_+(X)$ and $\mathcal{SC}(X^*) \neq P\Phi_-(X^*)$ that the perturbation classes $P\Phi_+(X,Y)$ and $P\Phi_-(X,Y)$ for operators acting between different spaces do not give operators ideals. See the Remark in section 26.6 of [Pietsch-Book:80].

Remark 4. It is not known if there exists an infinite dimensional Banach space X for which $\dim \mathcal{L}(X)/\mathcal{K}(X) < \infty$. This is a (probably hard) open problem in Banach space theory.

Remark 5. The definition of the inessential operators is not intrinsic. In order to check if $T \in \mathcal{I}n(X,Y)$, we have to consider the properties of the product ST of T with all the operators $S \in \mathcal{L}(Y,X)$.

Trying to obtain an intrinsic characterization of the inessential operators, Tarafdar introduced in [Tarafdar:72] the improjective operators. An operator $T \in \mathcal{L}(X,Y)$ is said to be *improjective*, $T \in \mathcal{I}mp(X,Y)$, if there is no infinite dimensional subspace M of X such that the restriction TJ_M is an isomorphism and $T(M)$ is complemented in Y. It is not difficult to show that $\mathcal{I}n(X,Y) \subset \mathcal{I}mp(X,Y)$.

The equality $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$ holds for many pairs of spaces (see [AienaG:98]). However, this inclusion is proper in general: an improjective not inessential operator was given in [AienaG:00]. The construction is based on the properties of the shift operator acting in the space described below as Example 6.1.2.

Remark 6. For classical Banach spaces the Calkin algebras are big spaces, in general. However we can give some interesting facts.

- For X one of the sequence spaces c_0 and ℓ_p, $1 \leq p < \infty$, there is only one Calkin algebra. Equivalently, $\overline{\mathcal{F}(X)} = \mathcal{K}(X) = \mathcal{I}n(X)$. Moreover, $\mathcal{L}(X)/\mathcal{K}(X)$ is nonseparable. In [CaradusPY:74] we can find a proof of this result and some additional details.
- For X a subprojective space, like the function spaces $L_p(0,1)$; $1 \leq p < 2$, $\mathcal{SS}(X) = \mathcal{I}n(X)$; and for X a superprojective space, like the function spaces $L_p(0,1)$; $2 \leq p < \infty$, $\mathcal{SC}(X) = \mathcal{I}n(X)$. See [CaradusPY:74] and [Weis:81].

4. Finitely decomposable spaces. In this section we study the relation between the decomposability of a Banach space X as a direct sum of closed infinite dimensional subspaces and the properties of the Calkin algebras $\mathcal{L}(X)/\mathcal{A}(X)$.

Definition 1. Let X be a Banach space.

- X is *indecomposable* if it cannot be written as the direct sum of two closed infinite dimensional subspaces.
- X is *n-decomposable* if $n = \max\{k \in \mathbb{N} : X = X_1 \oplus \cdots \oplus X_k\}$, where $X_1, \ldots, X_n$ are closed infinite dimensional subspaces of X.
- X is *hereditarily indecomposable* if all its closed subspaces are indecomposable.
- X is *quotient indecomposable* if all its quotients are indecomposable.

As an application of the analytic functional calculus for operators (see for example [TaylorL:80]) we can prove the following characterizations.

Theorem 1 ([GonzalezH:03]). *Let X be a complex Banach space. Then X is indecomposable if and only if $\sigma_e(T)$ is connected for all $T \in \mathcal{L}(X)$. Similarly, X is n-decomposable if and only if n is the maximum of the number of connected components of $\sigma_e(T)$ for $T \in \mathcal{L}(X)$.*

The first part of the following characterizations was proved in [Weis:81] in 1981. Observe that at that time no example of hereditarily indecomposable or quotient indecomposable space was known.

Theorem 2 (see [Ferenczi:97]). *A Banach space X is hereditarily indecomposable if and only if, for every Banach space Y,*

$$\mathcal{L}(X, Y) = \Phi_+(X, Y) \cup \mathcal{SS}(X, Y).$$

In the complex case, this is equivalent to the fact that for each closed subspace M of X and every $T \in \mathcal{L}(M, X)$ there exists $\lambda \in \mathcal{C}$ such that $T - \lambda J_M$ is strictly singular.

Theorem 3 (see [GonzalezH:01]). *A Banach space X is quotient indecomposable if and only if, for every Banach space Z,*

$$\mathcal{L}(Z, X) = \Phi_-(Z, X) \cup \mathcal{SC}(Z, X).$$

In the complex case, this is equivalent to the fact that for each quotient X/N of X and every $T \in \mathcal{L}(X, X/N)$ there exists $\lambda \in \mathcal{C}$ such that $T - \lambda Q_N$ is strictly cosingular.

Remark 1. The classical Banach spaces are far from being indecomposable. However, many examples of indecomposable spaces have been found by now.

- The space X_{GM}, which appears here as Example 6.1.1, is hereditarily indecomposable and quotient indecomposable. See [Ferenczi:99].
- The space X_s of Example 6.1.2 is indecomposable, but not hereditarily indecomposable or quotient indecomposable.
- Koszmider's space in Example 6.2 is an indecomposable $C(K)$ space, but contains subspaces isomorphic to c_0 and admits quotients isomorphic to ℓ_2.

It has been open for some time whether a hereditarily indecomposable space has to be separable. Finally, a negative answer has been obtained. However, some limitations apply.

THEOREM 4 ([ArgyrosT:04]). *There are nonseparable hereditarily indecomposable Banach space; however, each hereditarily indecomposable Banach space is isomorphic to a subspace of ℓ_∞.*

5. The "smallest" Calkin algebra. Here we deal with $\mathcal{L}(X)/\mathcal{I}n(X)$, which is the smallest Calkin algebra. Throughout this section, X is a complex Banach space.

REMARK 1. The inessential operators were introduced in [Kleinecke:63] as follows. Let $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X)$ denote the quotient map. Then $\mathcal{I}n(X)$ is the inverse under π of the radical of $\mathcal{L}(X)/\mathcal{K}(X)$.

The following classes of operators were introduced in [Atkinson:51].

DEFINITION 1. Let $T \in \mathcal{L}(X,Y)$.

- T is *left Atkinson*, denoted $T \in \Phi_l$, if $R(T)$ is complemented and $\dim N(T) < \infty$.
- T is *right Atkinson*, denoted $T \in \Phi_r$, if $N(T)$ is complemented $R(T)$ closed and $\dim Y/R(T) < \infty$.

REMARK 2. The relevance of the Atkinson operators in this section stems from the following facts. Let $\mathcal{L}(X)/\mathcal{A}(X)$ be a Calkin algebra.

1. $T \in \mathcal{L}(X)$ is left Atkinson if and only if the class of T in $\mathcal{L}(X)/\mathcal{A}(X)$ is left invertible.
2. $T \in \mathcal{L}(X)$ is right Atkinson if and only if the class of T in $\mathcal{L}(X)/\mathcal{A}(X)$ is right invertible.
3. $P\Phi_l(X) = P\Phi_r(X) = P\Phi(X) = \mathcal{I}n(X)$.
   We refer to [LebowS:71] or [CaradusPY:74] for proofs of these facts.

Let us see how the size of $\mathcal{L}(X)/\mathcal{I}n(X)$ is directly related to the size of the essential spectrum of the operators acting on X.

Let $|A|$ denote the cardinal of a set A. We introduce the class Σ_e^n as follows.

$$\Sigma_e^n := \{X \,:\, n = \max\{|\sigma_e(T)| \,:\, T \in \mathcal{L}(X)\}\,\}.$$

THEOREM 1 ([GonzalezH:05]). *The following assertions are equivalent:*

1. *For every $T \in \mathcal{L}(X)$, the essential spectrum $\sigma_e(T)$ is finite.*
2. $\dim \mathcal{L}(X)/\mathcal{I}n(X) < \infty$.
3. *The space X belongs to the class Σ_e^n, for some $n \in \mathbb{N}$.*

REMARK 3. It is not difficult to see that X belongs to the class Σ_e^1 if and only if $\dim \mathcal{L}(X)/\mathcal{I}n(X) = 1$.

DEFINITION 1. Let X and Y be Banach spaces.

- We say that X and Y are *essentially incomparable* when $\mathcal{L}(X,Y) = \mathcal{I}n(X,Y)$.
- We say that X and Y are *essentially isomorphic* when $\Phi(X,Y)$ is non-empty.

REMARK 4. The concept of essential incomparability is "symmetric" [Gonzalez:94]; i.e., $\mathcal{L}(X,Y) = \mathcal{I}n(X,Y)$ if and only if $\mathcal{L}(Y,X) = \mathcal{I}n(Y,X)$.

REMARK 5. X essentially isomorphic to Y means that X is isomorphic to Y up to a finite dimensional space.

There are many examples of pairs of Banach spaces which are essentially incomparable.

THEOREM 2 ([Gonzalez:94]). *In the following cases, the spaces X and Y are essentially incomparable.*

1. X *is reflexive and* Y *has the Dunford-Pettis property.*
2. X *has the reciprocal Dunford-Pettis property and* Y *has the Schur property.*
3. X *contains no copies of* ℓ_∞ *and* $Y = \ell_\infty$ *or* $C(K)$ *with* K *σ-stonian.*
4. X *contains no copies of* c_0 *and* $Y = C(K)$.
5. X *contains no complemented copies of* c_0 *and* $Y = C[0,1]$.
6. X *contains no complemented copies of* ℓ_1 *and* $Y = L_1[0,1]$.
7. X *contains no complemented copies of* ℓ_p *and* $Y = L_p[0,1]$, *or* ℓ_p; $1 < p < \infty$.

EXAMPLE 1. Let X be a hereditarily indecomposable space and let M be a subspace of X such that $\dim X/M = \infty$. It follows from Theorem 4.2 that M and X are essentially incomparable.

Similarly, let X be a quotient indecomposable space and let M be an infinite dimensional subspace of X. It follows from Theorem 4.3 that X and X/M are essentially incomparable.

The following result should be compared with the corresponding results for hereditarily indecomposable spaces (Theorem 4.2) and quotient indecomposable spaces (Theorem 4.3).

THEOREM 3 ([GonzalezH:05]). *For an infinite dimensional Banach space* X, *the following assertions are equivalent.*

1. $\dim \mathcal{L}(X)/\mathcal{I}n(X) = 1$; *i.e.,* $X \in \Sigma_e^1$.
2. $\mathcal{L}(X) = \Phi(X) \cup \mathcal{I}n(X)$.
3. $\mathcal{L}(X,Y) = \Phi_l(X,Y) \cup \mathcal{I}n(X,Y)$, *for all* Y.
4. $\mathcal{L}(Z,X) = \Phi_l(Z,X) \cup \mathcal{I}n(Z,X)$, *for all* Z.

Next we give a result that provides a precise description of the spaces X for which $\mathcal{L}(X)/\mathcal{I}n(X)$ is finite dimensional.

THEOREM 4 ([GonzalezH:05], Structure theorem). *Suppose that* $X \in \Sigma_e^n$. *Then there exist* k *integers* n_i *and* k *spaces* $X_1, \dots, X_k$ *satisfying the following properties.*

- $n = n_1 + \cdots + n_k$.
- X_i *and* X_j *are essentially incomparable for* $i \neq j$.
- X *is essentially isomorphic to* (n_1 *copies of* X_1) $\oplus \cdots \oplus$ (n_k *copies of* X_k).

As a consequence of Theorem 4 we obtain a description of $\mathcal{L}(X)/\mathcal{I}n(X)$.

THEOREM 5 ([GonzalezH:05]). *Suppose that* $X \in \Sigma_e^n$. *Then there exist* k *integers* $n_1, \dots, n_k$ *satisfying* $n = n_1 + \cdots + n_k$ *so that*

$$\mathcal{L}(X)/\mathcal{I}n(X) \text{ is isomorphic to } M_{n_1}(\mathcal{C}) \oplus \cdots \oplus M_{n_k}(\mathcal{C}).$$

Theorems 4 and 5 allow us to give examples of spaces with different Calkin algebras.

EXAMPLE 2. Let us consider the space X_{GM} in Example 6.1.1. Since X_{GM} is hereditarily indecomposable, taking infinite dimensional closed subspaces $M_1 \subset \cdots \subset M_n$ of X_{GM} such that $\dim M_{k+1}/M_k = \infty$ for $k = 1, \dots, n-1$ and denoting

$$Y := M_1 \oplus \cdots \oplus M_n,$$

then $\mathcal{L}(Y)/\mathcal{I}n(Y)$ can be identified with the diagonal $n \times n$ complex matrices, while $\mathcal{L}(Y)/\mathcal{SS}(Y)$ can be identified with the triangular $n \times n$ complex matrices.

Similarly, since X_{GM} is quotient indecomposable, taking closed subspaces $M_1 \supset \cdots \supset M_{n+1}$ of X_{GM} such that $\dim M_k/M_{k+1} = \infty$ for $k = 1, \dots, n$, and denoting

$$Z := M_1/M_2 \oplus \cdots \oplus M_n/M_{n+1},$$

then $\mathcal{L}(Z)/\mathcal{I}n(Z)$ can be identified with the diagonal $n \times n$ complex matrices, while $\mathcal{L}(Z)/\mathcal{SC}(Z)$ can be identified with the triangular $n \times n$ complex matrices.

6. The examples. In this section we present some examples of exotic Banach spaces that are relevant as examples or counterexamples in Fredholm theory.

As usual, we denote by $\mathbb{D}$ and $\mathbb{T}$ the unit disc and the unit circle in the complex field, respectively.

6.1. *The Gowers-Maurey examples* [Gowers-Maurey:97]

1. (The HI and QI space) There exists a reflexive, complex hereditarily indecomposable Banach space X_{GM} such that

$$\dim \mathcal{L}(X_{GM})/\mathcal{SS}(X_{GM}) = \dim \mathcal{L}(X_{GM}^*)/\mathcal{SC}(X_{GM}^*) = 1.$$

Even more, for every closed subspace Y of X_{GM} and every $T \in \mathcal{L}(Y, X_{GM})$, there exists a complex number λ such that $T - \lambda J_Y \in \mathcal{SS}(Y, X_{GM})$. It was shown in [Ferenczi:99] that the space X_{GM} is also quotient indecomposable. Thus

$$\dim \mathcal{L}(X_{GM})/\mathcal{SC}(X_{GM}) = \dim \mathcal{L}(X_{GM}^*)/\mathcal{SS}(X_{GM}^*) = 1.$$

Moreover, for every quotient Z of X_{GM} and every $T \in \mathcal{L}(X_{GM}, Z)$, there exists a complex number λ such that $T - \lambda Q \in \mathcal{SC}(X_{GM}, Z)$, where Q is the quotient map onto Z.

2. (The shift space) There exists a complex Banach space X_s which admits a Schauder basis and satisfies the following properties:
   (a) $\mathcal{L}(X_s)$ admits a closed ideal $\mathcal{A}(X_s)$ with $\mathcal{K}(X_s) \subset \mathcal{A}(X_s) \subset \mathcal{SS}(X_s)$.
   (b) Denoting by S and L the right shift and the left shift acting on X_s,

   $$\sigma(S) = \sigma(L) = \mathbb{D} \quad \text{and} \quad \sigma_e(S) = \sigma_e(L) = \mathbb{T}.$$

   (c) For every operator $T \in \mathcal{L}(X_s)$, there exists a sequence $(a_n)_{n=-\infty}^{\infty}$ of complex numbers such that $\sum_{n=-\infty}^{\infty} |a_n| < \infty$ and

   $$T - \sum_{n=0}^{\infty} a_n S^n - \sum_{n=1}^{\infty} a_{-n} L^n \in \mathcal{SS}(X_s).$$

   (d) Denoting by Ψ_T the function given by $\Psi_T(\lambda) := \sum_{n=-\infty}^{\infty} a_n \lambda^n$,

   $$\sigma(T) = \Psi_T(\mathbb{D}) \quad \text{and} \quad \sigma_e(T) = \Psi_T(\mathbb{T}).$$

(e) $\mathcal{L}(X_s)/\mathcal{A}(X_s)$ can be identified with the algebra $\ell_1(\mathbb{Z})$, endowed with the convolution product. Indeed, if $T = \sum_{n=0}^{\infty} a_n S^n - \sum_{n=1}^{\infty} a_{-n} L^n$ is strictly singular, then $\sigma_e(T) = \Psi_T(\mathbb{T}) = \{0\}$. Thus $T = 0$.

(f) Each pair of operators $A, B \in \mathcal{L}(X_s)$ commute modulo the strictly singular operators; i.e., $AB - BA \in \mathcal{SS}(X_s)$.

3. (The double shift space) There exists a complex Banach space X_{ds} which is isomorphic to its closed subspaces of even codimension while not being isomorphic to those of odd codimension. As a consequence, each Fredholm operator acting on X_{ds} has even index.
4. There exists a Banach space X which is isomorphic to $X \oplus X \oplus X$ but is not isomorphic to $X \oplus X$.
5. There exists a Banach space X_d which has an unconditional Schauder basis and satisfies the following properties.
    - Every operator on X_d is the sum of a diagonal operator and a strictly singular one.
    - $\mathcal{L}(X_d)/\mathcal{SS}(X_d)$ can be identified with ℓ_∞/c_0. See [Ferenczi:95].

Remark 1. The constructions of the previous Examples 1-5 admit modifications and extensions to obtain other Banach spaces satisfying nonclassical properties. We refer to the paper [GowersM:97] for further details.

6.2. *The Koszmider example* [Koszmider:04]. There exists a connected, separable compact space K such that the space $C(K)$ of real-valued continuous defined on K satisfies the following properties:

(a) $C(K)$ is indecomposable and is not isomorphic to any of its proper subspaces or quotients.

(b) It follows from property (a) that every semi-Fredholm operator on $C(K)$ is Fredholm with index zero.

(c) For every operator $T \in \mathcal{L}(C(K))$ there exists a function $\phi \in C(K)$ such that $T = M_\phi + S$, where M_ϕ is the multiplication operator by ϕ and S is a strictly singular operator.

(d) It follows from property (c) that the Calkin algebra $\mathcal{L}(C(K))/\mathcal{SS}(C(K))$ can be identified with the space $C(K)$. Indeed, since K is connected, if M_ϕ is strictly singular then $\phi(K) = \{0\}$, hence $M_\phi = 0$.

Remark 2. Koszmider has found some other examples of $C(K)$ spaces whose properties are remarkable from the point of view we consider here. See [Koszmider:05] and [Koszmider:05a].

6.3. *The Argyros et al. examples.* In the last few years, S. Argyros and coworkers have found plenty of examples of hereditarily indecomposable and quotient indecomposable Banach spaces.

1. There exists a reflexive, separable, hereditarily indecomposable Banach space X which admits a quotient isomorphic to ℓ_2 [ArgyrosF:00]. Hence, the dual space X^* is quotient indecomposable and contains a subspace isomorphic to ℓ_2.

2. There exists a Banach space X such that X, X^* and X^{**} are hereditarily indecomposable, X and X^* are separable but X^{**} is not separable [ArgyrosT:04].
   The quotient space X^{**}/X is isomorphic to $c_0(\Gamma)$, where Γ is an uncountable set.
   Every operator $T \in \mathcal{L}(X)$ can be written as $T = \lambda I + S$, where λ is a real number and S is strictly cosingular and weakly compact.
   Every operator $T \in \mathcal{L}(X^*)$ can be written as $T = \lambda I + S$, where λ is a real number and S is strictly singular and weakly compact.
   Every operator $T \in \mathcal{L}(X^{**})$ can be written as $T = S^* + K$, where $S \in \mathcal{L}(X^*)$ and K is compact.

We refer to [ArgyrosDKM:98], [ArgyrosLT:03], [ArgyrosM:03] and the above mentioned papers for further examples and additional information.

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