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EXISTENCE OF THE SMALLEST SELFADJOINT EXTENSION

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First of all we fix the notation and terminology. For further details see [DuSc] or [Rud].

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle *, * \rangle$. By an *operator* in $\mathcal{H}$ we mean a linear transformation, say T, defined on a linear subspace of $\mathcal{H}$, say $D(T)$, which is the *domain of definition* of T, with values in $\mathcal{H}$.

If $G(T) = \{(x, Tx); x \in D(T)\}$ is the *graph* of T, the *closure* of T is the operator whose graph is $\overline{G(T)}$, provided $\overline{G(T)}$ is a graph.

If T_1, T_2 are operators in $\mathcal{H}$, we write $T_1 \subset T_2$ (or $T_2 \supset T_1$) if T_2 extends T_1, that is, $D(T_1) \subset D(T_2)$ and $T_2|D(T_1) = T_1$.

A *symmetric operator* in $\mathcal{H}$ is an operator S such that $\langle Sx, y \rangle = \langle Sx, y \rangle$ for all $x, y \in D(S)$. A *selfadjoint* operator is an operator A such that $D(A)$ is dense in $\mathcal{H}$ and $A = A^*$, where A^* is the adjoint of A.

If A is selfadjoint, then the operator $i - A : D(A) \to \mathcal{H}$ is injective and surjective, and so its inverse $(i - A)^{-1}$ is everywhere defined and bounded.

We say that two selfadjoint operators A_1, A_2 *commute* if the bounded operators $(i - A_1)^{-1}$ and $(i - A_2)^{-1}$ commute. The next definition was introduced in [Vas].

Definition. Let $S = (S_1, \ldots, S_n)$ be a tuple consisting of symmetric operators in a Hilbert space $\mathcal{H}$. We say that S has a *smallest selfadjoint extension* if there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a tuple $A = (A_1, \ldots, A_n)$ consisting of commuting selfadjoint operators in $\mathcal{K}$ with the following properties:

(1) $A_j \supset S_j,\ j = 1, \ldots, n$;

(2) if $B = (B_1, \ldots, B_n)$ is a tuple consisting of commuting selfadjoint operators in a Hilbert space $\mathcal{L} \supset \mathcal{H}$ such that $B_j \supset S_j,\ j = 1, \ldots, n$, then $\mathcal{L} \supset \mathcal{K}$ and $B_j \supset A_j,\ j = 1, \ldots, n$.

Remarks. (i) In the previous definition, we write $\mathcal{K} \supset \mathcal{H}$ when there exists a linear isometry from $\mathcal{H}$ into $\mathcal{K}$, which allows the identification of $\mathcal{H}$ with a closed subspace of $\mathcal{K}$.

As a consequence of this identification, the smallest selfadjoint extension, when it exists, is uniquely determined.

(ii) Let $S = (S_1, \ldots, S_n)$ be a tuple consisting of symmetric operators in a Hilbert space $\mathcal{H}$ such that the closures $A_1, \ldots, A_n$ of $S_1, \ldots, S_n$ are commuting selfadjoint operators. Then $A = (A_1, \ldots, A_n)$ is the smallest selfadjoint extension of S.

If $n = 1$, and the deficiency indices are equal, the condition that the closure of $S = S_1$ be selfadjoint is also necessary for the existence of a smallest selfadjoint extension. Indeed, if $S = S_1$ is a closed symmetric operator whose deficiency indices are equal, then $D(S)$ equals the intersection of the domains of all selfadjoint extensions of S, as proved in the Appendix of [Dev]. Assuming that S has a smallest selfadjoint extension $A = A_1$, we infer readily that $S = A$.

For $n > 1$, the smallest selfadjoint extension, whose structure is not yet well understood, may have unexpected properties. For instance, it follows from Theorem 4.4 of [BeTh] (see also Theorem 2 below) that, for some tuples of symmetric operators, the smallest selfadjoint extension may exist in a Hilbert space strictly larger than the given one.

A characterization of the smallest selfadjoint extension for some tuples of symmetric operators is expressed by the following result (which is Theorem 3.3 from [Vas]).

THEOREM 1. *Let $S_1, \ldots, S_n$ be symmetric operators in a Hilbert space $\mathcal{H}$, such that $D = D(S_1) = \cdots = D(S_n)$, is invariant under $S_1, \ldots, S_n$. Let also $A_1, \ldots, A_n$ be commuting selfadjoint operators in a Hilbert space $\mathcal{K} \supset \mathcal{H}$, with $A_j \supset S_j$, $j = 1, \ldots, n$. Let*

$$\mathcal{K}_0 = \{(1 + A_1^2)^{-m} \cdots (1 + A_n^2)^{-m} x; x \in D, m \in \mathbf{Z}_+\},$$

which is a linear subspace of $\mathcal{K}$ invariant under $A_1, \ldots, A_n$.

The tuple $A = (A_1, \ldots, A_n)$ is the smallest selfadjoint extension of the tuple $S = (S_1, \ldots, S_n)$ if and only if

(1) *the subspace $\mathcal{K}_0$ is dense in $\mathcal{K}$;*

(2) *if $B_1, \ldots, B_n$ are commuting selfadjoint operators in a Hilbert space $\mathcal{L} \supset \mathcal{H}$, such that $B_j \supset S_j$, $j = 1, \ldots, n$, then*

$$\|(1 + B_1^2)^{-m} \cdots (1 + B_n^2)^{-m} x\| = \|(1 + A_1^2)^{-m} \cdots (1 + A_n^2)^{-m} x\|$$

for all $x \in D, m \in \mathbf{Z}_+$.

Although necessary for theoretical purposes, in particular for the proof of Theorem 2, the statement above is not explicit enough for some concrete problems, especially for moment problems. Specifically, condition (2) from Theorem 1 is not easily verified.

Knowing explicitly the structure of the smallest selfadjoint extension would have useful consequences concerning the uniqueness of the solution of the several variable Hamburger moment problem (see Theorem 2 below, which is Theorem 3.4 from [Vas]). Let us explain the meaning of this assertion.

Let us denote by $\mathbf{Z}_+^n$ the set of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, i.e., $\alpha_j \in \mathbf{Z}_+$ for all $j = 1, \ldots, n$. Let $\mathcal{P}_n$ be the algebra of all polynomial functions on $\mathbf{R}^n$, with complex coefficients. We shall denote by t^α the monomial $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$, where $t = (t_1, \ldots, t_n)$ is the current variable in $\mathbf{R}^n$, and $\alpha \in \mathbf{Z}_+^n$.

An n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ of real numbers is said to be *positive semi-definite* if the *associated linear map* $L_\gamma : \mathcal{P}_n \to \mathbf{C}$ is positive semi-definite, where $L_\gamma(t^\alpha) = \gamma_\alpha, \alpha \in \mathbf{Z}_+^n$.

The n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ is said to be a *moment sequence* if there exists a positive measure μ on $\mathbf{R}^n$ such that $\mathcal{P}_n \subset L^2(\mu)$, and $L_\gamma(p) = \int p d\mu$, $p \in \mathcal{P}_n$. In this case, the measure μ is said to be a *representing measure* for the n-sequence γ. The *Hamburger moment problem* means to characterize those n-sequences which are moment sequences.

Let $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ be a positive semi-definite n-sequence, and let $\mathcal{N} = \{p \in \mathcal{P}_n;\ L_\gamma(p\bar{p}) = 0\}$, which is an ideal in $\mathcal{P}_n$. Then L_γ induces a scalar product $\langle *, * \rangle$ on the quotient $\mathcal{P}_n/\mathcal{N}$, and let $\mathcal{H} = \mathcal{H}_\gamma$ be the completion of the quotient $\mathcal{P}_n/\mathcal{N}$ with respect to this scalar product.

We define in $\mathcal{H}$ the operators

$$T_j(p + \mathcal{N}) = t_j p + \mathcal{N}, p \in \mathcal{P}_n, j = 1, \dots, n,$$

which are symmetric and densely defined, with $D(T_j) = \mathcal{P}_n/\mathcal{N}$ for all j.

THEOREM 2. *Let $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ be a moment n-sequence. The representing measure of γ is unique if and only if the tuple $T = (T_1, \dots, T_n)$ has a smallest selfadjoint extension.*

The existence of general selfadjoint extensions (in the same basic Hilbert space) has been recently studied in [AlVa], using methods partially inspired from the theory of completely bounded maps developed in [Arv] and [Pow].

Let $\mathcal{P}_{n,\alpha}$ be the vector space generated by the monomials $t^\beta = t_1^{\beta_1} \cdots t_n^{\beta_n}$, with $\beta_j \le 2\alpha_j$, $j = 1, \dots, n$, $\alpha \in \mathbf{Z}_+^n$. For each integer $m \ge 1$, we denote by $M_m(\mathcal{P}_{n,\alpha})$ the space of all $m \times m$ matrices with entries from $\mathcal{P}_{n,\alpha}$.

We also consider the polynomials $p_\alpha(t) = (1 + t_1^2)^{\alpha_1} \cdots (1 + t_n^2)^{\alpha_n}$, as well as the rational functions $q_\alpha(t) = p_\alpha(t)^{-1}$, $t = (t_1, \dots, t_n) \in \mathbf{R}^n$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$ is arbitrary.

The next result is a particular case of Theorem 3.3 from [AlVa]. The symbol $\| * \|_m$ stands for the norm of a $m \times m$ complex matrix.

THEOREM 3. *Let $S = (S_1, \dots, S_n)$ be a tuple of symmetric, linear operators defined on a dense subspace $\mathcal{D}$ of the Hilbert space $\mathcal{H}$. Assume that $\mathcal{D}$ is invariant under $S_1, \dots, S_n$ and that $S_1, \dots, S_n$ commute on $\mathcal{D}$.*

The tuple S admits a selfadjoint extension in $\mathcal{H}$ if and only if for all $\alpha \in \mathbf{Z}_+^n$, $m \in \mathbf{N}$ and $x_1, \dots, x_m, y_1, \dots, y_m \in \mathcal{D}$ with

$$\sum_{j=1}^m \langle p_\alpha(S) x_j, x_j \rangle \le 1, \quad \sum_{j=1}^m \langle p_\alpha(S) y_j, y_j \rangle \le 1,$$

and for all $p = (p_{j,k}) \in M_m(\mathcal{P}_{n,\alpha})$ with $\sup_t \|q_\alpha(t) p(t)\|_m \le 1$, we have

$$\Big| \sum_{j,k=1}^m \langle p_{j,k}(S) x_k, y_j \rangle \Big| \le 1.$$

PROBLEM. Characterize, as explicitly as possible (i.e., only in terms of the given data), those tuples of symmetric operators having a smallest selfadjoint extension and describe the structure of such an object, when it exists.

As the symmetric operators in Theorem 2 have all equal deficiency indices, even a solution of the previous problem with this supplementary condition would be of interest.

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