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AFFINE ULTRAREGULAR GENERALIZED FUNCTIONS

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Abstract. Algebras of ultradifferentiable generalized functions satisfying some regularity assumptions are introduced. We give a microlocal analysis within these algebras related to the affine regularity type and the ultradifferentiability property. As a particular case we obtain new algebras of Gevrey generalized functions.

1. Introduction. Current research in the regularity problem in the Colombeau algebra $\mathcal{G}(\Omega)$ is based, e.g. see [10], on the Oberguggenberger subalgebra $\mathcal{G}^{\infty}(\Omega)$, which served as the first intrinsic measure of regularity within the Colombeau algebra. The subalgebra $\mathcal{G}^{\infty}(\Omega)$ plays the same role as $\mathcal{C}^{\infty}(\Omega)$ in $\mathcal{D}'(\Omega)$, and has indicated the importance of the asymptotic behavior of the representative nets of a Colombeau generalized function in studying regularity problems. However, the $\mathcal{G}^{\infty}$-regularity does not exhaust the regularity questions inherent to the Colombeau algebra, see [17]. Candidates proposed for measuring the regularity within $\mathcal{G}(\Omega)$ involve the growth in ε and the asymptotic behavior of the net of smooth functions representing a Colombeau generalized function.

The aim of this paper is to introduce and to study new classes of generalized functions measuring regularity both by the asymptotic behavior of the net of smooth functions representing a Colombeau generalized function and by their ultradifferentiable smoothness. We define the subalgebra $\mathcal{G}^{M,\mathcal{A}}(\Omega)$ of $\mathcal{G}(\Omega)$ representing classes of nets $(u_\varepsilon)_\varepsilon$ of smooth functions having simultaneously ultradifferentiable smoothness of Denjoy-Carleman type

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$M = (M_p)_{p\in\mathbf{Z}_+}$ and affine regular asymptotic behavior in ε. Elements of $\mathcal{G}^{M,\mathcal{A}}(\Omega)$ are called affine ultraregular generalized functions. The importance of ultradifferentiable functions in the study of partial differential equations is well established, see [12], [8], [19] and [3]. Affine ultraregular generalized functions of $(M,\mathcal{A})$ type will without doubt contribute to regularity theory in the Colombeau algebra.

Sections two and three recall respectively generalized functions of Colombeau type and ultradifferentiable functions and give some of their main properties. Section four introduces the algebras $\mathcal{G}^{M,\mathcal{A}}(\Omega)$ of affine ultraregular generalized functions and show their important properties. Section five is devoted to the $\mathcal{G}^{M,\mathcal{A}}$-microlocal analysis of Colombeau generalized functions. The last section is devoted to the extension of Hörmander's theorem on the product of distributions in the case of affine ultraregular generalized functions.

Let us mention the papers [1] and [18] where algebras of generalized ultradistributions are studied. One of the main purposes of these papers is to embed spaces of ultradistributions into algebras of generalized functions of Colombeau type, whereas our algebra $\mathcal{G}^{M,\mathcal{A}}(\Omega)$ is a subalgebra of the Colombeau algebra $\mathcal{G}(\Omega)$, and it is aimed to give new measures of regularity of Colombeau generalized functions.

2. Colombeau algebra. For a deep study of the Colombeau algebra see [4], [7] and [15]. Let Ω be a non-void open subset of $\mathbf{R}^n$, define $\mathcal{E}_m(\Omega)$ as the space of elements $(u_\varepsilon)_\varepsilon$ of $C^\infty(\Omega)^{]0,1]}$ such that for every compact $K \subset \Omega$, $\forall\alpha \in \mathbf{Z}_+^n, \exists m \in \mathbf{Z}_+$, $\exists C > 0, \exists\eta \in]0,1], \forall\varepsilon \in]0,\eta]$,

$$\sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| \leq C\varepsilon^{-m}.$$

By $\mathcal{N}(\Omega)$ we denote the elements $(u_\varepsilon)_\varepsilon \in \mathcal{E}_m(\Omega)$ such that for every compact $K \subset \Omega, \forall\alpha \in \mathbf{Z}_+^n, \forall m \in \mathbf{Z}_+$, $\exists C > 0, \exists\eta \in]0,1], \forall\varepsilon \in]0,\eta]$,

$$\sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| \leq C\varepsilon^{m}.$$

Definition 2.1. The *Colombeau algebra*, denoted $\mathcal{G}(\Omega)$, is the quotient algebra

$$\mathcal{G}(\Omega) = \frac{\mathcal{E}_m(\Omega)}{\mathcal{N}(\Omega)}.$$

$\mathcal{G}(\Omega)$ is a commutative and associative differential algebra containing $D'(\Omega)$ as a subspace and $C^\infty(\Omega)$ as a subalgebra. The subalgebra of generalized functions with compact support, denoted $\mathcal{G}_C(\Omega)$, is the space of elements f of $\mathcal{G}(\Omega)$ satisfying: there exist a representative $(f_\varepsilon)_{\varepsilon\in]0,1]}$ of f and a compact subset K of $\Omega, \forall\varepsilon \in]0,1]$, $\mathrm{supp} f_\varepsilon \subset K$.

One defines the subalgebra of regular elements $\mathcal{G}^\infty(\Omega)$, introduced by Oberguggenberger in [16], as the quotient algebra

$$\frac{\mathcal{E}_m^\infty(\Omega)}{\mathcal{N}(\Omega)},$$

where $\mathcal{E}_m^\infty(\Omega)$ is the space of elements $(u_\varepsilon)_\varepsilon$ of $C^\infty(\Omega)^{]0,1]}$ such that for every compact $K \subset \Omega$, $\exists m \in \mathbf{Z}_+, \forall\alpha \in \mathbf{Z}_+^n$, $\exists C > 0, \exists\eta \in]0,1], \forall\varepsilon \in]0,\eta]$,

$$\sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| \leq C\varepsilon^{-m}.$$

The following fundamental result is proved in [16]:

$$\mathcal{G}^{\infty}(\Omega) \cap \mathcal{D}'(\Omega) = C^{\infty}(\Omega).$$

This means that the subalgebra $\mathcal{G}^{\infty}(\Omega)$ plays in $\mathcal{G}(\Omega)$ the same role as $C^{\infty}(\Omega)$ in $\mathcal{D}'(\Omega)$, consequently one can introduce a local analysis by defining the generalized singular support of $u \in \mathcal{G}(\Omega)$. This was the first notion of regularity in the Colombeau algebra. Recently, different measures of regularity in algebras of generalized functions have been proposed, see [2], [6], [14] and [17].

3. Ultradifferentiable functions. We recall some classical results on ultradifferentiable function spaces. A sequence of positive numbers $(M_p)_{p \in \mathbf{Z}_+}$ is said to satisfy the following conditions:

$(H1)$ logarithmic convexity, if

$$M_p^2 \leq M_{p-1} M_{p+1}, \forall p \geq 1;$$

$(H2)$ stability under ultradifferential operators, if there exist $A > 0$ and $H > 0$ such that

$$M_p \leq A H^p M_q M_{p-q}, \forall p \geq q;$$

$(H3)'$ non-quasi-analyticity, if

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < +\infty.$$

Remark 3.1. Some results remain valid, see [11], when $(H2)$ is replaced by the following weaker condition:

$(H2)'$ stability under differential operators, if there exist $A > 0$ and $H > 0$ such that

$$M_{p+1} \leq A H^p M_p, \forall p \in \mathbf{Z}_+.$$

The associated function of the sequence $(M_p)_{p \in \mathbf{Z}_+}$ is the function $\widetilde{M}$, defined by

$$\widetilde{M}(t) = \sup_p \ln \frac{t^p}{M_p}, t \in \mathbf{R}_+^*.$$

Some results on the associated function are given in the following propositions proved in [11].

Proposition 3.2. *A positive sequence $(M_p)_{p \in \mathbf{Z}_+}$ satisfies condition $(H1)$ if and only if*

$$M_p = M_0 \sup_{t>0} [t^p \exp(-\widetilde{M}(t))], \quad p \in \mathbf{Z}_+.$$

Proposition 3.3. *A positive sequence $(M_p)_{p \in \mathbf{Z}_+}$ satisfies $(H2)$ if and only if $\exists A > 0, \exists H > 0, \forall t > 0$,*

$$2\widetilde{M}(t) \leq \widetilde{M}(Ht) + \ln(A M_0).$$

Remark 3.4. We will always suppose that the sequence $(M_p)_{p \in \mathbf{Z}_+}$ satisfies the condition $(H1)$ and $M_0 = 1$.

The space of ultradifferentiable functions of class M, denoted $E^M(\Omega)$, is the set of all $f \in C^\infty(\Omega)$ satisfying for every compact $K \subset \Omega$, $\exists c > 0, \forall \alpha \in \mathbf{Z}_+^n$,

$$\sup_{x \in K} |\partial^\alpha f(x)| \leq c^{|\alpha|+1} M_{|\alpha|}.$$

This space is also called the space of Denjoy-Carleman.

Example 3.5. If $(M_p)_{p\in\mathbf{Z}_+} = (p!^\sigma)_{p\in\mathbf{Z}_+}, \sigma > 1$, we obtain $E^\sigma(\Omega)$ the Gevrey space of order σ, and $\mathcal{A}(\Omega) := E^1(\Omega)$ is the space of real analytic functions defined on the open set Ω.

The basic properties of the space $E^M(\Omega)$ are summarized in the following proposition, for the proof see [13] and [11].

Proposition 3.6. *The space $E^M(\Omega)$ is an algebra. Moreover, if $(M_p)_{p\in\mathbf{Z}_+}$ satisfies $(H2)'$, then $E^M(\Omega)$ is stable by any differential operator of finite order with coefficients in $E^M(\Omega)$ and if $(M_p)_{p\in\mathbf{Z}_+}$ satisfies $(H2)$ then any ultradifferential operator of class M operates also as a sheaf homomorphism. The space $\mathcal{D}^M(\Omega) = E^M(\Omega) \cap \mathcal{D}(\Omega)$ is well defined and is not trivial if and only if the sequence $(M_p)_{p\in\mathbf{Z}_+}$ satisfies $(H3)'$.*

Remark 3.7. The strong dual of $\mathcal{D}^M(\Omega)$, denoted $\mathcal{D}'^M(\Omega)$, is called the space of Roumieu ultraditributions.

4. Affine ultraregular generalized functions. The purpose of this section is to introduce a notion of regularity within Colombeau algebra taking into account both the asymptotic growth and the smoothness property of generalized functions. We introduce algebras of ultradifferentiable regular generalized functions of class M, where the sequence $M = (M_p)_{p\in\mathbf{Z}_+}$satisfies the conditions $(H1)$ with $M_0 = 1, (H2)$ and $(H3)'$.

Definition 4.1. The space of *affine ultraregular moderate elements of class M*, denoted by $\mathcal{E}_m^{M,\mathcal{A}}(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{]0,1]}$ satisfying for every compact subset K of Ω, $\exists a \geq 0, \exists b \geq 0, \exists C > 0, \exists \varepsilon_0 \in]0,1], \forall \alpha \in \mathbf{Z}_+^n, \forall \varepsilon \leq \varepsilon_0$,

$$\sup_{x\in K} |\partial^\alpha f_\varepsilon(x)| \leq C^{|\alpha|+1} M_{|\alpha|} \varepsilon^{-a|\alpha|-b}.$$

The space of *null elements* is defined as $\mathcal{N}^{M,\mathcal{A}}(\Omega) := \mathcal{N}(\Omega) \cap \mathcal{E}_m^{M,\mathcal{A}}(\Omega)$.

The main properties of the spaces $\mathcal{E}_m^{M,\mathcal{A}}(\Omega)$ and $\mathcal{N}^{M,\mathcal{A}}(\Omega)$ are given in the following proposition.

Proposition 4.2.

1) *The space $\mathcal{E}_m^{M,\mathcal{A}}(\Omega)$ is a subalgebra of $\mathcal{E}_m(\Omega)$ stable under the action of differential operators.*
2) *The space $\mathcal{N}^{M,\mathcal{A}}(\Omega)$ is an ideal of $\mathcal{E}_m^{M,\mathcal{A}}(\Omega)$.*

Proof. 1) Let $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{E}_m^{M,\mathcal{A}}(\Omega)$ and K a compact subset of Ω,
$\exists a_1 \geq 0, \exists b_1 \geq 0, \exists C_1 > 0, \exists \varepsilon_1 \in]0,1]$, such that $\forall \beta \in \mathbf{Z}_+^n, \forall x \in K, \forall \varepsilon \leq \varepsilon_1$,

$$\left|\partial^\beta f_\varepsilon(x)\right| \leq C_1^{|\beta|+1} M_{|\beta|} \varepsilon^{-a_1|\beta|-b_1}$$

$\exists a_2 \geq 0, \exists b_2 \geq 0, \exists C_2 > 0, \exists \varepsilon_2 \in]0,1]$, such that $\forall \beta \in \mathbf{Z}_+^n, \forall x \in K, \forall \varepsilon \leq \varepsilon_2$,

$$\left|\partial^\beta g_\varepsilon(x)\right| \leq C_2^{|\beta|+1} M_{|\beta|} \varepsilon^{-a_2|\beta|-b_2}.$$

It is clear that $(f_\varepsilon + g_\varepsilon)_\varepsilon \in \mathcal{E}_m^{M,\mathcal{A}}(\Omega)$. Let $\alpha \in \mathbf{Z}_+^n$,

$$|\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| \leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \left|\partial^{\alpha-\beta} f_\varepsilon(x)\right| \left|\partial^\beta g_\varepsilon(x)\right|.$$

Take $a = \max(a_1, a_2)$ and $b = b_1 + b_2$. From $(H1)$, we have $M_p M_q \leq M_{p+q}$, so for $\varepsilon \leq \min\{\varepsilon_1, \varepsilon_2\}$ and $x \in K$

$$\begin{aligned}\frac{\varepsilon^{a|\alpha|+b}}{M_{|\alpha|}} |\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \frac{\varepsilon^{a_1|\alpha-\beta|+b_1}}{M_{|\alpha-\beta|}} \left|\partial^{\alpha-\beta} f_\varepsilon(x)\right| \frac{\varepsilon^{a_2|\beta|+b_2}}{M_{|\beta|}} \left|\partial^\beta g_\varepsilon(x)\right| \\ &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} C_1^{|\alpha-\beta|+1} C_2^{|\beta|+1} \leq C^{|\alpha|+1},\end{aligned}$$

where $C = \max\{C_1 C_2, C_1 + C_2\}$, which proves that $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{E}_m^{M,\mathcal{A}}(\Omega)$. Let now $\alpha, \beta \in \mathbf{Z}_+^n$, where $|\beta| = 1$, for $\varepsilon \leq \varepsilon_1$ and $x \in K$, we have

$$\left|\partial^\alpha (\partial^\beta f_\varepsilon)(x)\right| \leq C_1^{|\alpha|+2} M_{|\alpha|+1} \varepsilon^{-N_{|\alpha|+1}}.$$

From $(H2)'$, $\exists A > 0, H > 0$, such that $M_{|\alpha|+1} \leq A H^{|\alpha|} M_{|\alpha|}$, then

$$\left|\partial^\alpha (\partial^\beta f_\varepsilon)(x)\right| \leq A C_1^2 (C_1 H)^{|\alpha|} M_{|\alpha|} \varepsilon^{-a_1|\alpha|-a_1-b_1} \leq C^{|\alpha|+1} M_{|\alpha|} \varepsilon^{-a|\alpha|-b},$$

where $a = a_1$ and $b = a_1 + b_2$, which means $(\partial^\beta f_\varepsilon)_\varepsilon \in \mathcal{E}_m^{M,\mathcal{A}}(\Omega)$.

2) The fact that $\mathcal{N}^{M,\mathcal{A}}(\Omega) = \mathcal{N}(\Omega) \cap \mathcal{E}_m^{M,\mathcal{A}}(\Omega) \subset \mathcal{E}_m^{M,\mathcal{A}}(\Omega)$ and $\mathcal{N}(\Omega) = \mathcal{N}^{\mathcal{A}}(\Omega)$ is an ideal of $\mathcal{E}_m^{\mathcal{A}}(\Omega)$ implies that $\mathcal{N}^{M,\mathcal{A}}(\Omega)$ is an ideal of $\mathcal{E}_m^{M,\mathcal{A}}(\Omega)$. ■

The following definition introduces the algebra of affine ultraregular generalized functions.

Definition 4.3. The set of *affine ultraregular generalized functions* of class $(M_p)_{p\in\mathbf{Z}_+}$ is the quotient algebra

$$\mathcal{G}^{M,\mathcal{A}}(\Omega) := \frac{\mathcal{E}_m^{M,\mathcal{A}}(\Omega)}{\mathcal{N}^{M,\mathcal{A}}(\Omega)}.$$

The basic properties of $\mathcal{G}^{M,\mathcal{A}}(\Omega)$ are given in the following assertion.

Proposition 4.4. *$\mathcal{G}^{M,\mathcal{A}}(\Omega)$ is a differential subalgebra of $\mathcal{G}(\Omega)$.*

Proof. All algebraic properties hold from Proposition 4.2. ■

Example 4.5. If $a = 0$ we obtain as a particular case the algebra $\mathcal{G}^{M,\mathcal{B}}(\Omega)$ of [14] denoted there by $\mathcal{G}^L(\Omega)$.

Example 4.6. If we take $(M_p)_{p\in\mathbf{Z}_+} = (p!^\sigma)_{p\in\mathbf{Z}_+}$ we obtain a new subalgebra $\mathcal{G}^{\sigma,\mathcal{A}}(\Omega)$ of $\mathcal{G}(\Omega)$: the algebra of Gevrey affine regular generalized functions of order σ.

Example 4.7. If $a = 0$ and $(M_p)_{p\in\mathbf{Z}_+} = (p!^\sigma)_{p\in\mathbf{Z}_+}$ we obtain a new algebra, denoted $\mathcal{G}^{\sigma,\infty}(\Omega)$, that we will call the Gevrey-Oberguggenberger algebra of order σ.

The space $E^M(\Omega)$ is embedded into $\mathcal{G}^{M,\mathcal{A}}(\Omega)$ by the canonical map

$$\begin{array}{ccc} \sigma : E^M(\Omega) & \to & \mathcal{G}^{M,\mathcal{A}}(\Omega) \\ u & \to & [u_\varepsilon] \end{array},$$

where $u_\varepsilon = u$ for all $\varepsilon \in \left]0,1\right]$, which is an injective homomorphism of algebras.

Proposition 4.8. *The diagram*

$$\begin{array}{ccccc} E^M(\Omega) & \to & C^\infty(\Omega) & \to & \mathcal{D}'(\Omega) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G}^{M,\mathcal{B}}(\Omega) & \to & \mathcal{G}^{\mathcal{B}}(\Omega) & \to & \mathcal{G}(\Omega) \end{array}$$

is commutative, where $\mathcal{G}^{\mathcal{B}}(\Omega) = \mathcal{G}^\infty(\Omega)$.

Proof. The embeddings in the diagram are canonical except the embedding $\mathcal{D}'(\Omega) \to \mathcal{G}(\Omega)$, which is now well known in the framework of Colombeau generalized functions, see [7] for details. The commutativity of the diagram is then obtained easily from the commutativity of the classical diagram

$$\begin{array}{ccc} C^\infty(\Omega) & \to & \mathcal{D}'(\Omega) \\ & \searrow & \downarrow \\ & & \mathcal{G}(\Omega) \end{array}$$

which completes the proof. ∎

A fundamental result on regularity in $\mathcal{G}(\Omega)$ is the following.

Theorem 4.9. *We have $\mathcal{G}^{M,\mathcal{B}}(\Omega) \cap \mathcal{D}'(\Omega) = E^M(\Omega)$.*

Proof. Let $u = cl(u_\varepsilon)_\varepsilon \in \mathcal{G}^{M,\mathcal{B}}(\Omega) \cap C^\infty(\Omega)$, i.e. $(u_\varepsilon)_\varepsilon \in \mathcal{E}_m^{M,\mathcal{B}}(\Omega)$, then we have for every compact set $K \subset \Omega, \exists N \in \mathbf{Z}_+, \exists c > 0, \exists \eta \in \left]0,1\right], \forall \alpha \in \mathbf{Z}_+^n, \forall \varepsilon \in \left]0,\eta\right]$,

$$\sup_{x \in K} |\partial^\alpha u(x)| \le c^{|\alpha|+1} M_{|\alpha|} \varepsilon^{-N}.$$

When choosing $\varepsilon = \eta$, we obtain

$$\forall \alpha \in \mathbf{Z}_+^n, \sup_{x \in K} |\partial^\alpha u(x)| \le c^{|\alpha|+1} M_{|\alpha|} \eta^{-N} \le c_1^{|\alpha|+1} M_{|\alpha|},$$

where c_1 depends only on K. Then u is in $E^M(\Omega)$, this shows $\mathcal{G}^{M,\mathcal{B}}(\Omega) \cap C^\infty(\Omega) \subset E^M(\Omega)$. As the reverse inclusion is obvious, we have proved $\mathcal{G}^{M,\mathcal{B}}(\Omega) \cap C^\infty(\Omega) = E^M(\Omega)$. Consequently

$$\begin{aligned} \mathcal{G}^{M,\mathcal{B}}(\Omega) \cap \mathcal{D}'(\Omega) &= (\mathcal{G}^{M,\mathcal{B}}(\Omega) \cap \mathcal{G}^{\mathcal{B}}(\Omega)) \cap \mathcal{D}'(\Omega) \\ &= \mathcal{G}^{M,\mathcal{B}}(\Omega) \cap (\mathcal{G}^{\mathcal{B}}(\Omega) \cap \mathcal{D}'(\Omega)) \\ &= \mathcal{G}^{M,\mathcal{B}}(\Omega) \cap C^\infty(\Omega) = E^M(\Omega), \end{aligned}$$

which completes the proof. ∎

Proposition 4.10. *The algebra $\mathcal{G}^{M,\mathcal{A}}(\Omega)$ is a sheaf of subalgebras of $\mathcal{G}(\Omega)$.*

Proof. Let Ω be a non-void open of $\mathbf{R}^n$ and $(\Omega_\lambda)_{\lambda \in \Lambda}$ be an open covering of Ω. we have to show the properties

S1) If $f, g \in \mathcal{G}^{M,\mathcal{A}}(\Omega)$ such that $f_{/\Omega_\lambda} = g_{/\Omega_\lambda}, \forall \lambda \in \Lambda$, then $f = g$.

S2) If for each $\lambda \in \Lambda$, we have $f_\lambda \in \mathcal{G}^{M,\mathcal{A}}(\Omega_\lambda)$, such that for all $\lambda, \mu \in \Lambda$ with $\Omega_\lambda \cap \Omega_\mu \neq \phi$,

$$f_{\lambda/\Omega_\lambda \cap \Omega_\mu} = f_{\mu/\Omega_\lambda \cap \Omega_\mu},$$

then there exists a unique $f \in \mathcal{G}^{M,\mathcal{A}}(\Omega)$ with $f_{/\Omega_\lambda} = f_\lambda, \forall \lambda \in \Lambda$.

The property S1 is evident. To show S2, let $(\chi_j)_{j=1}^{\infty}$ be a E^M-partition of unity subordinate to the covering $(\Omega_\lambda)_{\lambda\in\Lambda}$. Define

$$f := (f_\varepsilon)_\varepsilon + \mathcal{N}^{M,\mathcal{A}}(\Omega),$$

where $f_\varepsilon = \sum_{j=1}^{\infty} \chi_j f_{\lambda_j\varepsilon}$ and $(f_{\lambda_j\varepsilon})_\varepsilon$ is a representative of f_{λ_j}. Moreover, we set $f_{\lambda_j\varepsilon} = 0$ on $\Omega \backslash \Omega_{\lambda_j}$, so that $\chi_j f_{\lambda_j\varepsilon}$ is C^∞ on all of Ω. First let K be compact subset of Ω, we have $K_j = K \cap \operatorname{supp}\chi_j$ is a compact subset of Ω_{λ_j} and $(f_{\lambda_j\varepsilon})_\varepsilon \in \mathcal{E}_m^{M,\mathcal{A}}\left(\Omega_{\lambda_j}\right)$, then $\left(\chi_j f_{\lambda_j\varepsilon}\right)$ satisfies $\mathcal{E}_m^{M,\mathcal{A}}$-estimate on each K_j, we have $\chi_j(x) \equiv 0$ on K except for a finite number of j, i.e. $\exists N > 0$, such that

$$\sum_{j=1}^{\infty} \chi_j f_{\lambda_j\varepsilon}(x) = \sum_{j=1}^{N} \chi_j f_{\lambda_j\varepsilon}(x), \forall x \in K.$$

So $(\sum \chi_j f_{\lambda_j\varepsilon})$ satisfies $\mathcal{E}_m^{M,\mathcal{A}}$-estimate on K, which means $(f_\varepsilon)_\varepsilon \in \mathcal{E}_m^{M,\mathcal{A}}(\Omega)$. It remains to show that $f_{/\Omega_\lambda} = f_\lambda, \forall \lambda \in \Lambda$. Let K be a compact subset of Ω_λ, choose $N > 0$ in such a way that $\sum_{j=1}^{N} \chi_j(x) \equiv 1$ on a neighborhood Ω' of K with $\overline{\Omega'}$ is compact of Ω_λ. For $x \in K$,

$$f_\varepsilon(x) - f_{\lambda\varepsilon}(x) = \sum_{j=1}^{N} \chi_j(x) \left(f_{\lambda_j\varepsilon}(x) - f_{\lambda\varepsilon}(x)\right).$$

Since $(f_{\lambda_j\varepsilon} - f_{\lambda\varepsilon}) \in \mathcal{N}^{M,\mathcal{A}}(\Omega_{\lambda_j} \cap \Omega_\lambda)$ and $K_j = K \cap \operatorname{supp}\chi_j$ is a compact subset of $\Omega \cap \Omega_{\lambda_j}$, then $(\sum_{j=1}^{N} \chi_j (f_{\lambda_j\varepsilon} - f_{\lambda\varepsilon}))$ satisfies the $\mathcal{N}^{M,\mathcal{A}}$-estimate on K. The uniqueness of such $f \in \mathcal{G}^{M,\mathcal{A}}(\Omega)$ follows from S1. ∎

Definition 4.11. The $(M, \mathcal{A})$-*singular support* of a generalized function $u \in \mathcal{G}(\Omega)$, denoted $\operatorname{sing\,supp}_{M,\mathcal{A}}(u)$, is the complement of the largest open set Ω' such that $u \in \mathcal{G}^{M,\mathcal{A}}(\Omega')$.

The basic property of $\operatorname{sing\,supp}_{M,\mathcal{A}}$ is summarized in the following, easy to prove, proposition.

Proposition 4.12. *Let $P(x, D)$ be a generalized linear partial differential operator with coefficients in $\mathcal{G}^{M,\mathcal{A}}(\Omega)$, then*

$$\operatorname{sing\,supp}_{M,\mathcal{A}}(P(x, D)\,u) \subset \operatorname{sing\,supp}_{M,\mathcal{A}}(u), \forall u \in \mathcal{G}(\Omega).$$

We introduce a local analysis within the Colombeau algebra; a generalized partial differential operator $P(x, D)$ with $\mathcal{G}^{M,\mathcal{A}}(\Omega)$ coefficients is called $(M, \mathcal{A})$-*hypoelliptic* in Ω, if

$$\operatorname{sing\,supp}_{M,\mathcal{A}}(P(x, D)\,u) = \operatorname{sing\,supp}_{M,\mathcal{A}}(u), \forall u \in \mathcal{G}(\Omega).$$

5. $(M,\mathcal{A})$**-Microlocal analysis.** We have defined the $(M,\mathcal{A})$-*singular support* of a generalized function $u\in\mathcal{G}(\Omega)$, we will now show how to microlocalize this concept.

A basic $(M,\mathcal{A})$-*microlocal analysis* in $\mathcal{G}(\Omega)$ can be developed thanks to the following result.

PROPOSITION 5.1. *Let $f=cl(f_\varepsilon)_\varepsilon\in\mathcal{G}_C(\Omega)$, then f is affine ultraregular of class $M=(M_p)_{p\in\mathbf{Z}_+}$ if and only if $\exists a\geq 0,\exists b\geq 0,\exists C>0,\exists k>0$, $\exists\varepsilon_0\in\,]0,1]\,,\forall\varepsilon\leq\varepsilon_0$, such that*

$$\left|\mathcal{F}(f_\varepsilon)\left(\xi\right)\right|\leq C\varepsilon^{-b}\exp-\widetilde{M}\left(k\varepsilon^a\left|\xi\right|\right),\forall\xi\in\mathbf{R}^n, \tag{1}$$

where $\mathcal{F}$ denotes the Fourier transform.

Proof. Suppose that $f=cl(f_\varepsilon)_\varepsilon\in\mathcal{G}_C(\Omega)\cap\mathcal{G}^{M,\mathcal{A}}(\Omega)$, then $\exists C>0,\exists a\geq 0,\exists b\geq 0,\exists\varepsilon_1>0,\forall\alpha\in\mathbf{Z}_+^n,\forall x\in K,\forall\varepsilon\leq\varepsilon_0$, $\operatorname{supp}f_\varepsilon\subset K$, such that

$$\left|\partial^\alpha f_\varepsilon\right|\leq C^{|\alpha|+1}M_{|\alpha|}\varepsilon^{-a|\alpha|-b},$$

so we have, $\forall\alpha\in\mathbf{Z}_+^n$,

$$\left|\xi^\alpha\right|\left|\mathcal{F}(f_\varepsilon)(\xi)\right|\leq\left|\int\exp\left(-ix\xi\right)\partial^\alpha f_\varepsilon(x)dx\right|$$

then, $\exists C>0,\forall\varepsilon\leq\varepsilon_0$,

$$\left|\xi\right|^{|\alpha|}\left|\mathcal{F}(f_\varepsilon)(\xi)\right|\leq C^{|\alpha|+1}M_{|\alpha|}\varepsilon^{--a|\alpha|-b},$$

which give

$$\begin{aligned}\left|\mathcal{F}(f_\varepsilon)\left(\xi\right)\right|&\leq C\frac{C^{|\alpha|}M_{|\alpha|}}{\left|\xi\right|^{|\alpha|}}\varepsilon^{-a|\alpha|-b}\leq C\varepsilon^{-b}\inf_\alpha\left\{\left(\frac{\varepsilon^{-a}C}{|\xi|}\right)^{|\alpha|}M_{|\alpha|}\right\}\\&\leq C\varepsilon^{-b}\exp-\widetilde{M}\left(\frac{\varepsilon^a\left|\xi\right|}{\sqrt{n}C}\right),\end{aligned}$$

i.e. we have (1).

Suppose now that (1) is valid, then $\exists C>0,\exists\varepsilon_0\in\,]0,1],\forall\varepsilon\leq\varepsilon_0$,

$$\begin{aligned}\left|\partial^\alpha f_\varepsilon(x)\right|&\leq C\varepsilon^{-b}\sup_\xi\left|\xi\right|^{|\alpha|}\exp(-\widetilde{M}(k\varepsilon^a|\xi|))\\&\leq C\varepsilon^{-b}(k\varepsilon^a)^{-|\alpha|}\sup_\xi\left|k\varepsilon^a\xi\right|^{|\alpha|}\exp(-\widetilde{M}(k\varepsilon^a|\xi|))\\&\leq C\varepsilon^{-b}(k\varepsilon^a)^{-|\alpha|}\sup_\eta\left|\eta\right|^{|\alpha|}\exp(-\widetilde{M}(|\eta|)).\end{aligned}$$

Proposition 3.2 gives $\exists C>0$ such that

$$\left|\partial^\alpha f_\varepsilon(x)\right|\leq C^{|\alpha|+1}M_{|\alpha|}\varepsilon^{-a|\alpha|-b},$$

where $C=\max\left(C,\frac{1}{k}\right)$, then $f\in\mathcal{G}^{M,\mathcal{A}}(\Omega)$. ∎

COROLLARY 5.2. *Let $f=cl(f_\varepsilon)_\varepsilon\in\mathcal{G}_C(\Omega)$, then f is a Gevrey affine ultraregular generalized function of order σ, i.e. $f\in\mathcal{G}^{\sigma,\mathcal{A}}(\Omega)$, if and only if $\exists a\geq 0,\exists b\geq 0,\exists C>0,\exists k>0$, $\exists\varepsilon_0>0,\forall\varepsilon\leq\varepsilon_0$, such that*

$$\left|\mathcal{F}(f_\varepsilon)(\xi)\right|\leq C\varepsilon^{-b}\exp(-k\varepsilon^a|\xi|^{\frac{1}{\sigma}}),\forall\xi\in\mathbf{R}^n.$$

In particular, f is a Gevrey generalized function of order σ, i.e. $f \in \mathcal{G}^{\sigma,\infty}(\Omega)$, if and only if $\exists b \geq 0, \exists C > 0, \exists k > 0$, $\exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0$, *such that*

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq C\varepsilon^{-b} \exp(-k|\xi|^{\frac{1}{\sigma}}), \forall \xi \in \mathbf{R}^n.$$

The above results permit to define the concept of $\mathcal{G}^{M,\mathcal{A}}$-wave front of $u \in \mathcal{G}(\Omega)$ and give the basic elements of a $(M,\mathcal{A})$-*generalized microlocal analysis* within the Colombeau algebra $\mathcal{G}(\Omega)$.

Definition 5.3. Define the cone $\Sigma_{\mathcal{A}}^M(f) \subset \mathbf{R}^n \backslash \{0\}, f \in \mathcal{G}_C(\Omega)$, as the complement of the set of points having a conic neighborhood Γ and $\exists a \geq 0, \exists b \geq 0, \exists C > 0, \exists k > 0$, $\exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0$, such that

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq C\varepsilon^{-b} \exp -\widetilde{M}(k\varepsilon^a|\xi|), \forall \xi \in \Gamma.$$

Proposition 5.4. *For every $f \in \mathcal{G}_C(\Omega)$, we have*

1) *The set $\Sigma_{\mathcal{A}}^M(f)$ is a closed subset;*
2) $\Sigma_{\mathcal{A}}^M(f) = \emptyset \Leftrightarrow f \in \mathcal{G}^{M,\mathcal{A}}(\Omega)$.

Proof. The proof of 1) is trivial, and 2) holds from Proposition 5.1. ■

Proposition 5.5. *For every $f \in \mathcal{G}_C(\Omega)$, we have*

$$\Sigma_{\mathcal{A}}^M(\psi f) \subset \Sigma_{\mathcal{A}}^M(f), \forall \psi \in E^M(\Omega).$$

Proof. Let $\xi_0 \notin \Sigma_{\mathcal{A}}^M(f)$, i.e. $\exists \Gamma$ a conic neighborhood of $\xi_0, \exists a \geq 0, \exists b \geq 0, \exists k_1 > 0, \exists c_1 > 0, \exists \varepsilon_1 \in \left]0,1\right], \forall \varepsilon \leq \varepsilon_1$,

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c_1\varepsilon^{-b} \exp -\widetilde{M}\left(k_1\varepsilon^a|\xi|\right), \forall \xi \in \Gamma.$$

Let $\chi \in \mathcal{D}^M(\Omega)$, $\chi = 1$ on a neighborhood of supp$f$, then $\chi\psi \in \mathcal{D}^M(\Omega), \forall \psi \in E^M(\Omega)$, hence, see [11], $\exists k_2 > 0, \exists c_2 > 0, \forall \xi \in \mathbf{R}^n$,

$$|\mathcal{F}(\chi\psi)(\xi)| \leq c_2 \exp -\widetilde{M}\left(k_2\,|\xi|\right).$$

Let Λ be a conic neighborhood of ξ_0 such that $\overline{\Lambda} \subset \Gamma$, then we have, for $\xi \in \Lambda$,

$$\begin{aligned}\mathcal{F}\left(\chi\psi f_\varepsilon\right)(\xi) &= \int_{\mathbf{R}^n} \mathcal{F}(f_\varepsilon)(\eta)\mathcal{F}(\chi\psi)(\eta-\xi)d\eta \\ &= \int_A \mathcal{F}(f_\varepsilon)(\eta)\mathcal{F}(\chi\psi)(\eta-\xi)d\eta + \int_B \mathcal{F}(f_\varepsilon)(\eta)\mathcal{F}(\chi\psi)(\eta-\xi)d\eta,\end{aligned}$$

where $A = \{\eta; |\xi-\eta| \leq \delta(|\xi|+|\eta|)\}$ and $B = \{\eta; |\xi-\eta| > \delta(|\xi|+|\eta|)\}$. Take $\delta > 0$ sufficiently small such that $\frac{|\xi|}{2} < |\eta| < 2|\xi|, \forall \eta \in A$, then $\exists c > 0, \forall \varepsilon \leq \varepsilon_1$,

$$\left|\int_A \mathcal{F}(f_\varepsilon)\,(\eta)\,\mathcal{F}(\chi\psi)\,(\eta-\xi)\,d\eta\right| \leq c\varepsilon^{-b} \exp -\widetilde{M}\left(k_1\varepsilon^a\frac{|\xi|}{2}\right)\int_A \exp -\widetilde{M}\left(k_2\,|\eta-\xi|\right)d\eta,$$

so $\exists c > 0, \exists k > 0$,

$$\left|\int_A \mathcal{F}(f_\varepsilon)\,(\eta)\,\mathcal{F}(\chi\psi)(\eta-\xi)d\eta\right| \leq c\varepsilon^{-b} \exp(-\widetilde{M}(k\varepsilon^a|\xi|)). \tag{2}$$

As $f \in \mathcal{G}_C(\Omega)$, then $\exists q \in \mathbf{Z}_+, \exists m > 0, \exists c > 0, \exists \varepsilon_2 > 0, \forall \varepsilon \leq \varepsilon_2$,

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c\varepsilon^{-q}\,|\xi|^m, \forall \xi \in \mathbf{R}^n,$$

hence for $\varepsilon \leq \min(\varepsilon_1, \varepsilon_2)$, $\exists c > 0$, such that we have

$$\left| \int_B \mathcal{F}(f_\varepsilon)(\eta)\, \mathcal{F}(\chi\psi)(\eta-\xi) d\eta \right| \leq c\varepsilon^{-q} \int_B |\eta|^m \exp(-\widetilde{M}(k_2|\eta-\xi|)) d\eta$$
$$\leq c\varepsilon^{-q} \int_B |\eta|^m \exp -\widetilde{M}(k_2\delta(|\xi|+|\eta|)) d\eta.$$

Proposition 3.3 gives $\exists H > 0, \exists A > 0, \forall t_1 > 0, \forall t_2 > 0,$

$$-\widetilde{M}(t_1+t_2) \leq -\widetilde{M}\left(\frac{t_1}{H}\right) - \widetilde{M}\left(\frac{t_2}{H}\right) + \ln A,$$

so

$$\left| \int_B \mathcal{F}(f_\varepsilon)(\eta)\mathcal{F}(\chi\psi)(\eta-\xi) d\eta \right| \leq cA\varepsilon^{-q} \exp -\widetilde{M}\left(\frac{k_2\delta}{H}|\xi|\right)$$
$$\cdot \int_B |\eta|^m \exp -\widetilde{M}\left(\frac{k_2\delta}{H}|\eta|\right) d\eta.$$

Hence $\exists c > 0, \exists k > 0$, such that

$$\left| \int_B \widehat{f}_\varepsilon(\eta)\widehat{\psi}(\eta-\xi) d\eta \right| \leq c\varepsilon^{-q} \exp -\widetilde{M}(k\varepsilon^a |\xi|), \tag{3}$$

consequently (2) and (3) give $\xi_0 \notin \Sigma_{\mathcal{A}}^M(\psi f)$. ■

We define the set $\Sigma_{\mathcal{A},x_0}^M(f)$ for a generalized function f and a point x_0 and the affine wave front set of class M in $\mathcal{G}(\Omega)$.

Definition 5.6. Let $f \in \mathcal{G}(\Omega)$ and $x_0 \in \Omega$, the cone of affine singular directions of class $M = (M_p)$ of f at x_0 is

$$\Sigma_{\mathcal{A},x_0}^M(f) := \bigcap \left\{ \Sigma_{\mathcal{A}}^M(\phi f) : \phi \in \mathcal{D}^M(\Omega) \text{ and } \phi = 1 \text{ on a neighborhood of } x_0 \right\}.$$

The following lemma indicates the relation between the local and microlocal $(M,\mathcal{A})$-analysis in $\mathcal{G}(\Omega)$.

Lemma 5.7. *Let $f \in \mathcal{G}(\Omega)$, then*

$$\Sigma_{\mathcal{A},x_0}^M(f) = \emptyset \Leftrightarrow x_0 \notin \operatorname{sing\,supp}_{M,\mathcal{A}}(f).$$

Proof. See the proof of the similar lemma 22 in [1]. ■

Definition 5.8. A point $(x_0, \xi_0) \notin WF_{\mathcal{A}}^M(f) \subset \Omega \times \mathbf{R}^n \backslash \{0\}$ if $\exists \phi \in \mathcal{D}^M(\Omega), \phi \equiv 1$ on a neighborhood of x_0, a conic neighborhood Γ of ξ_0, $a \geq 0, b \geq 0, k > 0, c > 0, \varepsilon_0 \in]0,1]$, such that $\forall \varepsilon \leq \varepsilon_0, \forall \xi \in \Gamma$,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi)| \leq c\varepsilon^{-b} \exp -\widetilde{M}(k\varepsilon^a|\xi|).$$

Remark 5.9. A point $(x_0, \xi_0) \notin WF_{\mathcal{A}}^M(f) \subset \Omega \times \mathbf{R}^n \backslash \{0\}$ means $\xi_0 \notin \Sigma_{\mathcal{A},x_0}^M(f)$.

The basic properties of $WF_{\mathcal{A}}^M$ are given in the following proposition.

Proposition 5.10. *Let $f \in \mathcal{G}(\Omega)$, then*

1) *The projection of $WF_{\mathcal{A}}^M(f)$ on Ω is the* $\operatorname{sing\,supp}_{M,\mathcal{A}}(f)$*;*
2) *If $f \in \mathcal{G}_C(\Omega)$, then the projection of $WF_{\mathcal{A}}^M(f)$ on $\mathbf{R}^n \backslash \{0\}$ is $\Sigma_{\mathcal{A}}^M(f)$;*
3) *$WF_{\mathcal{A}}^M(\partial^\alpha f) \subset WF_{\mathcal{A}}^M(f), \forall \alpha \in \mathbf{Z}_+^n$;*
4) *$WF_{\mathcal{A}}^M(gf) \subset WF_{\mathcal{A}}^M(f), \forall g \in \mathcal{G}^{M,\mathcal{A}}(\Omega)$.*

Proof. 1) and 2) hold by definition, Proposition 5.4 and Lemma 5.7.

3) Let $(x_0,\xi_0) \notin WF^M_{\mathcal{A}}(f)$, then $\exists \phi \in \mathcal{D}^M(\Omega), \phi \equiv 1$ on $\overline{U}$, where U is a neighborhood of x_0, there exists a conic neighborhood Γ of ξ_0, $\exists a \geq 0, \exists b \geq 0, \exists k_2 > 0, \exists c_1 > 0, \exists \varepsilon_0 \in]0,1]$, such that $\forall \xi \in \Gamma, \forall \varepsilon \leq \varepsilon_0$,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi)| \leq c_1 \varepsilon^{-b} \exp -\widetilde{M}(k_2 \varepsilon^a |\xi|). \tag{4}$$

We have, for $\psi \in \mathcal{D}^M(U)$ such that $\psi(x_0) = 1$,

$$\begin{aligned} |\mathcal{F}(\psi \partial f_\varepsilon)(\xi)| &= |\mathcal{F}(\partial(\psi f_\varepsilon))(\xi) - \mathcal{F}((\partial\psi) f_\varepsilon)(\xi)| \\ &\leq |\xi| \, |\mathcal{F}(\psi\phi f_\varepsilon)(\xi)| + |\mathcal{F}((\partial\psi)\phi f_\varepsilon)(\xi)| \,. \end{aligned}$$

As $WF^M_{\mathcal{A}}(\psi f) \subset WF^M_{\mathcal{A}}(f)$, so (4) holds for both $|\mathcal{F}(\psi\phi f_\varepsilon)(\xi)|$ and $|\mathcal{F}((\partial\psi)\phi f_\varepsilon)(\xi)|$. Then

$$\begin{aligned} |\xi| \, |\mathcal{F}(\psi\phi f_\varepsilon)(\xi)| &\leq c\varepsilon^{-b} |\xi| \exp -\widetilde{M}(k_2\varepsilon^a|\xi|) \\ &\leq c'\varepsilon^{-b-a} \exp -\widetilde{M}(k_3 \varepsilon^a |\xi|), \end{aligned}$$

with $c' > 0, k_3 > 0$ such that $\varepsilon^a |\xi| \leq c' \exp(\widetilde{M}(k_2\varepsilon^a|\xi|) - \widetilde{M}(k_3\varepsilon^a|\xi|))$ for ε sufficiently small. Hence (4) holds for $|\mathcal{F}(\psi\partial f_\varepsilon)(\xi)|$, which proves $(x_0,\xi_0) \notin WF^M_{\mathcal{A}}(\partial f)$.

4) Let $(x_0,\xi_0) \notin WF^M_{\mathcal{A}}(f)$, then there exist $\phi \in \mathcal{D}^M(\Omega), \phi(x) = 1$ on a neighborhood U of x_0, a conic neighborhood Γ of $\xi_0, a_1 \geq 0, b_1 \geq 0, k_1 > 0, c_1 > 0, \varepsilon_1 \in]0,1]$, such that $\forall \varepsilon \leq \varepsilon_1, \forall \xi \in \Gamma$,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi)| \leq c_1 \varepsilon^{-b_1} \exp -\widetilde{M}(k_1 \varepsilon^{a_1}|\xi|).$$

Let $\psi \in \mathcal{D}^M(\Omega)$ and $\psi = 1$ on $\operatorname{supp}\phi$, then $\mathcal{F}(\phi g_\varepsilon f_\varepsilon) = \mathcal{F}(\psi g_\varepsilon) * \mathcal{F}(\phi f_\varepsilon)$. We have $\psi g \in \mathcal{G}^{M,\mathcal{A}}(\Omega)$, then $\exists c_2 > 0, \exists a_2 \geq 0, \exists b_2 \geq 0, \exists k_2 > 0, \exists \varepsilon_2 > 0, \forall \xi \in \mathbf{R}^n, \forall \varepsilon \leq \varepsilon_2$,

$$|\mathcal{F}(\psi g_\varepsilon)(\xi)| \leq c_2 \varepsilon^{-b_2} \exp -\widetilde{M}(-k_2 \varepsilon^{a_2}|\xi|).$$

We have

$$\mathcal{F}(\phi g_\varepsilon f_\varepsilon)(\xi) = \int_A \mathcal{F}(\phi f_\varepsilon)(\eta)\mathcal{F}(\psi g_\varepsilon)(\eta - \xi) d\eta + \int_B \mathcal{F}(\phi f_\varepsilon)(\eta)\mathcal{F}(\psi g_\varepsilon)(\eta - \xi) d\eta,$$

where A and B are the same as in the proof of Proposition 5.5. By the same reasoning we obtain the proof. ∎

A microlocalization of Proposition 4.12 is given in the following result.

Corollary 5.11. *Let $P(x,D)$ be a generalized linear partial differential operator with $\mathcal{G}^{M,\mathcal{A}}(\Omega)$ coefficients, then*

$$WF^M_{\mathcal{A}}(P(x,D)f) \subset WF^M_{\mathcal{A}}(f), \forall f \in \mathcal{G}(\Omega).$$

The reverse inclusion will give a generalized microlocal affine ultraregularity of a generalized linear partial differential operator with $\mathcal{G}^{M,\mathcal{A}}(\Omega)$ coefficients. The first case of $\mathcal{G}^\infty$-microlocal hypoellipticity has been studied in [15], [5] and [10].

A general interesting problem of $(M,\mathcal{A})$-*generalized microlocal elliptic affine ultraregularity* is to prove the following inclusion

$$WF^M_{\mathcal{A}}(f) \subset WF^M_{\mathcal{A}}(P(x,D)f) \cup Char(P), \forall f \in \mathcal{G}(\Omega),$$

where $P(x,D)$ is a generalized partial differential operator with $\mathcal{G}^{M,\mathcal{A}}(\Omega)$ coefficients and $Char(P)$ is the set of generalized characteristic points of $P(x,D)$.

6. Generalized Hörmander's theorem. We extend Hörmander's result on the wave front set of the product of two distributions, the proof follows the same steps as the proof of theorem 26 in [1]. We recall the following fundamental lemma, see [9] for the proof.

LEMMA 6.1. *Let Σ_1 and Σ_2 be closed cones in $\mathbf{R}^n\backslash\{0\}$, such that $0 \notin \Sigma_1 + \Sigma_2$, then*

(i) *the closure of the set $\Sigma_1 + \Sigma_2$ in $\mathbf{R}^n\backslash\{0\}$ is $(\Sigma_1 + \Sigma_2) \cup \Sigma_1 \cup \Sigma_2$;*
(ii) *for any open conic neighborhood Γ of $\Sigma_1 + \Sigma_2$ in $\mathbf{R}^n\backslash\{0\}$, one can find open conic neighborhoods Γ_1, Γ_2 in $\mathbf{R}^m\backslash\{0\}$ of, respectively, Σ_1, Σ_2, such that $\Gamma_1 + \Gamma_2 \subset \Gamma$.*

Let us recall that

$$WF_{\mathcal{A}}^M(f) + WF_{\mathcal{A}}^M(g) = \left\{(x, \xi + \eta) : (x, \xi) \in WF_{\mathcal{A}}^M(f), (x, \eta) \in WF_{\mathcal{A}}^M(g)\right\}. \quad (5)$$

The principal result of this section is the following theorem.

THEOREM 6.2. *Let $f, g \in \mathcal{G}(\Omega)$ such that*

$$(x, 0) \notin WF_{\mathcal{A}}^M(f) + WF_{\mathcal{A}}^M(g), \forall x \in \Omega,$$

then

$$WF_{\mathcal{A}}^M(fg) \subseteq \left(WF_{\mathcal{A}}^M(f) + WF_{\mathcal{A}}^M(g)\right) \cup WF_{\mathcal{A}}^M(f) \cup WF_{\mathcal{A}}^M(g).$$

Proof. Let

$$(x_0, \xi_0) \notin \left(WF_{\mathcal{A}}^M(f) + WF_{\mathcal{A}}^M(g)\right) \cup WF_{\mathcal{A}}^M(f) \cup WF_{\mathcal{A}}^M(g),$$

then there exists $\phi \in D^M(\Omega)$ such that

$$\phi(x_0) = 1, \xi_0 \notin \left(\Sigma_{\mathcal{A}}^M(\phi f) + \Sigma_{\mathcal{A}}^M(\phi g)\right) \cup \Sigma_{\mathcal{A}}^M(\phi f) \cup \Sigma_{\mathcal{A}}^M(\phi g).$$

From (5) we have $0 \notin \Sigma_{\mathcal{A}}^M(\phi f) + \Sigma_{\mathcal{A}}^M(\phi g)$ then by lemma 6.1 (i), we have ξ_0 is not in the closure of the set $\Sigma_{\mathcal{A}}^M(\phi f) + \Sigma_{\mathcal{A}}^M(\phi g)$ in $\mathbf{R}^n\backslash\{0\}$. Let Γ_0 be an open conic neighborhood of $\Sigma_{\mathcal{A}}^M(\phi f) + \Sigma_{\mathcal{A}}^M(\phi g)$ in $\mathbf{R}^n\backslash\{0\}$ such that $\xi_0 \notin \overline{\Gamma}_0$, then thanks to lemma 6.1 (ii), there exist open cones Γ_1 and Γ_2 in $\mathbf{R}^n\backslash\{0\}$ such that

$$\Sigma_{\mathcal{A}}^M(\phi f) \subset \Gamma_1, \Sigma_{\mathcal{A}}^M(\phi g) \subset \Gamma_2 \quad \text{and} \quad \Gamma_1 + \Gamma_2 \subset \Gamma_0.$$

Define $\Gamma = \mathbf{R}^n\backslash\overline{\Gamma}_0$, so

$$\Gamma \cap \Gamma_2 = \emptyset \quad \text{and} \quad (\Gamma - \Gamma_2) \cap \Gamma_1 = \emptyset. \quad (6)$$

Let $\xi \in \Gamma$ and $\varepsilon \in]0, 1]$, we have

$$\begin{aligned}\mathcal{F}(\phi f_\varepsilon \phi g_\varepsilon)(\xi) &= (\mathcal{F}(\phi f_\varepsilon) * \mathcal{F}(\phi g_\varepsilon))(\xi) \\ &= \int_{\Gamma_2} \mathcal{F}(\phi f_\varepsilon)(\xi - \eta) \mathcal{F}(\phi g_\varepsilon)(\eta) d\eta + \int_{\Gamma_2^c} \mathcal{F}(\phi f_\varepsilon)(\xi - \eta) \mathcal{F}(\phi g_\varepsilon)(\eta) d\eta \\ &= I_1(\xi) + I_2(\xi).\end{aligned}$$

From (6), $\exists a_1 \geq 0, b_1 \geq 0, k_1 > 0, c_1 > 0, \varepsilon_1 \in]0, 1]$, such that $\forall \varepsilon \leq \varepsilon_1, \forall \xi \in \Gamma_2$,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi - \eta)| \leq c_1 \varepsilon^{-b_1} \exp -\widetilde{M}(k_1 \varepsilon^{a_1} |\xi|).$$

As $(\phi g_\varepsilon) \in \mathcal{G}_C(\Omega)$ we can show easily that $\forall a_2 \geq 0, \forall k_2 > 0, \exists b_2 \geq 0, \exists c_2 > 0, , \exists \varepsilon_2 \in]0, 1]$, such that $\forall \varepsilon \leq \varepsilon_2$,

$$|\mathcal{F}(\phi g_\varepsilon)(\eta)| \leq c_2 \varepsilon^{-b_2} \exp \widetilde{M}(k_2 \varepsilon^{a_2} |\eta|), \forall \eta \in \mathbf{R}^n.$$

Let $\gamma > 0$ sufficiently small such that $|\xi - \eta| \geq \gamma(|\xi| + |\eta|), \forall \eta \in \Gamma_2$, hence for $\varepsilon \leq \min(\varepsilon_1, \varepsilon_2)$, we have

$$|I_1(\xi)| \leq c_1 c_2 \varepsilon^{-b_1-b_2} \int_{\Gamma_2} \exp\left(-\widetilde{M}\left(k_1\varepsilon^{a_1}|\xi-\eta|\right) + \widetilde{M}\left(k_2\varepsilon^{a_2}|\eta|\right)\right) d\eta.$$

Proposition 3.3 gives $\exists H > 0, \exists A > 0, \forall t_1 > 0, \forall t_2 > 0$,

$$-\widetilde{M}(t_1+t_2) \leq -\widetilde{M}\left(\frac{t_1}{H}\right) - \widetilde{M}\left(\frac{t_2}{H}\right) + \ln A,$$

so

$$\begin{aligned}|I_1(\xi)| &\leq c_1c_2\varepsilon^{-b_1-b_2}\exp\left(-\widetilde{M}\left(\frac{k_1}{H}\varepsilon^{a_1}\gamma|\xi|\right)\right)\\ &\quad\cdot\int_{\Gamma_2}\exp\left(-\widetilde{M}\left(\frac{k_1}{H}\varepsilon^{a_1}\gamma|\eta|\right)+\widetilde{M}\left(k_2\varepsilon^{a_2}|\eta|\right)\right)d\eta\\ &\leq c_1c_2\varepsilon^{-b_1-b_2}\exp\left(-\widetilde{M}\left(\frac{k_1}{H}\varepsilon^{a_1}\gamma|\xi|\right)\right)\\ &\quad\cdot\int_{\Gamma_2}\exp\left(-\widetilde{M}\left(\left(\frac{k_1}{H^2}\varepsilon^{a_1}\gamma-k_2\varepsilon^{a_2}\right)|\eta|\right)\right)d\eta.\end{aligned}$$

Take $k = \frac{\gamma k_1}{H}$ and $\frac{k_1}{H^2}\varepsilon^{a_1}\gamma - k_2\varepsilon^{a_2} > 0$, then $\exists b = b(b1 + b_2, a_1, a_2, k_1, k_2, H), \exists c = c_1c_2$,

$$|I_1(\xi)| \leq c\varepsilon^{-b}\exp(-\widetilde{M}(k\varepsilon^{a_1}|\xi|)).$$

Let $r > 0$, then

$$I_2(\xi) = I_{21}(\xi) + I_{22}(\xi),$$

where

$$I_{21}(\xi) = \int_{\Gamma_2^c\cap\{|\eta|\leq r|\xi|\}} \mathcal{F}(\phi f_\varepsilon)(\xi-\eta)\mathcal{F}(\phi g_\varepsilon)(\eta)d\eta$$

and

$$I_{22}(\xi) = \int_{\Gamma_2^c\cap\{|\eta|\geq r|\xi|\}} \mathcal{F}(\phi f_\varepsilon)(\xi-\eta)\mathcal{F}(\phi g_\varepsilon)(\eta)d\eta.$$

Choose r sufficiently small such that if $|\eta| \leq r|\xi| \Rightarrow \xi - \eta \notin \Gamma_1$. Then $|\xi-\eta| \geq (1-r)|\xi| \geq (1-2r)|\xi| + |\eta|$, consequently $\exists c > 0, \exists a_1, a_2, b_1, k_1, k_2 > 0, \exists \varepsilon_1 > 0$ such that $\forall \varepsilon \leq \varepsilon_1$,

$$\begin{aligned}|I_{21}(\xi)| &\leq c\varepsilon^{-b}\int_{\Gamma_2}\exp(-\widetilde{M}(k_1\varepsilon^{a_1}|\xi-\eta|)-\widetilde{M}(k_2\varepsilon^{a_2}|\eta|))\\ &\leq c\varepsilon^{-b}\exp(-\widetilde{M}(k_1'\varepsilon^{a_1}|\xi|))\int\exp(-\widetilde{M}(k_1\varepsilon^{a_1}|\eta|)-\widetilde{M}(k_2\varepsilon^{a_2}|\eta|))d\eta\\ &\leq c'\varepsilon^{-b'}\exp(-\widetilde{M}(k_1'\varepsilon^{a_1}|\xi|)).\end{aligned}$$

If $|\eta| \geq r|\xi|$ we have $|\eta| \geq \frac{|\eta|+r|\xi|}{2}$, then $\exists c > 0$, $\exists a_1, b_1, k_1 > 0$, $\forall a_2, k_2 > 0$, $\exists b_2 > 0$,

$\exists \varepsilon_2 > 0$ such that $\forall \varepsilon \leq \varepsilon_2$,

$$\begin{aligned}
|I_{21}(\xi)| &\leq c\varepsilon^{-b_1-b_2} \int_{\Gamma_2} \exp(\widetilde{M}(k_2\varepsilon^{a_2}|\xi-\eta|) - \widetilde{M}(k_1\varepsilon^{a_1}|\eta|))d\eta \\
&\leq c\varepsilon^{-b_1-b_2} \int_{\Gamma_2} \exp\left(\widetilde{M}\left(k_2\varepsilon^{a_2}|\xi-\eta|\right) - \widetilde{M}\left(\frac{k_1}{2}\varepsilon^{a_1}|\eta| + \frac{k_1 r}{2}\varepsilon^{a_1}|\xi|\right)\right) d\eta \\
&\leq c\varepsilon^{-b_1-b_2} \exp\left(-\widetilde{M}\left(\frac{k_1 r}{2H}\varepsilon^{a_1}|\xi|\right)\right) \\
&\quad \cdot \int_{\Gamma_2} \exp\left(\widetilde{M}\left(k_2\varepsilon^{a_2}|\xi-\eta|\right) - \widetilde{M}\left(\frac{k_1}{2H}\varepsilon^{a_1}|\eta|\right)\right) d\eta.
\end{aligned}$$

If we take k_2 and $\frac{1}{a_2}$ sufficiently small, we obtain $\exists a > 0, \exists b > 0, \exists c > 0, \exists \varepsilon_3 > 0$, such that $\forall \varepsilon \leq \varepsilon_3$,

$$|I_{21}(\xi)| \leq c\varepsilon^{-b} \exp(-\widetilde{M}(k\varepsilon^{a}|\xi|)),$$

which finishes the proof. ∎

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