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DISTRIBUTIONS THAT ARE FUNCTIONS

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Abstract. It is well-known that any locally Lebesgue integrable function generates a unique distribution, a so-called regular distribution. It is also well-known that many non-integrable functions can be regularized to give distributions, but in general not in a unique fashion. What is not so well-known is that to many distributions one can associate an ordinary function, the function that assigns the distributional point value of the distribution at each point where the value exists, and that in many cases this ordinary function determines the distribution in a unique fashion. In this talk we consider several classes of distributions that are given in terms of the ordinary function of their point values. In particular, we consider distributions that have a point value everywhere, those that have lateral limits at each point, and then introduce the class of distributionally integrable functions. We study several properties of distributions of these classes and apply these ideas to study the boundary behavior of solutions of partial differential equations.

1. Introduction. Distributions are classified as *regular distributions,* which are those generated by locally Lebesgue integrable functions, and *singular distributions,* which are the rest. It is clear that one can treat the regular distributions as ordinary functions, but this does not mean that all singular distributions are not ordinary functions. In this article we consider whether singular distributions are ordinary functions.

The importance of these questions can be seen by the construction of a distributional integral of ordinary functions. Indeed, integration of *functions* is a very important and much studied topic, even for functions defined on the real line. On the other hand, the problem of the construction of primitives of distributions of one real variable is basically trivial, since any distribution of the space $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ has primitives $\mathsf{F} \in \mathcal{D}'(\mathbb{R})$, and all primitives are of the form $\mathsf{F} = \mathsf{F}_0 + C$, where F_0 is a particular primitive and C is a constant. Therefore one could try to define a new integration process for functions as

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follows: Start with a function f, to it associate a corresponding distribution f, find a primitive of this distribution, F, and then find the function F that corresponds to F. Then the function F would be a primitive of the initial function f, and we could compute integrals as

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{1}$$

This would be a very powerful integration process, but for it to work first we need to understand whether there is a way to associate, in a unique fashion, ordinary functions f to distributions f and conversely. A second problem is that we need to make sure that the function F is defined everywhere on $\mathbb{R}$, so that the right side of (1) makes sense for all a and b; however, once the first problem is solved, the second will be solved automatically.

We emphasize that we want to integrate *functions,* not distributions. The definite integral of a distribution f over $[a,b]$ can be computed as $\mathsf{F}(b) - \mathsf{F}(a)$, if those values exist [2]. This a well understood subject. The distributional integration that we describe, however, is a quite different construction.

Therefore our aim is to identify classes of distributions that correspond to ordinary functions in a unique, natural way. It turns out that while distributions are defined by their action on test functions and do not have values at points, in general, there is a rather useful notion of distributional point value, introduced by Łojasiewicz [18], that allows us to assign an ordinary function to a distribution in many cases, even if the distribution is singular.

We start our tour in Section 2 by considering a basic but surprisingly confusing issue, namely, what is the meaning of the concept of "function" in mathematical analysis. Indeed, can one say what the value of a function $f \in L^1(\mathbb{R})$ at a particular given point x_0 is?, or can we modify f on a set of measure zero, such as $\{x_0\}$, to obtain that the value $f(x_0)$ is arbitrary? Next we move to the corresponding question for distributions, giving several examples where the distribution is or is not an ordinary function.

In Section 3 we explain the ideas of point values in the sense of Łojasiewicz, the notion of the Cesàro behavior of distributions, and the various methods of summability of distributional evaluations.

In [18] Łojasiewicz proved that a distribution that has distributional point values equal to zero at *all* points must vanish. This means, precisely, that a distribution that has values everywhere is determined by the ordinary function that gives those values. We explain these matters in Section 4. Recently this analysis was extended to the distributionally regulated functions [26]; we give those ideas in Section 5. In this article, in Section 6 we shall further generalize these ideas to the locally distributionally integrable functions which are determined by the ordinary function that gives their point values, but now those point values exist just *almost everywhere.*

We study several properties of distributions of these classes. In particular, in Section 7 we apply these ideas to study the boundary behavior of solutions of partial differential equations. Our basic tool for this analysis is the ϕ-transform, also studied in Section 7. We also consider in Section 8 the Fourier transform of distributions that correspond to ordinary functions and use the characterization of point values of Fourier series [8] and

Fourier transforms [26, 25] in order to give the pointwise inversion formulas. Finally, in Section 9 we give an illustration of these ideas in the interpretation of the Poisson summation formula in terms of distributional point values and use this to derive a direct extension to Łojasiewicz functions of an uncertainty principle result for Fourier transforms which is valid for continuous integrable functions.

2. What is a function? Let us start with a simple question: What is an ordinary function?

This seems like a silly question, one whose solution is well-known by everyone, but keep reading! The usual definition that one teaches in elementary schools is the following: If X and Y are two sets, then a function f from X to Y is a correspondence $f : X \to Y$ with the property that $\forall x \in X\ \exists! y \in Y$ such that $f(x) = y$. However, is this what we use in analysis?

In complex variables one uses "functions" like $\sqrt{z}$ or $\ln z$ that are "multivalued." Of course one can think of such functions as defined in an appropriate Riemann surface, but that is not usually said.

A more serious situation is found when working with Lebesgue integrable functions. Indeed, the elements of the space $L^p(X)$ that we use so frequently in analysis are not ordinary functions but equivalence classes of them. Nevertheless, if $f \in L^p(\mathbb{R})$ most mathematicians would think of f as a "function," despite the fact that it is not clear what $f(x_0)$ is if $x_0 \in \mathbb{R}$. Observe that trying to assign a value $f(x_0)$ seems like an impossible task since one can modify f on a set of measure 0, such as $\{x_0\}$ for instance, without changing its equivalence class.

Therefore, it seems like calling an element $f \in L^p(\mathbb{R})$ an "ordinary function" is completely wrong since the value of f at any $x_0 \in \mathbb{R}$ is never clearly defined. The interesting thing, however, is that if the concept of the value $f(x_0)$ is properly defined then the value will exist almost everywhere and thus an element of $L^p(\mathbb{R})$ is really an ordinary function defined in a set of the form $\mathbb{R} \setminus Z$, where Z has measure 0. The key question is then how $f(x_0)$ is defined, and in this case we could ask, for example, that x_0 be a Lebesgue point of f in the sense that $v = f(x_0)$ is the only number v that satisfies

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{|x-x_0|<\varepsilon} |f(x) - v| \, \mathrm{d}x = 0.$$

It is well-known that almost all points are Lebesgue points. Later on we shall need more general definitions of value at a point, but this suffices in this case.

Let us now turn our attention to distributions. If f is a locally Lebesgue integrable function, $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, then to f there correspond a unique distribution $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$, given by

$$\langle \mathsf{f}, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x) \, \mathrm{d}x,$$

for any test function $\phi \in \mathcal{D}(\mathbb{R})$. We say that f is a *regular* distribution in this case. In the texts one immediately identifies f and f, and starts using the same notation for both objects, but for the purposes of this talk it is important to distinguish between them, so that we may write $f \leftrightarrow \mathsf{f}$, but consider f and f as different objects. We shall always denote the evaluation of a distribution f on a test function ϕ as $\langle \mathsf{f}, \phi \rangle$ or as $\langle \mathsf{f}(x), \phi(x) \rangle$.

The distributions that are not regular are called *singular*. It is now clear that any regular distribution is an ordinary function (defined almost everywhere). The question is then if a singular distribution can be an ordinary function. Naturally if $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ is a distribution then $\langle \mathsf{f}, \phi \rangle$ is defined for all test functions $\phi \in \mathcal{D}(\mathbb{R})$, but this is a "global" definition and it is not clear if one can define the value of f at any point in $\mathbb{R}$.

Let us consider some examples.

Example 2.1. Let us start with the most famous distribution, namely, the Dirac delta "function" $\delta(x)$ whose action on a test function $\phi \in \mathcal{D}(\mathbb{R})$ is given by

$$\langle \delta(x), \phi(x) \rangle = \phi(0).$$

Is $\delta(x)$ an *ordinary* function? It will follow from our definition of point values to be given in Section 3 that the value $\delta(x_0)$ exists for any $x_0 \in \mathbb{R}\backslash\{0\}$ and actually $\delta(x_0) = 0$ for any $x_0 \neq 0$. But for an *ordinary* function the value at just one point, $x_0 = 0$ in this case, is irrelevant and thus one would obtain that if $\delta(x)$ were an ordinary function then it would vanish. Consequently, $\delta(x)$ is not an ordinary function.

Example 2.2. Let us now consider a non-integrable function, again one of the first examples that one finds in any textbook in the theory of distributions. Indeed, is $1/x$ an ordinary function? The fact that $1/x$ is not defined at $x = 0$ is irrelevant, since one just needs it to be defined almost everywhere. The important question is different: we want to know if a distribution is an ordinary function or not, and thus we first need to know if $1/x$ is a distribution.

If one goes to the textbooks one finds that it is possible to "regularize" the non locally integrable function $1/x$, that is, we can construct distributions $\mathsf{g} \in \mathcal{D}'(\mathbb{R})$, called regularizations of $1/x$, that satisfy

$$\langle \mathsf{g}(x), \phi(x) \rangle = \int_{-\infty}^{\infty} \frac{\phi(x)}{x} \, \mathrm{d}x,$$

whenever the integral converges. We apologize for using the standard but so overused term "regularization" which is employed with so many different meanings in several other areas, even in the theories of generalized functions. Unfortunately, there is no unique way to define the *right* regularization of $1/x$. One may consider the principal value distribution p.v.$(1/x)$ defined by the principal value integral,

$$\begin{aligned}\left\langle \mathrm{p.v.}\left(\frac{1}{x}\right), \phi(x) \right\rangle &= \mathrm{p.v.} \int_{-\infty}^{\infty} \frac{\phi(x)}{x} \, \mathrm{d}x \\ &= \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\phi(x)}{x} \, \mathrm{d}x,\end{aligned}$$

or the distributions $1/(x + i0)$ and $1/(x - i0)$, defined as

$$\left\langle \frac{1}{x \pm i0}, \phi(x) \right\rangle = \lim_{y \to 0^+} \int_{-\infty}^{\infty} \frac{\phi(x)}{x \pm iy} \, \mathrm{d}x.$$

Therefore, the proper questions one could ask are the following. Is p.v.$(1/x)$ an ordinary function?; is $1/(x+i0)$ an ordinary function?; or is $1/(x-i0)$ an ordinary function? The answer is no in the three cases because of the non-uniqueness: In the three cases the "ordinary function" would be given by $1/x$ for $x \neq 0$.

It is very interesting that many trivial but annoying problems with the use of distributions in several applied areas are related to problems similar to that of $1/x$. Indeed, $1/x$ is an ordinary function, defined in $\mathbb{R}\backslash\{0\}$; there is no question about that. However, $1/x$ is *not* a uniquely defined distribution and thus it is not a distribution corresponding to an ordinary function. Therefore, if different authors obtain "different" formulas that seem to contradict each other it is, in many cases, because they use different definitions for the regularizations of non-integrable functions.

EXAMPLE 2.3. The trigonometric functions $\tan x$, $\cot x$, $\sec x$, and $\csc x$ are studied starting in elementary school, and thus it is easy to forget that they are not uniquely defined distributions. They do have *standard* regularizations, the principal value ones, but these distributions are not ordinary functions. A very interesting related case is that of the hyperbolic cotangent and its distributional derivatives [16], [11].

EXAMPLE 2.4. The well-known distributions x_+^α are defined if $\alpha \in \mathbb{C}\backslash\{-1,-2,-3,\ldots\}$. They are regular distributions for $\Re e\,\alpha > -1$, but if $\Re e\,\alpha \leq -1$ they are not ordinary functions.

EXAMPLE 2.5. Let us now consider the distribution $\mathsf{s}_\alpha(x) = |x|^\alpha \sin(1/x)$ for $\alpha \in \mathbb{C}$. If $\Re e\,\alpha > -1$ then the function $|x|^\alpha \sin(1/x)$ is locally Lebesgue integrable and thus it is a regular distribution given by

$$\langle \mathsf{s}_\alpha(x), \phi(x)\rangle = \int_{-\infty}^{\infty} |x|^\alpha \sin(1/x)\,\phi(x)\,\mathrm{d}x. \tag{2}$$

It is easy to show that s_α admits an analytic continuation from the right side half-plane $\Re e\,\alpha > -1$ to the whole complex plane. If $-1 \geq \Re e\,\alpha > -2$ then the function $|x|^\alpha \sin(1/x)$ is not Lebesgue integrable near $x = 0$ but it is Denjoy integrable and $\mathsf{s}_\alpha(x)$ can still be defined by (2). Is $\mathsf{s}_\alpha(x)$ an ordinary function when $-1 \geq \alpha > -2$? It seems fair to say that the answer is yes! Interestingly, using the notion of distributional point value [18], Łojasiewics defined a new integral for which the right side of (2) is defined for all $\alpha \in \mathbb{C}$, and gives the evaluation $\langle \mathsf{s}_\alpha(x), \phi(x)\rangle$ for all $\alpha \in \mathbb{C}$. We can then say that $\mathsf{s}_\alpha(x)$ is a distribution that corresponds to an ordinary function for *all* complex numbers α.

EXAMPLE 2.6. Let $C \subset [0,1]$ be the Cantor set and let $\psi : [0,1] \longrightarrow [0,1]$ be the Cantor function, so that ψ is continuous, increasing, onto, and satisfies $\psi'(x) = 0$ for $x \in [0,1]\backslash C$. Extend ψ to $\mathbb{R}$ by setting $\psi(x) = 0$ if $x < 0$, and $\psi(x) = 1$ if $x > 1$. Let $\nu = \psi'$, the derivative in the distributional sense. Then ν is a distribution that vanishes on $\mathbb{R}\backslash C$. The distribution ν does not correspond to an ordinary function.

After looking at these examples our problem is clearer. Identify classes of distributions f for which there is an "ordinary" function f, defined on a set of the form $\mathbb{R}\backslash Z$, where Z has measure 0, such that there is a unique correspondence $\mathsf{f} \leftrightarrow f$. One expects the values of f and of f to be the same on $\mathbb{R}\backslash Z$. Observe that in a sense these ordinary functions are actually equivalence classes of measurable functions equal almost everywhere. If f is locally Lebesgue integrable then we obtain the regular distributions, but there are many distributions that correspond to ordinary functions that are not locally Lebesgue integrable and, quite clearly, there are many distributions that are not given by ordinary functions.

3. Point values. In this section we make a parenthesis in our study of distributions that are ordinary functions in order to give the definition of distributional point values [18] that we need to employ in our analysis. We will also recall the basic ideas involved in the Cesàro behavior of distributions [9].

The spaces of test functions $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{S}$ and the corresponding spaces of distributions are well-known [17, 23, 24]. In general [30], we call a topological vector space $\mathcal{A}$ a space of test functions if $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{E}$, where the inclusions are continuous, and if $\frac{d}{dx}$ is a continuous operator of $\mathcal{A}$. An useful space, particularly in the study of distributional asymptotic expansions [13, 14, 22, 27] is $\mathcal{K}'$, the dual of $\mathcal{K}$. The space $\mathcal{K}'$ plays a fundamental role in the theory of summability of distributional evaluations [9].

In a seminal article, Łojasiewicz [18] defined the value of a distribution $\mathsf{f} \in \mathcal{D}'$ at the point x_0 as the limit

$$\mathsf{f}(x_0) = \lim_{\varepsilon \to 0} \mathsf{f}(x_0 + \varepsilon x),$$

if the limit exists in $\mathcal{D}'(\mathbb{R})$, that is if

$$\lim_{\varepsilon \to 0} \langle \mathsf{f}(x_0 + \varepsilon x), \phi(x) \rangle = \mathsf{f}(x_0) \int_{-\infty}^{\infty} \phi(x)\, \mathrm{d}x, \tag{3}$$

for each $\phi \in \mathcal{D}(\mathbb{R})$. It was shown by Łojasiewicz [18] that the existence of the distributional point value $\mathsf{f}(x_0) = \gamma$ is equivalent to the existence of $n \in \mathbb{N}$, and a primitive of order n of f, that is $F^{(n)} = \mathsf{f}$, which is continuous and satisfies

$$\lim_{x \to x_0} \frac{n! F(x)}{(x - x_0)^n} = \gamma.$$

Since (3) is only supposed to hold for $\phi \in \mathcal{D}(\mathbb{R})$, we emphasize this fact saying that $\mathsf{f}(x_0) = \gamma$ in $\mathcal{D}'$, in case that (3) is satisfied. Suppose now that $\mathsf{f} \in \mathcal{S}'$ and $\mathsf{f}(x_0) = \gamma$ in $\mathcal{D}'$; initially, (3) does not have to be true for $\phi \in \mathcal{S}$. However, it is shown in [10, Corollary 1] that if (3) holds for $\phi \in \mathcal{D}$, it will remain true for $\phi \in \mathcal{S}$; so we can say $\mathsf{f}(x_0) = \gamma$ in $\mathcal{S}'$, and this is equivalent to the existence of $\mathsf{f}(x_0)$ in $\mathcal{D}'$. Actually using the notion of the Cesàro behavior of a distribution at infinity [9] explained bellow, (3) will hold if $f(x) = O(|x|^\beta)$ (C), as $|x| \to \infty$, $\phi(x) = O(|x|^\alpha)$, strongly $|x| \to \infty$, and $\alpha < -1$, $\alpha + \beta < -1$. An asymptotic estimate is *strong* if it remains valid after differentiation of any order.

The notion of distributional point value introduced by Łojasiewicz has been shown to be of fundamental importance in analysis [2, 8, 20, 21, 26, 25, 28, 29]. It seems to be originated in the idea of generalized differentials introduced by Denjoy [4]. There are other notions of distributional point values as that of Campos Ferreira [2], who also introduced the very useful concept of *bounded* distributions.

Notice that the distributional limit $\lim_{x\to x_0} \mathsf{f}(x)$ can be defined for $\mathsf{f} \in \mathcal{D}'(\mathbb{R} \setminus \{x_0\})$. If the point value $\mathsf{f}(x_0)$ exists distributionally then the distributional limit $\lim_{x\to x_0} \mathsf{f}(x)$ exists and equals $\mathsf{f}(x_0)$. On the other hand, if $\lim_{x\to x_0} \mathsf{f}(x) = L$ distributionally then there exist constants $a_0, \ldots, a_n$ such that $\mathsf{f}(x) = \mathsf{f}_0(x) + \sum_{j=0}^{n} a_j \delta^{(j)}(x - x_0)$, where the distributional point value $\mathsf{f}_0(x_0)$ exists and equals L.

We may also consider lateral limits. We say that the distributional lateral value $\mathsf{f}(x_0^+)$ exists if $\mathsf{f}(x_0^+) = \lim_{\varepsilon\to 0^+} \mathsf{f}(x_0 + \varepsilon x)$ in $\mathcal{D}'(0,\infty)$, that is,

$$\lim_{\varepsilon\to 0^+} \langle \mathsf{f}(x_0 + \varepsilon x), \phi(x)\rangle = \mathsf{f}(x_0^+) \int_0^\infty \phi(x)\,\mathrm{d}x \ , \quad \phi \in \mathcal{D}(0,\infty).$$

Similar definitions apply to $\mathsf{f}(x_0^-)$. Notice that the distributional limit $\lim_{x\to x_0} \mathsf{f}(x)$ exists if and only if the distributional lateral limits $\mathsf{f}(x_0^-)$ and $\mathsf{f}(x_0^+)$ exist and coincide.

The Cesàro behavior of a distribution at infinity is studied by using the order symbols $O(x^\alpha)$ and $o(x^\alpha)$ in the Cesàro sense. If $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ and $\alpha \in \mathbb{R}\backslash\{-1,-2,-3,...\}$, we say that $\mathsf{f}(x) = O(x^\alpha)$ as $x \to \infty$ in the Cesàro sense and write

$$\mathsf{f}(x) = O(x^\alpha)\ (\mathrm{C}, N),\ \text{ as } x \to \infty,$$

if there exists $N \in \mathbb{N}$ such that every primitive F of order N, i.e., $F^{(N)} = \mathsf{f}$, is an ordinary function for large arguments and satisfies the ordinary order relation

$$F(x) = p(x) + O(x^{\alpha+N}),\ \text{ as } x \to \infty,$$

for a suitable polynomial p of degree at most $N-1$. Note that if $\alpha > -1$, then the polynomial p is irrelevant. A similar definition applies to the little o symbol. The definitions when $x \to -\infty$ are clear. One can also consider the case when $\alpha = -1, -2, -3, ...$ [14, Definition 6.3.1]. When the value of N is not important we shall use the simpler notation $\mathsf{f}(x) = O(x^\alpha)$ (C) as $x \to \infty$.

The elements of $\mathcal{S}'$ can be characterized by their Cesàro behavior at $\pm\infty$, in fact, $\mathsf{f} \in \mathcal{S}'$ if and only if there exists $\alpha \in \mathbb{R}$ such that $\mathsf{f}(x) = O(x^\alpha)$ (C), as $x \to \infty$, and $\mathsf{f}(x) = O(|x|^\alpha)$ (C), as $x \to -\infty$. On the other hand, this is true for all $\alpha \in \mathbb{R}$ if and only if $\mathsf{f} \in \mathcal{K}'$.

Using these ideas, one can define the limit of a distribution at ∞ in the Cesàro sense. We say that $\mathsf{f} \in \mathcal{D}'$ has a limit L at infinity in the Cesàro sense and write

$$\lim_{x\to\infty} \mathsf{f}(x) = L\ (\mathrm{C}),$$

if $\mathsf{f}(x) = L + o(1)$ (C), as $x \to \infty$.

The Cesàro behavior of a distribution f at infinity is related to the parametric behavior of $\mathsf{f}(\lambda x)$ as $\lambda \to \infty$. In fact, one can show [14, Theorem 6.5.1] that if $\alpha > -1$, then $\mathsf{f}(x) = O(x^\alpha)$ (C) as $x \to \infty$ and $\mathsf{f}(x) = O(|x|^\alpha)$ (C) as $x \to -\infty$ if and only if

$$\mathsf{f}(\lambda x) = O(\lambda^\alpha) \text{ as } \lambda \to \infty,$$

where the last relation holds weakly in $\mathcal{D}'$, i.e., for all $\phi \in \mathcal{D}$ fixed, $\langle \mathsf{f}(\lambda x), \phi(x)\rangle = O(\lambda^\alpha)$, $\lambda \to \infty$. On the other hand, a distribution f belongs to the space $\mathcal{K}'$ if and only if it satisfies the moment asymptotic expansion [13].

Let $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ with support bounded on the left. If $\phi \in \mathcal{E}(\mathbb{R})$ then the evaluation $\langle \mathsf{f}(x), \phi(x)\rangle$ will not be defined, in general. We say that the evaluation exists in the

Cesàro sense and equals L, written as

$$\langle \mathsf{f}(x), \phi(x)\rangle = L \quad (\mathrm{C}), \tag{4}$$

if $\mathsf{g}(x) = L + o(1)$ (C) as $x \to \infty$, where g is the primitive of $\mathsf{f}\phi$ with support bounded on the left. A similar definition applies if $\operatorname{supp}\mathsf{f}$ is bounded on the right. Observe that if f corresponds to a locally integrable function f with $\operatorname{supp}\mathsf{f} \subset [a, \infty)$ then (4) means that

$$\int_a^\infty f(x)\phi(x)\,\mathrm{d}x = L \quad (\mathrm{C}).$$

Naturally, this will hold for *any* integration method we use. If $\mathsf{f}(x) = \sum_{n=0}^\infty a_n \delta(x-n)$ then (4) tells us that

$$\sum_{n=0}^\infty a_n \phi(n) = L \quad (\mathrm{C}).$$

In the general case when the support of f extends to both $-\infty$ and $+\infty$, there are various different but related notions of evaluations in the Cesàro sense (or in any other summability sense, in fact). If f admits a representation of the form $\mathsf{f} = \mathsf{f}_1 + \mathsf{f}_2$, with $\operatorname{supp}\mathsf{f}_1$ bounded on the left and $\operatorname{supp}\mathsf{f}_2$ bounded on the right, such that $\langle \mathsf{f}_j(x), \phi(x)\rangle = L_j$ (C) exist, then we say that the (C) evaluation $\langle \mathsf{f}(x), \phi(x)\rangle$ (C) exists and equals $L = L_1 + L_2$. This is clearly independent of the decomposition. The notation (4) is used in this situation.

It happens many times that $\langle \mathsf{f}(x), \phi(x)\rangle$ (C) does not exist, but the symmetric limit, $\lim_{x\to\infty}\{\mathsf{g}(x) - \mathsf{g}(-x)\} = L$, where g is any primitive of $\mathsf{f}\phi$, exists in the (C) sense. Then we say that the evaluation $\langle \mathsf{f}(x), \phi(x)\rangle$ exists in the principal value Cesàro sense, and write

$$\mathrm{p.v.}\langle \mathsf{f}(x), \phi(x)\rangle = L \quad (\mathrm{C}).$$

Observe that p.v. $\sum_{n=-\infty}^\infty a_n\phi(n) = L$ (C) if and only if $\sum_{-N}^N a_n\phi(n) \to L$ (C) as $N \to \infty$ while p.v. $\int_{-\infty}^\infty f(x)\phi(x)\,\mathrm{d}x = L$ (C) if and only if $\int_{-A}^A f(x)\phi(x)\,\mathrm{d}x \to L$ (C) as $A \to \infty$.

A very useful intermediate notion is the following. If there exists k such that

$$\lim_{x\to\infty}\{\mathsf{g}(ax) - \mathsf{g}(-x)\} = L \quad (\mathrm{C}, k), \quad \forall a > 0,$$

we say that the distributional evaluation exists in the e.v. Cesàro sense and write

$$\mathrm{e.v.}\langle \mathsf{f}(x), \phi(x)\rangle = L \quad (\mathrm{C}, k),$$

or just e.v.$\langle \mathsf{f}(x), \phi(x)\rangle = L$ (C) if there is no need to call the attention to the value of k. We shall also use the notation

$$\mathrm{s.v.}\langle \mathsf{f}(x), \phi(x)\rangle = L \quad (\mathrm{C}, k),$$

if we have that

$$\lim_{x\to\infty}\{\mathsf{g}(a + x) - \mathsf{g}(-x)\} = L \quad (\mathrm{C}, k), \quad \forall a \in \mathbb{R}.$$

4. Łojasiewicz distributions. In this section we introduce a large class of distributions that correspond to ordinary functions, the class of Łojasiewicz distributions. In general Łojasiewicz distributions are not regular distributions, that is, they correspond to ordinary functions that are not locally Lebesgue integrable functions.

The simplest class of distributions that correspond to functions are those that come from continuous functions. If $\mathsf{f} \leftrightarrow f$ and f is continuous then it is an ordinary function: We can always say what $f(x_0)$ is for any x_0. The function f is not just defined almost everywhere but it is actually defined *everywhere.*

Interestingly, there is another case when f corresponds in a unique way to an ordinary function, one that is defined everywhere, the case of the Łojasiewicz distributions and the corresponding Łojasiewicz functions.

DEFINITION 4.1. A distribution f is a *Łojasiewicz distribution* if the distributional point value $\mathsf{f}(x_0)$ exists for every $x_0 \in \mathbb{R}$.

DEFINITION 4.2. A function f defined in $\mathbb{R}$ is called a *Łojasiewicz function* if there exists a Łojasiewicz distribution f such that

$$\mathsf{f}(x) = f(x) \qquad \forall x \in \mathbb{R}.$$

The correspondence $f \leftrightarrow \mathsf{f}$ is clear in the case of Łojasiewicz functions and distributions. The Łojasiewicz functions can be considered as a distributional generalization of continuous functions. They are defined at all points, and furthermore the value at each given point is not arbitrary but the (distributional) limit of the function as one approaches the given point. The Łojasiewicz functions and distributions were introduced in [18] but the name, which is only quite natural, has been used by this writer for some time in order to call attention to the work of Łojasiewicz.

4.1. Properties. The most important properties of these distributions and functions are the following.

1. $\mathsf{f} \leftrightarrow f$, i.e., $\mathsf{f} = 0 \Leftrightarrow f = 0$.

2. If f_0 is a Łojasiewicz distribution, and f_1 is a primitive, $\mathsf{f}_1' = \mathsf{f}_0$, then f_1 is also a Łojasiewicz distribution.

3. If f is a Łojasiewicz distribution and ψ is a smooth function, then $\psi\mathsf{f}$ is a Łojasiewicz distribution and

$$(\psi\mathsf{f})(x) = \psi(x)\mathsf{f}(x).$$

4. If f is Łojasiewicz function ($f \leftrightarrow \mathsf{f}$), then we can *define* its definite integral as

$$\int_a^b f(x)\,\mathrm{d}x = \mathsf{f}_1(b) - \mathsf{f}_1(a), \tag{5}$$

where $\mathsf{f}_1' = \mathsf{f}$.

5. If $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ is a Łojasiewicz distribution ($f \leftrightarrow \mathsf{f}$), then the evaluation of f on a test function ϕ, $\langle \mathsf{f}, \phi \rangle$, can actually be given as an integral, namely,

$$\langle \mathsf{f}, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x = \int_a^b f(x)\phi(x)\,\mathrm{d}x, \qquad \phi \in \mathcal{D}(\mathbb{R}) \ , \tag{6}$$

where $\operatorname{supp}\phi \subset [a, b]$.

6. If $\mathsf{f} \in \mathcal{S}'(\mathbb{R})$ is a Łojasiewicz distribution ($f \leftrightarrow \mathsf{f}$), then the evaluation of f on a test function $\phi \in \mathcal{S}(\mathbb{R})$ is given by

$$\langle \mathsf{f}, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x \qquad (\mathrm{C}).$$

This formula also holds if the Łojasiewicz distribution belongs to $\mathcal{K}'(\mathbb{R})$ and the test function ϕ belongs to $\mathcal{K}(\mathbb{R})$.

It seems as if (6) is the key relation between a distribution and an ordinary function, $\mathsf{f} \leftrightarrow f$, for a suitable integration method.

A typical example of a Łojasiewicz function is

$$s_{\alpha,\beta}(x) = \begin{cases} |x|^{\alpha} \sin |x|^{-\beta}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

for $\alpha \in \mathbb{C}$ and $\beta > 0$. If H is the Heaviside function, then the functions $H(\pm x)s_{\alpha,\beta}(x)$ and their linear combinations are also Łojasiewicz functions. It is not hard to see that this implies that derivatives of arbitrary order of $\mathsf{s}_{\alpha,\beta}$, where $\mathsf{s}_{\alpha,\beta} \leftrightarrow s_{\alpha,\beta}$, are also Łojasiewicz distributions.

These are rapidly oscillating functions. However, not all fast oscillating functions are Łojasiewicz functions. Curiously, the regular distribution $\sin(\ln|x|)$ is not a Łojasiewicz distribution since the distributional value at $x = 0$ does not exist in the Łojasiewicz sense (even though it exists and equals 0 in the Campos Ferreira sense [2]).

5. Distributionally regulated functions.

Another case when a distribution is an "ordinary function" is the case of regulated distributions, introduced and studied in [26]. They are generalizations of the ordinary regulated functions [5], which are functions whose lateral limits exist at all points, although they may be different. They are related to the recently introduced "thick" points [12].

Definition 5.1. A distribution f is called a *regulated distribution* if the distributional lateral limits

$$\mathsf{f}(x_0 + 0) \quad \text{and} \quad \mathsf{f}(x_0 - 0),$$

exist $\forall x_0 \in \mathbb{R}$, and there are no delta functions at any point.

If $\mathsf{f}(x_0 + 0) = \mathsf{f}(x_0 - 0)$ then $\mathsf{f}(x_0)$ exists, since these distributions do not have delta functions, and therefore we can define the function

$$f(x_0) = \mathsf{f}(x_0),$$

for these x_0. Then f is a distributionally regulated function. The function f is defined in the set $\mathbb{R} \setminus \mathfrak{S}$, where $\mathfrak{S}$ is the set of points x_0 where $\mathsf{f}(x_0 + 0) \neq \mathsf{f}(x_0 - 0)$. The set $\mathfrak{S}$ has measure zero since in fact it is countable at the most [26].

One can actually define

$$f(x_0) = \frac{\mathsf{f}(x_0 + 0) + \mathsf{f}(x_0 - 0)}{2},$$

and this is defined everywhere.

EXAMPLE 5.2. If a, b, c, d are constants, and H is the Heaviside function, then

$$f_0(x) = \left(a + b\sin\frac{1}{x}\right)H(x) + \left(c + d\sin\frac{1}{x}\right)H(-x),$$

is a distributionally regulated function; it is not a classical regulated function and it is not a function of bounded variation. One can use some condensation of singularities technique to obtain examples that show this behavior not only at $x = 0$ but over a dense set. For instance, if $\{\omega_n\}_{n=0}^{\infty}$ is dense in $\mathbb{R}$, and if $\sum_{n=0}^{\infty}|a_n| < \infty$, then

$$f_1(x) = \sum_{n=0}^{\infty} a_n f_0(x - \omega_n),$$

is a distributionally regulated function with distributional jumps at the points $x = \omega_n$. Similarly, if $q > 1$, the function

$$f_2(x) = \sum_{n=1}^{\infty} \frac{f_0(\sin nx)}{n^q},$$

is continuous at all the irrational points and has distributional jump discontinuities at each rational number.

5.1. Properties. The basic properties of the distributionally regulated functions and the corresponding regulated distributions are the following.

1. $\mathsf{f} \leftrightarrow f$, i.e., $\mathsf{f} = 0 \Longleftrightarrow f = 0$.

2. If f_0 is a regulated distribution, and f_1 is a primitive, $\mathsf{f}_1' = \mathsf{f}_0$, then f_1 is a Łojasiewicz distribution.

3. If f is a regulated distribution and ψ is a smooth function, then $\psi\mathsf{f}$ is a regulated distribution and

$$(\psi\mathsf{f})(x) = \psi(x)\mathsf{f}(x).$$

4. If f is regulated function ($f \leftrightarrow \mathsf{f}$), then we can *define* its definite integral as

$$\int_a^b f(x)\,\mathrm{d}x = \mathsf{f}_1(b) - \mathsf{f}_1(a), \tag{7}$$

where $\mathsf{f}_1' = \mathsf{f}$.

5. If $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ is a regulated distribution ($f \leftrightarrow \mathsf{f}$), then

$$\langle \mathsf{f}, \phi\rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x, \qquad \phi \in \mathcal{D}(\mathbb{R}).$$

6. If f is a regulated distribution ($f \leftrightarrow \mathsf{f}$), and $\mathsf{f} \in \mathcal{S}'(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$, or $\mathsf{f} \in \mathcal{K}'(\mathbb{R})$ and $\phi \in \mathcal{K}(\mathbb{R})$, then the evaluation of f on the test function ϕ is given by

$$\langle \mathsf{f}, \phi\rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x \quad \text{(C)}.$$

6. Integrable distributions. We have considered two classes of distributions that correspond to ordinary functions. The Łojasiewicz functions are a generalization of the continuous functions, while the distributionally regulated functions are a generalization of the ordinary regulated functions; both classes consist of functions that can be defined everywhere in $\mathbb{R}$. We shall now consider a class of distributions and a corresponding class of ordinary functions that are just defined almost everywhere, more like the locally Lebesgue integrable functions. In order to introduce these classes we need to define a new integral.

Let f be a function defined almost everywhere in $\mathbb{R}$, that is, in a set of the form $\mathbb{R}\setminus Z$, where Z has measure zero. We say that f is *distributionally (locally) integrable* if there exist two distributions f_0 and f_1, of the space $\mathcal{D}'(\mathbb{R})$ such that:

a) $\mathsf{f}_1' = \mathsf{f}_0$.

b) $\mathsf{f}_0(x)$ exist for almost every $x \in \mathbb{R}$ and

$$f(x) = \mathsf{f}_0(x) \qquad \text{(a.e.)}.$$

c) $\mathsf{f}_1(x) = f_1(x)$ exist for every $x \in \mathbb{R}$.

d) The function f_1 is absolutely continuous in the following generalized sense: for each interval $[a,b]$, there exists a sequence of subsets $\{E_n\}_{n=1}^{\infty}$ such that $[a,b] = \bigcup_{n=1}^{\infty} E_n$, and f_1 is absolutely continuous in E_n.

In this case we write:

$$\text{(dist)} \int_a^b f(x)\,\mathrm{d}x = \mathsf{f}_1(b) - \mathsf{f}_1(a), \tag{8}$$

and call (dist) $\int_a^b f(x)\,\mathrm{d}x$ the distributional integral of the function f. When there is no possibility of confusion one may write just $\int_a^b f(x)\,\mathrm{d}x$, but for the purposes of this article we will not do so.

Naturally one can consider distributional integration for functions defined just in an interval, not on the whole real line.

Observe that the distribution f_0 is unique, if it exists, while f_1 is unique up to an additive constant. We say that f_0 is a *(locally) integrable distribution.*

6.1. Properties. The basic properties of the locally integrable distributions, f, and the distributionally locally integrable functions, f, are the following.

1. $\mathsf{f} \leftrightarrow f$, i.e., $\mathsf{f} = 0 \iff f = 0$ (a.e.).

2. If f_0 is a locally integrable distribution, and f_1 is a primitive, $\mathsf{f}_1' = \mathsf{f}_0$, then f_1 is a Łojasiewicz distribution.

3. If f is a locally integrable distribution and ψ is a smooth function, then $\psi\mathsf{f}$ is a locally integrable distribution and

$$(\psi\mathsf{f})(x) = \psi(x)\mathsf{f}(x) \qquad \text{(a.e.)}.$$

4. If $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ is a locally integrable distribution ($f \leftrightarrow \mathsf{f}$), then

$$\langle \mathsf{f}, \phi\rangle = \text{(dist)} \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x\ , \qquad \phi \in \mathcal{D}(\mathbb{R}).$$

5. If f is a locally integrable distribution ($f \leftrightarrow \mathsf{f}$), and $\mathsf{f} \in \mathcal{S}'(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$, or $\mathsf{f} \in \mathcal{K}'(\mathbb{R})$ and $\phi \in \mathcal{K}(\mathbb{R})$, then the evaluation of f on the test function ϕ is given by

$$\langle \mathsf{f}, \phi \rangle = (\mathrm{dist}) \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x \qquad (\mathrm{C}).$$

Observe that any locally Lebesgue integrable function is distributionally locally integrable. More generally, if f is Denjoy-Perron-Henstock locally integrable, then it is distributionally locally integrable; in this case

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is a continuous function. If f is distributionally locally integrable then in general F will not be a continuous function but rather a Łojasiewicz function.

In general if f is locally distributionally integrable and $E \subset \mathbb{R}$ is measurable with characteristic function χ_E, then $\chi_E f$ is not locally distributionally integrable. Thus one cannot talk about

$$(\mathrm{dist}) \int_E f(x)\,\mathrm{d}x.$$

This is also true for any non-absolute integral such as the Denjoy-Perron-Henstock integral. However, if $m(E) = 0$, then $\chi_E f$ is locally distributionally integrable and has integral 0, so that

$$(\mathrm{dist}) \int_E f(x)\,\mathrm{d}x = 0 \qquad \text{if} \quad m(E) = 0.$$

Furthermore, if f a locally distributionally integrable and $E = \bigcup_{n=1}^N I_n$ is a finite union of non-overlapping intervals then $\chi_E f$ is locally distributionally integrable and

$$(\mathrm{dist}) \int_{\bigcup_{n=1}^N I_n} f(x)\,\mathrm{d}x = \sum_{n=1}^N (\mathrm{dist}) \int_{I_n} f(x)\,\mathrm{d}x.$$

It follows from the results of [18] that if f is a Łojasiewicz function then it is locally distributionally integrable and actually the definition of the integral (5) is the same as the definition (8). Similarly, if f is a distributionally regulated function then it is locally distributionally integrable and (7) coincides with (8).

Distributional integration satisfies many of the usual properties of non-absolute integrals, such as mean value theorems, integration by parts formulas, etc. Strikingly, these properties are all that one needs to obtain many important results in analysis, and thus several classical results that hold for Lebesgue integrable functions remain true for distributionally integrable functions.

EXAMPLE 6.1. The unique Łojasiewicz distribution $\mathsf{s}_{\alpha,\beta}$ for $\alpha \in \mathbb{C}$, $\beta > 0$, $\mathsf{s}_{\alpha,\beta} \leftrightarrow s_{\alpha,\beta}$, such that

$$s_{\alpha,\beta}(x) = |x|^\alpha \sin x^{-\beta}, \quad x \neq 0,$$

and $s_{\alpha,\beta}(0) = 0$, is always locally integrable. The reader may want to analyze when $|x|^\alpha \sin x^{-\beta}$ is regular or when it is Denjoy integrable.

EXAMPLE 6.2. The regular distribution $x^i H(x)$ is not a Łojasiewicz distribution, therefore its distributional derivative, a regularization of $ix^{i-1}H(x)$, is not distributionally integrable.

EXAMPLE 6.3. Examples of distributions that are not integrable include the following. The Dirac delta function $\delta(x)$ is not integrable because its primitives, $H(x)+C$, are not Łojasiewicz distributions. The same can be said of *any* regularization of the function $1/x$. Another interesting example is provided by $\nu = \psi'$, the derivative of the Cantor function: In fact a distribution concentrated on a set of measure 0 cannot be integrable.

7. The ϕ-transform. Following [26], we introduce the ϕ-transform, a function of two variables that we now define. Let $\phi \in \mathcal{D}(\mathbb{R})$ be a fixed test function that satisfies

$$\int_{-\infty}^{\infty} \phi(x)\,\mathrm{d}x = 1. \tag{9}$$

If $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ we introduce the function of two variables $F = F_\phi\{\mathsf{f}\}$ by the formula

$$F(x,y) = \langle \mathsf{f}(x+y\xi), \phi(\xi)\rangle\ , \quad x \in \mathbb{R}, \quad y > 0,$$

the distributional evaluation with respect to the variable ξ. We call F the ϕ-transform of f. Observe that if f is locally integrable, $\mathsf{f} \leftrightarrow f$, then

$$F(x,y) = (\mathrm{dist}) \int_{-\infty}^{\infty} f(x+y\xi)\phi(\xi)\,\mathrm{d}\xi. \tag{10}$$

The ϕ-transform can also be defined if ϕ does not belong to $\mathcal{D}(\mathbb{R})$ as long as we consider only distributions f of a more restricted class. Indeed, we can consider the case when $\phi \in \mathcal{A}(\mathbb{R})$ and $\mathsf{f} \in \mathcal{A}'(\mathbb{R})$ for any suitable space of test functions $\mathcal{A}(\mathbb{R})$, such as $\mathcal{S}(\mathbb{R})$, $\mathcal{K}(\mathbb{R})$, or $\mathcal{E}(\mathbb{R})$. It would be needed to evaluate (10) in the (C) sense in some cases. Observe that we assume (9) in every case. The ϕ-transform has been studied by various authors, under different names, such as "standard average with kernel ϕ" [6].

The basic property of the ϕ-transform is that $\mathsf{f}(x)$ is the *distributional* boundary value of $F(x,y)$ as $y \to 0$, that is, if $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ then

$$\lim_{y\to 0^+} F(x,y) = \mathsf{f}(x),$$

distributionally in the space $\mathcal{D}'(\mathbb{R})$, namely,

$$\lim_{y\to 0} \langle F(x,y), \psi(x)\rangle = \langle \mathsf{f}(x), \psi(x)\rangle\ , \quad \forall \psi \in \mathcal{D}(\mathbb{R}).$$

The result will also hold when $\mathsf{f} \in \mathcal{E}'(\mathbb{R})$ and $\phi \in \mathcal{E}(\mathbb{R})$ if $\phi \in L^1(\mathbb{R})$. Another case when $\mathsf{f}(x)$ is the distributional boundary value of $F(x,y)$ as $y \to 0$ is if

$$\mathsf{f}(x) = O(|x|^\beta) \quad (\mathrm{C})\ , \quad \text{as } |x| \to \infty, \tag{11}$$

$$\phi(x) = O(|x|^\alpha)\ , \quad \text{strongly as } |x| \to \infty, \tag{12}$$

and

$$\alpha < -1\ , \qquad \alpha + \beta < -1, \tag{13}$$

as follows from [10, Theorem 1]. It is true in particular if $\mathsf{f} \in \mathcal{S}'(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$.

For future reference, we say that if $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ and $\phi \in \mathcal{D}(\mathbb{R})$ we are in Case I. If (11), (12), and (13) are satisfied, we say that we are in Case II. When $\mathsf{f} \in \mathcal{S}'(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$ we say that we are in Case III. Most results will hold in any of these three cases. However, the results are usually false when we just assume that $\mathsf{f} \in \mathcal{E}'(\mathbb{R})$ and $\phi \in \mathcal{E}(\mathbb{R})$.

Suppose now that $\mathsf{f}(x_0) = \gamma$, distributionally. In any of the cases I, II, or III, we have [26]

$$\lim_{(x,y)\to(x_0,0)} F(x,y) = \gamma,$$

in any sector $y \geq m|x - x_0|$ for any $m > 0$.

THEOREM 7.1. *If* f *is a locally integrable distribution,* $\mathsf{f} \leftrightarrow f$*, then*

$$\lim_{\substack{(x,y)\to(w,0)\\ \text{angularly}}} F(x,y) = f(w),$$

almost everywhere with respect to w*, and if* f *is a Łojasiewicz distribution, for all* $w \in \mathbb{R}$.

These results apply to general distributions and test functions. When the test function ϕ is of certain special forms, however, the ϕ-transform becomes a particular solution of a partial differential equation, and those results become results on the boundary behavior of solutions of partial differential equations.

Suppose first that $\phi = \phi_1$ where

$$\phi_1(x) = \frac{p(x)}{q(x)},$$

p and q are polynomials, $\alpha = \deg q - \deg p \geq 2$, q does not have real zeros, and $\int_{-\infty}^{\infty} \phi_1(x)\,\mathrm{d}x = 1$. Let

$$q(x) = \sum_{k=0}^{n} a_k x^k.$$

Then if $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ satisfies the estimate $\mathsf{f}(x) = O(|x|^\beta)$ (C), $|x| \to \infty$, where $\alpha + \beta < -1$, then the ϕ-transform

$$F_1(x,y) = \langle \mathsf{f}(x+y\xi), \phi_1(\xi)\rangle, \quad x \in \mathbb{R},\, y > 0,$$

is a solution of the partial differential equation

$$\sum_{k=0}^{n} a_{n-k} \frac{\partial^n F}{\partial x^k \partial y^{n-k}} = 0,$$

with $F(x,0^+) = \mathsf{f}(x)$ distributionally, since

$$\begin{aligned}\sum_{k=0}^{n} a_{n-k} \frac{\partial^n F}{\partial x^k \partial y^{n-k}} &= \sum_{k=0}^{n} a_{n-k} \langle \mathsf{f}^{(n)}(x+y\xi)\xi^{n-k}, \phi_1(\xi)\rangle \\ &= \langle \mathsf{f}^{(n)}(x+y\xi)q(\xi), \phi_1(\xi)\rangle = \langle \mathsf{f}^{(n)}(x+y\xi), p(\xi)\rangle = 0.\end{aligned}$$

Observe that if f is a locally integrable distribution, $\mathsf{f} \leftrightarrow f$, then

$$F_1(x,y) = (\text{dist}) \int_{-\infty}^{\infty} f(x+y\xi)\phi_1(\xi)\,\mathrm{d}\xi \qquad \text{(C)}.$$

In the particular case when $q(x) = x^2 + 1$, $p(x) = 1/\pi$, we obtain

$$\phi_2(x) = \frac{1}{\pi(x^2+1)},$$

and $F_2(x, y)$ is the Poisson "integral" of f, which in case $\mathsf{f}(x) = O(|x|^\beta)$ (C), $|x| \to \infty$, for some $\beta < 1$, is the harmonic function with $F_2(x, 0^+) = \mathsf{f}(x)$ distributionally that satisfies $F_2(x, y) = O(|x|^\beta)$ (C), $|x| \to \infty$, for each fixed $y > 0$. Observe that

$$F_2(x, y) = \frac{y}{\pi}(\text{dist}) \int_{-\infty}^{\infty} \frac{f(\xi)\, d\xi}{(x-\xi)^2 + y^2},$$

if f is a locally integrable distribution, $\mathsf{f} \leftrightarrow f$.

THEOREM 7.2. *If f is a distributionally integrable function then $F_1(x, y)$, and in particular $F_2(x, y)$, satisfies that $F_1(x, y) \to f(w)$ as $(x, y) \to (w, 0)$ in any sector $y \geq m|x - w|$ for $m > 0$, almost everywhere with respect to w, and for all w if f is a Łojasiewicz function.*

Let us now take $\phi = \varphi_\nu$ where its Fourier transform is given by

$$\widehat{\varphi_\nu}(u) = e^{-u^\nu},$$

where $\nu = 2p$ is an even positive integer. Alternatively, φ_ν is the only solution in $\mathcal{S}$ of the ordinary differential equation

$$\varphi^{(\nu-1)}(\xi) = (-1)^p \frac{\xi}{\nu} \varphi(\xi),$$

with $\int_{-\infty}^{\infty} \varphi(\xi)\, d\xi = 1$. Then if $\mathsf{f} \in \mathcal{S}'(\mathbb{R})$, and F is the ϕ-transform corresponding to φ_ν, the function

$$G_\nu(x, t) = F(x, t^{1/\nu}), \quad x \in \mathbb{R},\ t > 0,$$

is a solution of the initial value problem

$$\frac{\partial G}{\partial t} = (-1)^{p-1} \frac{\partial^\nu G}{\partial x^\nu},$$
$$G(x, 0^+) = \mathsf{f}(x), \text{ distributionally.}$$

In particular, if $\nu = 2$, then

$$\widehat{\varphi_\nu}(u) = e^{-u^2}, \quad \varphi_\nu(\xi) = \frac{1}{2\sqrt{\pi}} e^{-\xi^2/4},$$

and $G_2(x, t)$ is the solution of the heat equation $G_{,t} = G_{,xx}$ that satisfies $G(x, 0^+) = f(x)$, distributionally, and with $G(x, t) \in \mathcal{S}'(\mathbb{R})$ for each fixed $t > 0$.

THEOREM 7.3. *If f is a locally integrable distribution, $\mathsf{f} \leftrightarrow f$ then $G_2(x, t)$ takes the form*

$$G_2(x, t) = \frac{1}{2\sqrt{\pi t}}(dist) \int_{-\infty}^{\infty} f(\xi) e^{-\frac{(\xi - x)^2}{4t}}\, d\xi\ .$$

If f is a distributionally integrable function, then $G_\nu(x, t) \to f(w)$ in any region of the form $t \geq m|x - w|^\nu$ almost everywhere with respect to w. If f is a Łojasiewicz function this holds for all $w \in \mathbb{R}$.

8. The Fourier transform of locally integrable distributions. We shall now give the characterization of the Fourier transform of tempered locally integrable distributions.

The characterization of the Fourier series of those periodic distributions that have a distributional point value was given in [8]: If $\mathsf{f}(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ in the space $\mathcal{D}'(\mathbb{R})$ then

$$\mathsf{f}(\theta_0) = \gamma, \quad \text{distributionally,}$$

if and only if there exists k such that

$$\lim_{x\to\infty} \sum_{-x\le n\le ax} a_n e^{in\theta_0} = \gamma \quad (\mathrm{C}, k), \quad \forall a > 0.$$

Therefore, if f is a periodic locally distributionally integrable function, of period 2π, then the coefficients

$$a_n = (\mathrm{dist}) \int_0^{2\pi} f(\theta) e^{-in\theta} \, \mathrm{d}\theta,$$

are well-defined for all $n \in \mathbb{Z}$, and

$$\lim_{x\to\infty} \sum_{-x\le n\le ax} a_n e^{in\theta} = f(\theta) \quad (\mathrm{C}, k), \quad \forall a > 0, \tag{14}$$

almost everywhere with respect to θ. If f is a Łojasiewicz function (14) holds for all $\theta \in \mathbb{R}$.

Recently a similar result for general Fourier transforms was obtained [26, 25]. Indeed, let $\mathsf{f} \in \mathcal{S}'(\mathbb{R})$, and let $x_0 \in \mathbb{R}$, then

$$\mathsf{f}(x_0) = \gamma, \quad \text{distributionally,}$$

if and only if

$$\mathrm{e.v.}\langle \widehat{\mathsf{f}}(u), e^{-iux_0} \rangle = 2\pi\gamma \quad (\mathrm{C}).$$

We have chosen the constants in the Fourier transform in such a way that

$$\widehat{f}(u) = \int_{-\infty}^{\infty} f(x) e^{ixu} \mathrm{d}x,$$

if the integral makes sense. In case $\widehat{f}$ is locally distributionally integrable this means that

$$\mathrm{e.v.}(\mathrm{dist}) \int_{-\infty}^{\infty} \widehat{f}(u) e^{-iux_0} \mathrm{d}u = 2\pi\gamma \quad (\mathrm{C}).$$

Suppose now that f is a locally integrable tempered distribution, $\mathsf{f} \leftrightarrow f$. Then

$$\mathrm{e.v.}\langle \widehat{\mathsf{f}}(u), e^{-iux_0} \rangle = 2\pi f(x_0) \quad (\mathrm{C}), \tag{15}$$

almost everywhere with respect to x_0, and actually everywhere if f is a Łojasiewicz function. When $\widehat{\mathsf{f}}$ is also locally integrable, $\widehat{\mathsf{f}} \leftrightarrow \widehat{f}$, then (15) becomes

$$\mathrm{e.v.}(\mathrm{dist}) \int_{-\infty}^{\infty} \widehat{f}(u) e^{-iux_0} \mathrm{d}u = 2\pi f(x_0) \quad (\mathrm{C}).$$

9. The Poisson summation formula. The Poisson summation formula

$$\sum_{k=-\infty}^{\infty} f(k) = \sum_{m=-\infty}^{\infty} \widehat{f}(2\pi m),$$

is a rather useful tool in many areas of mathematics. Unfortunately, one needs strong convergence assumptions for it to hold. Furthermore, it cannot hold in this form for general distributions because one needs the point values of f and its Fourier transform. There are generalizations where the convergence is replaced by summability and there is actually a *parametric* version that holds for *all* tempered distributions [7]. Here we shall consider a different, *pointwise* distributional version.

DEFINITION 9.1. Let $\mathsf{f} \in \mathcal{S}'(\mathbb{R})$. We shall say that f satisfies *condition (B)* if f satisfies that for each smooth function ϕ such that $\phi^{(k)}(x) = O(1)$ as $|x| \to \infty$ for all k, the evaluation s.v.$\langle \mathsf{f}(x), \phi(x)\rangle$ (C) exists.

If f satisfies condition (B) then for each $p > 0$ we can define a periodic distribution $\mathsf{g}_p \in \mathcal{S}'(\mathbb{R})$ by putting

$$\langle \mathsf{g}_p(x), \phi(x)\rangle_{\text{per}} = \text{s.v.}\langle \mathsf{f}(x), \phi(x)\rangle \quad (\text{C}), \tag{16}$$

for any smooth periodic function of period p, where $\langle \mathsf{g}_p(x), \phi(x)\rangle_{\text{per}}$ is the evaluation in the space of periodic distributions and periodic test functions of period p, or in the circle, if one prefers. Notice that

$$\mathsf{g}_p(x) = \sum_{k=-\infty}^{\infty} \mathsf{f}(x + kp) \quad (\text{C}), \tag{17}$$

in the space $\mathcal{S}'(\mathbb{R})$. Observe also that if $\widehat{\mathsf{f}}$ is a Łojasiewicz distribution then

$$\widehat{\mathsf{f}}\left(\frac{2\pi m}{p}\right) = \langle \mathsf{g}_p(x), e^{2\pi i m x/p}\rangle_{\text{per}}.$$

Now, since g_p is a periodic distribution, it has a distributionally convergent Fourier series, which because of (16) and (17) becomes

$$\sum_{k=-\infty}^{\infty} \mathsf{f}(x + kp) = \sum_{m=-\infty}^{\infty} \widehat{\mathsf{f}}\left(-\frac{2\pi m}{p}\right) e^{2\pi i m x/p} \quad (\text{C}), \tag{18}$$

in $\mathcal{S}'(\mathbb{R})$; the (C) corresponds to the series on the left, the one on the right is convergent. Moreover, if g_p is an ordinary function, $\mathsf{g}_p \leftrightarrow g_p$, then the series on the right side of (18) is Cesàro summable almost everywhere with respect to x, to the value $g_p(x)$, i.e., (18) would become valid almost everywhere with respect to x, but in that case both series would need to be understood in the (C) sense.

As an application of this version of the Poisson summation formula we shall consider an interesting "uncertainty principle" result for Fourier transforms. There is vast literature on uncertainty principles in Fourier analysis [15], that basically say that both a function and its Fourier transform cannot both have certain "nice" properties. A very simple example is the fact that if both f and $\widehat{\mathsf{f}}$ have compact support, then they must vanish. A similar but much more interesting result is the Theorem of Benedicks [1] according to which if both f and $\widehat{\mathsf{f}}$ correspond to continuous Lebesgue integrable functions in $\mathbb{R}^n$ and if both sets

$$\Sigma(f) = \{x \in \mathbb{R}^n : f(x) \neq 0\},$$

and $\Sigma(\widehat{f})$ have finite measure, then f must vanish. Actually the theorem can be stated by just requiring that $f \in L^1(\mathbb{R}^n)$ since this immediately yields that $f, \widehat{f} \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

Benedicks' result is plainly false for distributions, as consideration of a Dirac comb $\sum_{k=-\infty}^{\infty} \delta(x-k)$ shows. Naturally, such Dirac combs do not correspond to ordinary functions. However, as we shall presently see, the result is valid for Łojasiewicz distributions that satisfy condition (B). Strikingly, the proof is basically the same as that of Benedicks [1, 15]. We will present the proof in the one variable case because in this article we have considered only distributions of one variable, but employing the corresponding ideas in several variables [3, 19] the same proof can be used in $\mathbb{R}^n$.

THEOREM 9.2. *Let* $\mathsf{f} \in \mathcal{S}'(\mathbb{R})$ *satisfy condition (B). Suppose both* f *and* $\widehat{\mathsf{f}}$ *are Łojasiewicz distributions and that both sets* $\Sigma(f)$ *and* $\Sigma(\widehat{f})$ *have finite measure. Then* $\mathsf{f} = 0$.

Proof. Indeed, suppose $\mathsf{f} \in \mathcal{S}'(\mathbb{R})$ is a Łojasiewicz distribution that satisfies condition (B). Then so is $\mathsf{f}(x)e^{iax}$ for any $a \in \mathbb{R}$. Let w be the distribution of two variables given by

$$\mathsf{w}(x,a) = \sum_{m=-\infty}^{\infty} \widehat{\mathsf{f}}(2\pi(m+a))e^{-2\pi i m x/p}, \tag{19}$$

or, because of (18), by

$$\mathsf{w}(x,a) = \sum_{k=-\infty}^{\infty} \mathsf{f}(x+k)e^{ia(x+k)}. \tag{20}$$

Performing a dilation, if needed, we may assume that the set $\Sigma(f)$ has measure smaller than 1. Then we can find subsets A, B, and F of $\mathbb{R}$ such that A and B have full measure, and have the following properties. First, the sum in (19) is finite whenever $a \in A$, and thus that sum is a trigonometric polynomial in x, $w_a(x)$, if $a \in A$. Similarly, the sum in (20) is finite whenever $x \in B$, and thus that sum is a trigonometric polynomial in a, $w^x(a)$, if $x \in B$. Finally, because $m(\Sigma(f)) < 1$, we can choose $F \subset [0,1]$ with positive measure, $m(F) > 0$, such that $\mathsf{f}(x+k) = 0$ for all $x \in F$ and all $k \in \mathbb{Z}$, so that $w^x(a) = 0$ for all $(x,a) \in F \times \mathbb{R}$.

Now, if $a \in A$ is fixed, we have that the trigonometric polynomial $\mathsf{w}_a(x)$, $\mathsf{w}_a \leftrightarrow w_a$, is equal distributionally in x to a distribution corresponding to the measurable function $w^x(a)$, and thus $w_a(x) = w^x(a)$ almost everywhere in x. Therefore $w_a(x) = 0$ on a set of positive measure, and thus $w_a = 0$. Hence $\widehat{\mathsf{f}}(2\pi(m+a)) = 0$ for all $(a,m) \in A \times \mathbb{Z}$, and this yields that $\widehat{\mathsf{f}}(u) = 0$ almost everywhere in u, and since $\widehat{\mathsf{f}}$ is a Łojasiewicz distribution it follows that $\widehat{\mathsf{f}} = 0$. ∎

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