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THE ALGEBRA OF POLYNOMIALS ON THE SPACE OF ULTRADIFFERENTIABLE FUNCTIONS

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Abstract. We consider the space $\mathcal{D}^{\mathcal{M}}$ of ultradifferentiable functions with compact supports and the space of polynomials on $\mathcal{D}^{\mathcal{M}}$. A description of the space $\mathcal{P}(\mathcal{D}^{\mathcal{M}})$ of polynomial ultradistributions as a locally convex direct sum is given.

1. Introduction. Roumieu and Beurling ultradistributions are meant as elements of the dual space to a non-quasi analytic class of infinitely differentiable functions equipped with a natural locally convex topology (see e.g. [9]). In this paper, we will consider the space $\mathcal{P}(\mathcal{D}^{\mathcal{M}})$ of polynomial ultradistributions, where $\mathcal{D}^{\mathcal{M}}$ denotes the space of ultradifferentiable functions (for the definition see Section 2). The space $\mathcal{P}(\mathcal{D}^{\mathcal{M}})$ contains the space of ultradistributions as a proper subspace and it is the smallest space, which is stable under tensor multiplication. We shall describe the space $\mathcal{P}(\mathcal{D}^{\mathcal{M}})$ in terms of the direct sums of symmetric tensor powers of the space $\mathcal{D}'_{\mathcal{M}}$, dual to $\mathcal{D}^{\mathcal{M}}$; we prove that such a direct sum is a convolution algebra. In physics such algebras are known as Borcher's algebras (cp. [1]). It is widely known that the space $\mathcal{D}^{\mathcal{M}}(R^n)$ of ultradifferentiable functions equipped with a natural locally convex topology is topologically isomorphic to the space $E(C^n)$ of entire functions of exponential type [4], via the Fourier-Laplace transformation; we shall prove, however, that this isomorphism can be extended to the corresponding spaces of polynomials.

2. Polynomials on locally convex spaces. In this paper the symbol N_1 denotes the set $N \setminus \{0\}$ of strictly positive integer numbers.

Let $\mathcal{L}^n(X, C)$ denote the space of n-linear, continuous forms defined on a locally

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convex space X

$$F_n : \prod_{\imath=1}^{n} X := \underbrace{X \times \ldots \times X}_{n} \ni (x_1, \ldots, x_n) \mapsto F_n(x_1, \ldots, x_n) \in C.$$

With any n-linear, continuous form $F_n \in \mathcal{L}^n(X, C)$ we can associate the composition

$$P_n = F_n \circ \Delta_n, \qquad \Delta_n : X \ni x \mapsto {}^n x := (x, \ldots, x) \in \prod_{\imath=1}^{n} X,$$

which, according to [2], we shall call *a homogenous polynomial* of degree n on the space X. The linear space of all homogenous polynomials of degree n will be denoted by $\mathcal{P}_n(X)$.

When we have a polynomial $P_n \in \mathcal{P}_n(X)$ we can get back the linear symmetric form F_n, associated to P_n, by the following polarization formula (comp. i.e [2])

$$F_n(x_1, \ldots, x_n) = \frac{1}{2^n\, n!} \sum_{e_\imath = \pm 1} e_1 \ldots e_n\, P_n\Big(\sum_{\imath=1}^{n} e_\imath x_\imath \Big). \tag{1}$$

On the space $\mathcal{L}^n(X, C)$ we will consider the locally convex topology of uniform convergence on bounded, absolutely convex subsets of $\prod_{\imath=1}^{n} X$, this topology will be denoted by τ_β. By τ_β we will also denote the topology on the space $\mathcal{P}_n(X)$ of uniform convergence on bounded, absolutely convex subsets of X.

By the algebra of polynomials on the space X, we mean the locally convex direct sum

$$\mathcal{P}(X) := \sum_{n \in N_1} \mathcal{P}_n(X) = \Big\{ P(x) = \sum_{n=1}^{m} P_n(x) : \; P_n \in \mathcal{P}_n(X); \; m \in N_1 \Big\}.$$

It is obvious that $\mathcal{P}(X)$ is an algebra with respect to multiplication

$$\mathcal{P}(X) \times \mathcal{P}(X) \ni (P, Q) \mapsto PQ \in \mathcal{P}(X),$$

$$P(x)\, Q(x) = \sum_{n \in N_1} \sum_{m=1}^{n} P_m(x)\, Q_{n-m+1}(x), \qquad x \in X.$$

Now, we would like to introduce some notations, connected with tensor products. Let $\otimes^n X := X \otimes \ldots \otimes X$ denote the algebraic tensor product of n copies of the space X, and let $\widehat{\otimes}_p^n X$ denote its completion in the projective topology. In the space $\otimes^n X$ we consider the operation of symmetrization

$$\varsigma_n : \otimes^n X \ni x_1 \otimes \ldots \otimes x_n \mapsto x_1 \odot \ldots \odot x_n := \frac{1}{n!} \sum_{\varsigma \in G_n} x_{\varsigma(1)} \otimes \ldots \otimes x_{\varsigma(n)},$$

where G_n is the group of permutations.

The operator ς_n is a projection in the space $\otimes^n X$, continuous with respect to the given topology τ [2], hence it can be extended onto the completion of $\otimes^n X$. This extension will be also denoted by ς_n. In our paper by

$$(x_1 \otimes \ldots \otimes x_m) \odot (x_{m+1} \otimes \ldots \otimes x_n), \qquad 1 \leq m \leq n,$$

we shall understand $x_1 \odot x_2 \odot \ldots \odot x_n$ and the operator $\odot$ can be extended by linearity and continuity to an operator $(\widehat{\otimes}_p^m X) \times (\widehat{\otimes}_p^{n-m} X) \to \widehat{\odot}_p^n X$.

We shall use the following notation: $\odot^n X := \varsigma_n(\otimes^n X)$, and $\widehat{\odot}_p^n X := \varsigma_n(\widehat{\otimes}_p^n X\ )$.

Let χ_n denote the canonical inclusion of the cartesian product into the tensor product

$$\chi_n : \prod_{\imath=1}^{n} X \ni (x_1, \dots, x_n) \mapsto x_1 \otimes \dots \otimes x_n \in \otimes^n X.$$

3. The space $D^{\mathcal{M}}$ and its properties. Let us consider N_1^n with lexicographical order and by $\overline{k}$, $\widehat{k}$ we will denote the predecessor and the successor of k for $k \in N_1^n$.

Let $\mathcal{M} \equiv \{\mu_k\}_{k \in N_1^n}$ denote a sequence of positive numbers with the following properties:

(1M) $\mu_k^2 \le \mu_{\overline{k}} \mu_{\widehat{k}}$, (logarithmic convexity);

(2M) $\displaystyle\sum_{k \in N_1^n} \frac{\mu_k}{\mu_{\widehat{k}}} < \infty$ (non-quasi analyticity);

(3M) there are $c > 0$ and $d_j > 0$ $(j = 1, \dots, n)$ such that $\mu_{\widehat{k}} \le c d^k \mu_k$, where $d = (d_1, \dots, d_n)$ (stability under differential operators)

If for $a, b \in R^n$ such that $a_j < b_j\, (j = 1, \dots, n)$, $[a, b]$ denotes the n-dimensional interval $\prod_{j=1}^n [a_j, b_j]$ and $\nu \in \operatorname{int} R_+^n$ is any vector with positive coordinates, then we will consider the following space

$$D^M_{[a,b],\nu}(R^n) := \big\{\varphi \in C^\infty(R^n)\colon\ \operatorname{supp}\varphi \subset [a, b],\ \|\varphi\|_{[a,b],\nu} < \infty\big\},$$

where

$$\|\varphi\|_{[a,b],\nu} := \sup_{t \in [a,b]} \sup_{k \in N_1^n} \left|\frac{D^k \varphi(t)}{\nu^k \mu_k}\right|$$

with $D^k = D_1^{k_1} \dots D_n^{k_n}$, $D_j^{k_j} = \Big(-i\frac{\partial}{\partial t_j}\Big)^{k_j}$ and $\nu^k = {\nu_1}^{k_1} \dots {\nu_n}^{k_n}$.

Let us define an order relation between vectors of R^n, namely $a \succ b$ if and only if $a_j < b_j$, $j = 1, \dots, n$.

By $D^{\mathcal{M}}(R^n)$ we mean the inductive limit of the spaces $D^M_{[a,b],\nu}(R^n)$, i.e.

$$D^{\mathcal{M}}(R^n) = \liminf_{\nu \succ 0,\ a \succ b} D^M_{[a,b],\nu}(R^n),$$

with the inductive limit topology.

The Denjoy-Carleman Theorem implies that $D^{\mathcal{M}}(R^n)$ is nontrivial. If by $D'_{\mathcal{M}}(R^n)$ we denote the dual space for $D^{\mathcal{M}}(R^n)$ then the following properties of $D^{\mathcal{M}}(R^n)$ and $D'_{\mathcal{M}}(R^n)$ hold (see [4, Theorem 2.6])

THEOREM 3.1.

(i) *Every $D^{\mathcal{M}}_{[a,b],\nu}(R^n)$ is a Banach space.*

(ii) *The inclusions*

$$D^M_{[a,b],\nu}(R^n) \mapsto D^M_{[c,d],\mu}(R^n), \quad [a, b] \subset [c, d], \quad \nu \prec \mu$$

are compact.

(iii) *$D^{\mathcal{M}}(R^n)$ is a nuclear, reflexive, (DF)-space.*

(iv) *$D'_{\mathcal{M}}(R^n)$ is a nuclear, reflexive, (F)-space and an (M^*) space in the sense of Silva.*

4. The description of the space of polynomials on the space of ultradifferentiable functions. In order to simplify the notation by $\mathcal{D}^{\mathcal{M}}$ we will denote the space $D^{\mathcal{M}}(R^1)$ and by $\mathcal{D}'_{\mathcal{M}}$ the dual space for $\mathcal{D}^{\mathcal{M}}$.

In $D^{\mathcal{M}}(R^n)$ we consider the following operator

$$\varsigma_n^* : D^{\mathcal{M}}(R^n) \ni \varphi(t) \mapsto (\varsigma_n^* \circ \varphi)(t) := \frac{1}{n!} \sum_{\varsigma \in G_n} \varphi(t_{\varsigma(1)}, \dots, t_{\varsigma(n)}),$$

where $t = (t_1, \dots, t_n) \in R^n$. The operator ς_n^* is a projection on the closed subspace of $D^{\mathcal{M}}(R^n)$ of symmetric functions

$$\mathcal{D}^{\mathcal{M}}(R^n) := \mathcal{R}(\varsigma_n^*) \subset D^{\mathcal{M}}(R^n).$$

We would like to describe the dual space for $\prod_{n\in N} \mathcal{D}^{\mathcal{M}}(R^n)$. Let $\mathcal{D}'_{\mathcal{M}}(R^n)$ denote the dual space for $\mathcal{D}^{\mathcal{M}}(R^n)$ with the strong topology $\beta\langle \mathcal{D}'_{\mathcal{M}}(R^n) \mid \mathcal{D}^{\mathcal{M}}(R^n)\rangle$. We shall prove the following theorem (comp. [3]):

THEOREM 4.1. *The following mappings:*

$$\begin{array}{ccccc} \mathcal{D}'_{\mathcal{M}}(R^n) & \xrightarrow{\varrho} & \widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}} & \xrightarrow{\vartheta} & \mathcal{P}_n(\mathcal{D}^{\mathcal{M}}) \\ T_n & \xmapsto{\varrho} & \varrho(T_n) = f_n & \xmapsto{\vartheta} & F_n \end{array}$$

are topological isomorphisms. Moreover the second of them

$$\vartheta : \widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}} \ni f_n \mapsto F_n := f_n \circ \chi_n \circ \Delta_n \in \mathcal{P}_n(\mathcal{D}^{\mathcal{M}})$$

is given by the formula

$$F_n(\varphi) = \langle f_n \mid \otimes^n \varphi \rangle, \qquad \varphi \in \mathcal{D}^{\mathcal{M}}$$

and is an extension $\chi_n \circ \Delta_n$ of the superposition of canonical mappings

$$\begin{array}{ccccc} \mathcal{D}^{\mathcal{M}} & \xrightarrow{\Delta_n} & \prod_{\iota=1}^{n} \mathcal{D}^{\mathcal{M}} & \xrightarrow{\chi_n} & \otimes^n \mathcal{D}^{\mathcal{M}} \\ \varphi & \xmapsto{\Delta_n} & {}^n\varphi & \xmapsto{\chi_n} & \otimes^n \varphi \end{array}$$

where $\otimes^n \varphi$ is the scalar function of n real variables,

$$\otimes^n \varphi(t) := \varphi(t_1) \cdot \ldots \cdot \varphi(t_n), \qquad t = (t_1, \dots, t_n) \in R^n.$$

Proof. Let operators ς_n and ς'_n be mutually adjoint with respect to the dual pair $\langle \otimes^n \mathcal{D}'_{\mathcal{M}} \mid \otimes^n \mathcal{D}^{\mathcal{M}} \rangle$ given by the bilinear form

$$\langle u_1 \otimes \ldots \otimes u_n | \varphi_1 \otimes \ldots \otimes \varphi_n \rangle = \langle u_1 | \varphi_1 \rangle \ldots \langle u_n | \varphi_n \rangle. \tag{2}$$

Then for any $u_1, \dots, u_n \in \mathcal{D}'_{\mathcal{M}}$ and $\varphi_1, \dots, \varphi_n \in \mathcal{D}^{\mathcal{M}}$ the operator ς_n satisfies:

$$\langle u_1 \odot \ldots \odot u_n \mid \varphi_1 \otimes \ldots \otimes \varphi_n \rangle = \langle u_1 \otimes \ldots \otimes u_n \mid \varphi_1 \odot \ldots \odot \varphi_n \rangle,$$

$$\varsigma_n : \varphi_1 \otimes \ldots \otimes \varphi_n \mapsto \varphi_1 \odot \ldots \odot \varphi_n := \frac{1}{n!} \sum_{\varsigma \in G_n} \varphi_{\varsigma(1)} \otimes \ldots \otimes \varphi_{\varsigma(n)} .$$

Let $\mathcal{R}(\varsigma_n)$ be denoted by $\odot^n \mathcal{D}^{\mathcal{M}}$.

If the set of seminorms $\{p_i\}_{i\in I}$ defines the topology in $\mathcal{D}^{\mathcal{M}}$, then the set of seminorms

$$(p_{i_1} \otimes \ldots \otimes p_{i_n})(\psi) = \inf_{\psi = \sum_{m\in {N_1}^n} \varphi_{m_1} \otimes \ldots \otimes \varphi_{m_n} \in \otimes^n \mathcal{D}^{\mathcal{M}}} \sum_{m \in {N_1}^n} p_{i_1}(\varphi_{m_1}) \ldots p_{i_n}(\varphi_{m_n})$$

defines the projective topology in $\otimes^n\mathcal{D}^{\mathcal{M}}$. We have the following

$$\begin{aligned}(p_{i_1}\otimes\ldots\otimes p_{i_n})(\varsigma_n\circ\psi) &\le \inf\sum_{m\in {N_1}^n}\frac{1}{n!}\sum_{\varsigma\in G_n} p_{i_1}(\varphi_{m_{\varsigma(1)}})\ldots p_{i_n}(\varphi_{m_{\varsigma(n)}})\\ &= \inf\sum_{m\in {N_1}^n}\frac{1}{n!}\sum_{\varsigma\in G_n} p_{i_{\varsigma(1)}}(\varphi_{m_1})\ldots p_{i_{\varsigma(n)}}(\varphi_{m_n})\\ &= \varsigma_n\circ(p_{i_1}\otimes\ldots\otimes p_{i_n})(\psi),\end{aligned}$$

where

$$\varsigma_n\circ(p_{i_1}\otimes\ldots\otimes p_{i_n}) := \frac{1}{n!}\sum_{\sigma\in G_n} p_{i_{\varsigma(1)}}\otimes\ldots\otimes p_{i_{\varsigma(n)}}$$

is a seminorm in $\otimes^n\mathcal{D}^{\mathcal{M}}$, continuous in the projective topology. Hence the projection ς_n is continuous. The continuity of ς_n and the fact that the subspace $\otimes^n\mathcal{D}^{\mathcal{M}}$ is dense in $\widehat{\otimes}_p^n\mathcal{D}^{\mathcal{M}}$ imply that there exists a continuous extension of ς_n on the completions of the spaces $\otimes^n\mathcal{D}^{\mathcal{M}}$ and $\odot^n\mathcal{D}^{\mathcal{M}}$ respectively, namely $\varsigma_n : \widehat{\otimes}_p^n\mathcal{D}^{\mathcal{M}} \longrightarrow \widehat{\odot}_p^n\mathcal{D}^{\mathcal{M}}$. Hence we can represent the space $\widehat{\otimes}_p^n\mathcal{D}^{\mathcal{M}}$ as locally convex direct sum

$$\widehat{\otimes}_p^n\mathcal{D}^{\mathcal{M}} = \widehat{\odot}_p^n\mathcal{D}^{\mathcal{M}} \dotplus \mathcal{N}(\varsigma_n). \tag{3}$$

Let us remark that $(\widehat{\otimes}_p^n\mathcal{D}^{\mathcal{M}})'_\beta$ denotes the dual of $\widehat{\otimes}_p^n\mathcal{D}^{\mathcal{M}}$, endowed with the topology of uniform convergence on bounded, absolutely convex subsets, therefore one can replace the notation $(\widehat{\otimes}_p^n\mathcal{D}^{\mathcal{M}})'_\beta$ with the notation $((\widehat{\otimes}_p^n\mathcal{D}^{\mathcal{M}})', \tau_\beta)$.

Theorem 3.1 implies that $\mathcal{D}^{\mathcal{M}}$ is a nuclear (DF)-space and $\mathcal{D}'_{\mathcal{M}}$ is a (F)-space. For such spaces the following isomorphism

$$(\widehat{\otimes}_p^n\mathcal{D}^{\mathcal{M}})'_\beta \simeq \widehat{\otimes}_p^n\mathcal{D}'_{\mathcal{M}}. \tag{4}$$

holds. The isomorphism (4) implies that the bilinear form (2) defines the dual pair $\langle\widehat{\otimes}_p^n\mathcal{D}'_{\mathcal{M}} \mid \widehat{\otimes}_p^n\mathcal{D}^{\mathcal{M}}\rangle$ and the operator ς'_n is adjoint to ς_n with respect to this duality. In particular ς'_n is continuous in the strong topology.

Hence, and also from the equality (3) we obtain that the space $\widehat{\otimes}_p^n\mathcal{D}'_{\mathcal{M}}$ can be represented as the locally convex space

$$\widehat{\otimes}_p^n\mathcal{D}'_{\mathcal{M}} = \widehat{\odot}_p^n\mathcal{D}'_{\mathcal{M}} \dotplus \mathcal{N}(\varsigma'_n). \tag{5}$$

When in the space $\mathcal{L}^n(\mathcal{D}^{\mathcal{M}}, C)$ the topology τ_β of uniform convergence on bounded absolutely convex subsets is considered we have that

$$\big(\mathcal{L}^n(\mathcal{D}^{\mathcal{M}}, C), \tau_\beta\big) \simeq (\widehat{\otimes}_p^n\mathcal{D}^{\mathcal{M}})'_\beta. \tag{6}$$

Then from (4), (5) and (6) we get that

$$\big(\mathcal{L}^n(\mathcal{D}^{\mathcal{M}}, C), \tau_\beta\big) \simeq \widehat{\otimes}_p^n\mathcal{D}'_{\mathcal{M}} \simeq \widehat{\odot}_p^n\mathcal{D}'_{\mathcal{M}} \dotplus \mathcal{N}(\varsigma'_n). \tag{7}$$

The first isomorphism in (7) implies, in particular, that any form $\bar{f}_n \in \mathcal{L}^n(\mathcal{D}^{\mathcal{M}}, C)$ can be represented as $\bar{f}_n = \bar{F}_n\circ\chi_n$ for some $\bar{F}_n \in \widehat{\otimes}_p^n\mathcal{D}'_{\mathcal{M}}$, and there exists the representation in the form of absolutely convergent series $\bar{F}_n = \sum_{l\in {N_1}^n} u_{l_1}\otimes\ldots\otimes u_{l_n} \in \widehat{\otimes}_p^n\mathcal{D}'_{\mathcal{M}}$, where $u_{l_\iota} \in \mathcal{D}'_{\mathcal{M}}$, $\iota = 1,\ldots,n$ [10, Theorem 6.4]. Hence for any $\varphi_1,\ldots,\varphi_n \in \mathcal{D}^{\mathcal{M}}$ the operator

ς_n' satisfies the following equalities

$$(\varsigma_n' \circ \bar{f}_n)(\varphi_1, \dots, \varphi_n) = \frac{1}{n!} \sum_{\varsigma \in G_n} \sum_{l \in {N_1}^n} \langle u_{l_{\varsigma(1)}} \mid \varphi_1 \rangle \dots \langle u_{l_{\varsigma(n)}} \mid \varphi_n \rangle$$
$$= \frac{1}{n!} \sum_{\varsigma \in G_n} \sum_{l \in {N_1}^n} \langle u_{l_1} \mid \varphi_{\varsigma(1)} \rangle \dots \langle u_{l_n} \mid \varphi_{\varsigma(n)} \rangle,$$

which means that the composition $f_n^\varsigma := \varsigma_n' \circ \bar{f}_n$ belongs to the space $\mathcal{L}_\varsigma^n(\mathcal{D}^\mathcal{M}, C)$ of symmetric continuous n-linear forms on $\mathcal{D}^\mathcal{M}$. The second isomorphism in (7) implies

$$\mathcal{R}(\varsigma_n') = \big(\mathcal{L}_\varsigma^n(\mathcal{D}^\mathcal{M}, C), \tau_\beta\big) \simeq \widehat{\odot}_p^n \mathcal{D}'_\mathcal{M}. \tag{8}$$

Now we shall prove that the following topological isomorphism

$$\big(\mathcal{L}_\varsigma^n(\mathcal{D}^\mathcal{M}, C), \tau_\beta\big) \simeq \mathcal{P}_n(\mathcal{D}^\mathcal{M}) \tag{9}$$

holds. For any symmetric form $f_n^\varsigma \in \mathcal{L}_\varsigma^n(\mathcal{D}^\mathcal{M}, C)$ we have the polarization formula (1). Hence its restriction to the diagonal of cartesian product

$$\Delta_n' : \big(\mathcal{L}_\varsigma^n(\mathcal{D}^\mathcal{M}, C), \tau_\beta\big) \ni f_n^\varsigma \longrightarrow f_n^\varsigma \circ \Delta_n \in \mathcal{P}_n(\mathcal{D}^\mathcal{M})$$

should be the isomorphism (9) we are looking for. Since Δ_n' is surjective, then it is enough to prove its continuity. Any continuous seminorm on $\mathcal{P}_n(\mathcal{D}^\mathcal{M})$ has the form

$$p_S(f_n^\varsigma \circ \Delta_n) = \sup_{\varphi \in S} \big|(f_n^\varsigma \circ \Delta_n)(\varphi)\big|, \qquad f_n^\varsigma \circ \Delta_n \in \mathcal{P}_n(\mathcal{D}^\mathcal{M}),$$

where S is a bounded absolutely convex subset of $\mathcal{D}^\mathcal{M}$. The polarization formula (1) implies that

$$p_{S_1 \dots S_n}(f_n^\varsigma) \le \frac{1}{2^n \cdot n!} \sum_{e_\iota = \pm 1} \sup_{\iota \in \{1,\dots,n\}} \sup_{\varphi_\iota \in S_\iota} \Big|(f_n^\varsigma \circ \Delta_n)\Big(\sum_{\iota=1}^n e_\iota \varphi_\iota\Big)\Big|$$
$$= \frac{n^n}{2^n \cdot n!} \sum_{e_\iota = \pm 1} \sup_{\iota \in \{1,\dots,n\}} \sup_{x_\iota \in S_\iota} \Big|(f_n^\varsigma \circ \Delta_n)\Big(\frac{1}{n}\sum_{\iota=1}^n e_\iota \varphi_\iota\Big)\Big|$$
$$\le \frac{n^n}{n!} p_S(f_n^\varsigma \circ \Delta_n).$$

Hence Δ_n' is the required isomorphism (9).

By combining the isomorphisms (8) and (9) we obtain that the mapping

$$\widehat{\odot}_p^n \mathcal{D}'_\mathcal{M} \ni f_n \mapsto \big\langle f_n \mid \otimes^n \varphi \big\rangle = (f_n \circ \chi_n \circ \Delta_n)(\varphi) := F_n(\varphi), \tag{10}$$

given for any $\varphi \in \mathcal{D}^\mathcal{M}$, is the second isomorphism ϑ. It is known [5, Theorem 2.1] that

$$D^\mathcal{M}(R^n) \simeq \widehat{\otimes}_p^n \mathcal{D}^\mathcal{M}. \tag{11}$$

Isomorphism (11) implies that the functions of the form $\varphi(t) = \sum_{l \in N_1^n} \varphi_{l_1}(t_1) \dots \varphi_{l_n}(t_n)$, where $\varphi_{l_1}, \dots, \varphi_{l_n} \in \mathcal{D}^\mathcal{M}$, form a dense subspace $\otimes^n \mathcal{D}^\mathcal{M}$ of the space $D^\mathcal{M}(R^n)$. Since

$$(\varsigma_n^* \circ \varphi)(t) = \frac{1}{n!} \sum_{l \in {N_1}^n} \sum_{\varsigma \in G_n} \varphi_{l_1}(t_{\varsigma(1)}) \dots \varphi_{l_n}(t_{\varsigma(n)})$$
$$= \frac{1}{n!} \sum_{l \in {N_1}^n} \sum_{\varsigma \in G_n} \varphi_{l_{\varsigma(1)}}(t_1) \dots \varphi_{l_{\varsigma(n)}}(t_n) = (\varsigma_n \circ \varphi)(t),$$

the continuity of projections implies that the algebraic equality

$$\mathcal{D}^{\mathcal{M}}(R^n) = \widehat{\odot}_p^n \mathcal{D}^{\mathcal{M}}$$

holds. The topological isomorphism $\mathcal{D}^{\mathcal{M}}(R^n) \simeq \widehat{\odot}_p^n \mathcal{D}^{\mathcal{M}}$ is a corollary of (11) and the adjoint topological isomorphism

$$\varrho : \mathcal{D}'_{\mathcal{M}}(R^n) \longrightarrow \widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}}$$

is obvious. ∎

Let us denote

$$\mathcal{D}'_S := \sum_{n\in N_1} \mathcal{D}'_{\mathcal{M}}(R^n), \qquad \mathcal{D}^{\mathcal{M}}_S := \prod_{n\in N_1} \mathcal{D}^{\mathcal{M}}(R^n).$$

Notice that $\langle \mathcal{D}'_S \mid \mathcal{D}^{\mathcal{M}}_S\rangle$ is a dual pair according to its canonical bilinear form

$$\langle T \mid \bar{\varphi}\rangle = \sum_{n\in N_1} \langle T_n \mid \varphi_n\rangle \qquad \text{for} \quad T = \sum_{n\in N_1} T_n \in \mathcal{D}'_S, \quad \bar{\varphi} = \prod_{n\in N_1} \varphi_n \in \mathcal{D}^{\mathcal{M}}_S,$$

where $T_n \in \mathcal{D}'_{\mathcal{M}}(R^n)$ and $\varphi_n \in \mathcal{D}^{\mathcal{M}}(R^n)$. Let us remark that if $\bar{\varphi} \in \mathcal{D}^{\mathcal{M}}_S$, then $\bar{\varphi} = (\varphi_n)$ and, for different k and n, φ_k is a function of $({x^k}_1, {x^k}_2, \ldots, {x^k}_k)$ and φ_n is a function of $({x^n}_1, {x^n}_2, \ldots, {x^n}_n)$, where $({x^k}_1, {x^k}_2, \ldots, {x^k}_{\min(k,n)})$ and $({x^n}_1, {x^n}_2, \ldots, {x^n}_{\min(k,n)})$ can be different.

We shall prove the following

Theorem 4.2.

(i) *The locally convex space* $\sum_{n\in N_1} \widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}}$ *is a topological algebra with respect to convolution, given by the formula*

$$f * h := \sum_{n\in N_1} \Big(\sum_{m=1}^{n} f_m \odot h_{n-m+1} \Big),$$

where $f = \sum_{n\in N_1} f_n$, $h = \sum_{n\in N_1} h_n \in \sum_{n\in N_1} \widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}}$ *and* $f_n, h_n \in \widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}}$.

(ii) *The following mappings:*

$$\begin{array}{ccccc} \mathcal{D}'_S & \overset{\varrho}{\longrightarrow} & \sum_{n\in N_1} \widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}} & \overset{\vartheta}{\longrightarrow} & \mathcal{P}(\mathcal{D}^{\mathcal{M}}) \\ T = \sum_{n\in N_1} T_n & \overset{\varrho}{\longmapsto} & f = \sum_{n\in N_1} f_n & \overset{\vartheta}{\longmapsto} & F = \sum_{n\in N_1} F_n \end{array}$$

where $f_n := \varrho(T_n)$ *and* $F_n := f_n \circ \chi_n \circ \Delta_n = \vartheta(f_n)$, *are surjective topological isomorphisms.*

(iii) *The convolution in the algebra* $\sum_{n\in N_1} \widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}}$ *is transformed by the isomorphism* ϑ *into the product of polynomials in the algebra* $\mathcal{P}(\mathcal{D}^{\mathcal{M}})$, *i.e.*

$$\vartheta(f * h) = F \cdot H, \qquad F = \vartheta(f), \quad H = \vartheta(h) \in \mathcal{P}(\mathcal{D}^{\mathcal{M}}).$$

Proof. If we put

$$\varrho(T) = \sum_{n\in N_1} \varrho(T_n) = \sum_{n\in N_1} f_n = f, \qquad \vartheta(f) = \sum_{n\in N_1} \vartheta(f_n) = \sum_{n\in N_1} F_n = F,$$

then Theorem 4.1 implies that there exist isomorphisms ϱ and ϑ.

Now we prove (i); for any $f_n \in \widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}}$ and $h_m \in \widehat{\odot}_p^m \mathcal{D}'_{\mathcal{M}}$ we have that

$$f_n \odot h_m \in (\widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}}) \odot (\widehat{\odot}_p^m \mathcal{D}'_{\mathcal{M}}) \subset \widehat{\odot}_p^{n+m} \mathcal{D}'_{\mathcal{M}}$$

and the convolution " $\ast$ " in the direct sum $\sum_{n\in N_1} \widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}}$ is well defined. Its continuity follows from the continuity of the canonical mapping in the symmetric tensor product

$$(\widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}}) \times (\widehat{\odot}_p^m \mathcal{D}'_{\mathcal{M}}) \ni (f_n, h_m) \to f_n \odot h_m \in \widehat{\odot}_p^{n+m} \mathcal{D}'_{\mathcal{M}}.$$

From the formula (10) we obtain that

$$\begin{aligned} F_n(\varphi) \cdot H_m(\varphi) &= \langle f_n \mid \otimes^n \varphi \rangle \cdot \langle h_m \mid \otimes^m \varphi \rangle = \langle f_n \otimes h_m \mid \otimes^{n+m} \varphi \rangle \\ &= \langle f_n \odot h_m \mid \otimes^{n+m} \varphi \rangle = (f_n \odot h_m) \circ \chi_{n+m} \circ \Delta_{n+m}(\varphi). \end{aligned}$$

Hence $F_n \cdot H_m \in \mathcal{P}_{n+m}(\mathcal{D}^{\mathcal{M}})$ and for any polynomial ultradistributions $F = \sum_{n\in N_1} F_n$ and $H = \sum_{n\in N_1} H_n$ belonging to the space $\mathcal{P}(\mathcal{D}^{\mathcal{M}})$ we get that

$$\begin{aligned} F(\varphi) \cdot H(\varphi) &= \sum_{n\in N_1} \sum_{m=1}^{n} F_m(\varphi) \cdot H_{n-m+1}(\varphi) \\ &= \sum_{n\in N_1} \sum_{m=1}^{n} (f_m \odot h_{n-m+1}) \circ \chi_{n+1} \circ \Delta_{n+1}(\varphi) \\ &= (f \ast h) \circ \chi_{n+1} \circ \Delta_{n+1}(\varphi). \end{aligned}$$

Therefore the mapping

$$\sum_{n\in N_1} \widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}} \ni f = \sum_{n\in N_1} f_n \overset{\vartheta}{\longmapsto} F = \sum_{n\in N_1} f_n \circ \chi_n \circ \Delta_n \in \mathcal{P}(\mathcal{D}^{\mathcal{M}})$$

transforms the convolution into the product of polynomial ultradistributions. ■

5. Entire functions of exponential type. Let now $\nu = (\nu_1, \ldots, \nu_n)$ be an arbitrarily chosen vector with positive coordinates and let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in R^n$ be such that $b \succ a$. Let $\zeta = (\zeta_1, \ldots, \zeta_n) \in C^n$ and $\zeta = \xi + \mathrm{i}\tau$, where $\xi = (\xi_1, \ldots, \xi_n)$, $\tau = (\tau_1, \ldots, \tau_n) \in R^n$. In the space of entire functions we introduce the subspace of functions of exponential type in the following way

$$E_\nu[a,b] = \big\{\Phi : C^n \ni \zeta \mapsto \Phi(\zeta) \in C, \quad \|\Phi\|_{E_\nu[a,b]} < \infty\big\},$$

with the norm given by the formula

$$\|\Phi\|_{E_\nu[a,b]} = \sup_{k\in N^n} \sup_{\zeta\in C^n} \frac{\big|\zeta^k \Phi(\zeta) \exp\big(-H_{[a,b]}(\tau)\big)\big|}{\nu^k \mu_k},$$

where for $t = (t_1, \ldots, t_n)$, $\tau = (\tau_1, \ldots, \tau_n) \in R^n$

$$H_{[a,b]}(\tau) = \sup_{t\in[a,b]} (t, \tau), \qquad (t, \tau) = \sum_{\iota=1}^{n} t_\iota \tau_\iota$$

is the supporting function of the n-dimensional cube $[a,b] \subset R^n$. We call

$$E(C^n) := \bigcup \big\{E_\nu[a,b] : \nu \in \mathrm{int}\ R^n_+, \quad [a,b] \subset R^n\big\}$$

the space of ultraincreasing functions of exponential type.

Let us notice that $E(C^n)$ is contained in the known locally convex space of entire functions of exponential type described for example in [6, 9.1] and [8].

The following theorem gives some connection between $D^{\mathcal{M}}(R^n)$ and $E(C^n)$.

THEOREM 5.1 ([4, Theorem 9.1]). *The Fourier transform is a surjective topological isomorphism*

$$\mathcal{F} : D^{\mathcal{M}}(R^n) \to E(C^n).$$

The adjoint Fourier transform is a topological isomorphism of dual spaces, endowed with their strong topologies

$$\mathcal{F}' : E'(C^n) \to D'_{\mathcal{M}}(R^n).$$

Theorems 3.1 and 5.1 imply in particular that the spaces $E(C^n)$ and $E'(C^n)$ are nontrivial, nuclear, reflexive, locally convex. Moreover $E(C^n)$ is a (LN^*)-space in the sense of Silva and (DF)-space and $E'(C^n)$ is a (M^*)-space in the sense of Silva.

Let $E = E(C)$ denote the space of ultraincreasing entire functions of exponential type of one complex variable. We shall prove a Paley–Wiener type theorem for polynomial ultradistributions.

THEOREM 5.2. *The Fourier transform*

$$\mathcal{F} : \mathcal{D}^{\mathcal{M}} \to E$$

can be unambiguously extended to the topological isomorphism

$$\overline{\mathcal{F}}'_{\mathcal{P}} : \mathcal{P}(E) \to \mathcal{P}(\mathcal{D}^{\mathcal{M}}).$$

The proof of this theorem is postponed after Theorem 5.3.

The range of the projection

$$(\sigma_n \circ \Phi)(z) = \frac{1}{n!} \sum_{\varsigma \in G_n} \Phi(z_{\varsigma(1)}, \ldots, z_{\varsigma(n)}), \qquad \Phi \in E(C^n)$$

is denoted by $\mathcal{E}(C^n) = \mathcal{R}(\sigma_n)$. Obviously $\sigma_1(E) = E$. Let $\mathcal{E}'(C^n)$ be the strong dual space to $\mathcal{E}(C^n)$.

From Theorems 4.1 and 5.1 the linear topological isomorphisms

$$\mathcal{E}'(C^n) \simeq \widehat{\odot}_p^n E' \simeq \mathcal{P}_n(E) \tag{12}$$

follow. The symmetric projective tensor product $\odot_p^n E'$ and the space of polynomials $\mathcal{P}_n(E)$ are understood in the same way as $\odot_p^n \mathcal{D}'_{\mathcal{M}}$ and $\mathcal{P}_n(\mathcal{D}^{\mathcal{M}})$. The second of these isomorphisms is determined by the formula

$$\odot_p^n E' \ni \widetilde{P}_n \mapsto \langle \widetilde{P}_n \mid {}^n\Phi \rangle = P_n(\Phi) \in \mathcal{P}_n(E), \qquad \Phi \in E.$$

Let us denote

$$\mathcal{E}' = \sum_{n \in N_1} \mathcal{E}'(C^n), \quad \mathcal{E} = \prod_{n \in N_1} \mathcal{E}(C^n).$$

The following theorem is true:

THEOREM 5.3. *The following mappings:*

$$\begin{array}{ccccc} \mathcal{E}' & \xrightarrow{\widetilde{\varrho}} & \sum\limits_{n\in N_1} \widehat{\odot}_p^n E' & \xrightarrow{\widetilde{\vartheta}} & \mathcal{P}(E) \\ T = \sum\limits_{n\in N_1} T_n & \overset{\widetilde{\varrho}}{\longmapsto} & f = \sum\limits_{n\in N_1} f_n & \overset{\widetilde{\vartheta}}{\longmapsto} & F = \sum\limits_{n\in N_1} F_n, \end{array}$$

where $f_n = \widetilde{\varrho}(T_n)$ *and* $F_n = f_n \circ \widetilde{\chi}_n \circ \widetilde{\Delta}_n = \widetilde{\vartheta}(f_n)$, *are topological isomorphisms.*

Proof. The isomorphisms $\widetilde{\varrho}$, $\widetilde{\vartheta}$ exist from (12) if we put

$$\widetilde{\vartheta}(f) = \sum_{n\in N_1} \widetilde{\vartheta}(f_n) = \sum_{n\in N_1} F_n = F$$

which completes the proof. ∎

Proof of Theorem 5.2. Let $\mathcal{F}' : E' \longrightarrow \mathcal{D}'_{\mathcal{M}}$ denote the adjoint Fourier transform with respect to dual pairs $\langle E' \mid E\rangle$ and $\langle \mathcal{D}'_{\mathcal{M}} \mid \mathcal{D}^{\mathcal{M}}\rangle$ and let the operator $\overline{\mathcal{F}}'$ be defined in the following way

$$\begin{array}{rcl} \overline{\mathcal{F}}'_{\Pi} := \prod\limits_{n=1}^{\infty} {}^n\mathcal{F}' : \quad \sum\limits_{n\in N_1} \widehat{\otimes}_p^n E' & \to & \sum\limits_{n\in N_1} \widehat{\otimes}_p^n \mathcal{D}'_{\mathcal{M}} \\ f = \sum\limits_{n\in N_1} f_n & \mapsto & \overline{\mathcal{F}}' f = \sum\limits_{n\in N_1} {}^n\mathcal{F}' f_n, \end{array}$$

where

$$\begin{array}{rcl} {}^n\mathcal{F}' := \underbrace{\mathcal{F}' \otimes \ldots \otimes \mathcal{F}'}_{n} : \quad \widehat{\otimes}_p^n E' & \to & \widehat{\otimes}_p^n \mathcal{D}'_{\mathcal{M}}, \\ v_1 \otimes \ldots \otimes v_n & \mapsto & \mathcal{F}'v_1 \otimes \ldots \otimes \mathcal{F}'v_n. \end{array}$$

Theorem 5.1 implies that

$$\mathcal{N}(\mathcal{F}') = \{0\}, \qquad \mathcal{R}(\mathcal{F}') = \mathcal{D}'_{\mathcal{M}}. \tag{13}$$

For nuclear spaces X and Y and for a linear, continuous operator $A : X \to X$ it is true that $\mathcal{N}(A \otimes I_Y) = \mathcal{N}(A)\widehat{\otimes}_p Y$ (comp. Lemma 9 of [7]). One can also prove that $\mathcal{N}\ (I_X \otimes B) = X\widehat{\otimes}_p\mathcal{N}(B)$, when B is a linear continuous operator in Y. Since the spaces considered are nuclear and (13) holds, then

$$\mathcal{N}({}^n\mathcal{F}') = \{0\}, \qquad \overline{\mathcal{R}({}^n\mathcal{F}')} \simeq \widehat{\otimes}_p^n \mathcal{D}'_{\mathcal{M}},$$

hence the mapping ${}^n\mathcal{F}'$ is a continuous isomorphism with dense image. The inverse mapping is of the form

$$({}^n\mathcal{F}')^{-1} := \underbrace{(\mathcal{F}')^{-1} \otimes \ldots \otimes (\mathcal{F}')^{-1}}_{n} : \widehat{\otimes}_p^n \mathcal{D}'_{\mathcal{M}} \longrightarrow \widehat{\otimes}_p^n E'$$

and it is continuous as a tensor product of continuous operators. Therefore its domain is equal to $\overline{\mathcal{R}({}^n\mathcal{F}')} = \mathcal{R}({}^n\mathcal{F}') = \widehat{\otimes}_p^n \mathcal{D}'_{\mathcal{M}}$. Hence we obtain that $\overline{\mathcal{F}}'$ is also a topological isomorphism.

From the definitions of the relevant mappings we have isomorphisms

$$\widehat{\odot}_p^n E' \overset{{}^n\mathcal{F}'}{\simeq} \widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}}, \qquad \sum_{n\in N_1} \widehat{\odot}_p^n E' \overset{\overline{\mathcal{F}}'}{\simeq} \sum_{n\in N_1} \widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}}.$$

Indeed, the composition $\varsigma_n' \circ {}^n\mathcal{F}'$ (where ς_n denotes the symmetrization operator) transforms $\widehat{\odot}_p^n E'$ into $\widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}}$ and because $\odot^n \mathcal{D}'_{\mathcal{M}} \subset \mathcal{R}(\varsigma_n' \circ {}^n\mathcal{F}')$, the image of this composition is dense in $\widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}}$. The inverse mapping has the form $({}^n\mathcal{F}')^{-1} \circ \varsigma_n'$ and it is defined on $\mathcal{R}(\varsigma_n' \circ {}^n\mathcal{F}')$. The continuity of this mapping implies that there exists an extension of it on $\widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}}$, hence $\mathcal{R}(\varsigma_n' \circ {}^n\mathcal{F}') = \widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}}$.

Since the diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{D}^{\mathcal{M}}) & \xrightarrow{\vartheta^{-1}} & \sum \widehat{\odot}_p^n \mathcal{D}'_{\mathcal{M}} \\ \overline{\mathcal{F}}'_{\mathcal{P}} \downarrow & & \downarrow \overline{\mathcal{F}}'^{-1} \\ \mathcal{P}(E) & \xrightarrow{\widetilde{\vartheta}^{-1}} & \sum \widehat{\odot}_p^n E' \end{array}$$

should commute, the operator $\mathcal{F}'_{\mathcal{P}}$ is unambiguously defined and it is a topological isomorphism. ■

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