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SCHWARTZ KERNEL THEOREM IN ALGEBRAS OF GENERALIZED FUNCTIONS

VINCENT VALMORIN

Département de Mathématiques et Informatique
Université des Antilles et de la Guyane
Campus de Fouillole
97159 Pointe à Pitre Cedex, France
E-mail: vincent.valmorin@univ-ag.fr

Abstract. A new approach to the generalization of Schwartz's kernel theorem to Colombeau algebras of generalized functions is given. It is based on linear maps from algebras of classical functions to algebras of generalized ones. In particular, this approach enables one to give a meaning to certain hypotheses in preceding similar work on this theorem. Results based on the properties of G^{∞}-generalized functions class are given. A straightforward relationship between the classical and the generalized versions of Schwartz's kernel theorem is established.

1. Introduction. Let Y and X denote pure finite dimensional manifolds. With standard notation, as given by [6], the Schwartz kernel theorem states that to every linear map T from $\mathcal{D}(Y)$ to $\mathcal{D}'(X)$ whose restriction to each $\mathcal{D}_M(Y)$ is continuous from $\mathcal{D}_M(Y)$ to $\mathcal{D}'(X)$ where $\mathcal{D}'(X)$ is endowed with its weak topology, is associated a unique kernel distribution $K \in \mathcal{D}'(Y \times X)$ such that for every $u \in \mathcal{D}(Y)$ and every $v \in \mathcal{D}(X)$, one has $\langle K, u \otimes v\rangle = \langle T(u), v\rangle$.

The purpose of this paper is to complement a recent work of [5] on Schwartz's kernel theorem in the framework of Colombeau algebras of generalized functions.

In [5] Schwartz kernel type theorems are considered through the notion of *continuously moderate* net $(L_\varepsilon)_\varepsilon$ of linear maps $L_\varepsilon : \mathcal{D}(\mathbb{R}^p) \to \mathcal{E}(\mathbb{R}^m), 0 < \varepsilon \leq 1$, involving growth properties of $L_\varepsilon(f)$ as $\varepsilon \to 0$ (a precise definition is given in section 4). It is shown that a continuously moderate net has a linear extension $L : \mathcal{G}_c(\mathbb{R}^p) \to \mathcal{G}(\mathbb{R}^m)$ where $\mathcal{G}_c(\mathbb{R}^p)$ and $\mathcal{G}(\mathbb{R}^m)$ are the respective Colombeau algebras. Moreover, under additional growth properties with respect to the parameter ε, L can be represented by a kernel which is a

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generalized function H_L on $\mathbb{R}^n \times \mathbb{R}^m$. A relationship with the classical Schwartz's kernel theorem is also established in a weak sense. Similar results, using the notion of $\mathcal{L}_{b,c}$-*strongly continuously moderate* net which is associated to certain subalgebras of $\mathcal{G}_c(\mathbb{R}^p)$ containing the subalgebra $\mathcal{G}_c^\infty(\mathbb{R}^p)$, are also given.

Our approach to Schwartz's kernel theorem is the converse of that of [5]: we take a linear map T from $\mathcal{D}(Y)$ to $\mathcal{G}(X)$ where Y and X are open sets of $\mathbb{R}^p$ and $\mathbb{R}^m$ respectively and seek conditions for a kernel representation of T. Here we use a sequential construction of Colombeau algebras but this is not important in the comparison of our method with that of [5]. Since T is necessarily defined by a (at most one) sequence (T_n) of linear maps $T_n : \mathcal{D}(Y)$ to $\mathcal{E}(X)$, this leads, due to the hypothesis of continuity in the classical Schwartz's kernel theorem, to the notion of r-continuity which means that there exists such a sequence where each T_n is continuous. Moreover, for each given $f \in \mathcal{D}(Y)$ the sequence $(T_n(f))_n$ must be moderate, that is $(T_n(f))_n \in \mathcal{E}_M(X)$ (see section 2), this fact introduces a quite natural notion of a *moderateness*. Now the question of a possible relationship between r-continuous moderate sequences and continuously moderate ones becomes evident. In fact we prove that these two notions are equivalent by using a standard Baire space argument. In some sense, this justifies the notion of continuously moderate nets.

In the sequel, we first provide background material on Colombeau algebras, then we give results on extension of linear maps which are applicable in our setting. In particular it is shown that every linear map from $\mathcal{D}(Y)$ to $\mathcal{G}(X)$ can be extended to a linear map from $\mathcal{G}_c(Y)$ to $\mathcal{G}(X)$. Next we give details on continuity and moderateness as previously mentioned. The last section is devoted to our version of the Schwartz kernel theorem in the framework of Colombeau algebras and related results. We prove a strong relationship of our Schwartz's kernel theorem with its classical version, results involving properties of $\mathcal{G}^\infty$ type are also given.

2. Basic definitions and notation. Here we give background material on Colombeau's theory of generalized functions, more details can be found e.g. in [3, 4, 7, 10].

Let Ω be an open set in $\mathbb{R}^d$ and $\mathcal{E}(\Omega)$ be the space of smooth functions on Ω with its usual topology. The notation $K \Subset \Omega$ means that K is a compact set in Ω. Then the set $\mathcal{E}_M(\Omega)$ of moderate sequences consists of sequences $(f_n)_n \in \mathcal{E}(\Omega)^{\mathbb{N}}$ such that

$$\forall K \Subset \Omega,\ \forall \alpha \in \mathbb{N}^d,\ \exists r \in \mathbb{R},\ \exists C > 0,\ \|\partial^\alpha f_n\|_{L^\infty(K)} \le Cn^r, n \ge 1.$$

The set $\mathcal{N}(\Omega)$ of negligible sequences consists of sequences $(f_n)_n \in \mathcal{E}(\Omega)^{\mathbb{N}}$ satisfying

$$\forall K \Subset \Omega,\ \forall \alpha \in \mathbb{N}^d,\ \forall q \in \mathbb{R},\ \exists C > 0,\ \|\partial^\alpha f_n\|_{L^\infty(K)} \le Cn^q, n \ge 1.$$

$\mathcal{E}_M(\Omega)$ is an algebra and moreover $\mathcal{N}(\Omega)$ is an ideal of $\mathcal{E}_M(\Omega)$. The simplified Colombeau algebra $\mathcal{G}(\Omega)$ is defined as the quotient $\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$. The class of an element $(f_n)_n \in \mathcal{E}_M(\Omega)$ will be denoted by $\mathrm{cl}(f_n)$. It is seen that $\mathcal{E}_M(\Omega)$ is a differential algebra. If one considers sequences $(f_n)_n$ consisting of constant functions on Ω, then one obtains the corresponding spaces $\mathcal{E}_M$ and $\mathcal{N}_0$. Then, the Colombeau algebra of generalized complex numbers is $\overline{\mathbb{C}} = \mathcal{E}_M/\mathcal{N}_0$. We notice that $\overline{\mathbb{C}}$ is a ring but not a field, see e.g. [1] and [2].

The subalgebra of generalized functions with compact support will be denoted by $\mathcal{G}_c(\Omega)$. In [2] it is proved that $\mathcal{G}_c(\Omega)$ is a dense ideal. The subset of $\mathcal{G}(\Omega)$ consisting of elements having a representative (and then all) $(f_n)_n$ such that:

$$\forall K \Subset \Omega,\ \exists r \in \mathbb{R},\ \forall \alpha \in \mathbb{N}^d,\ \exists C > 0,\ \|\partial^\alpha f_n\|_{L^\infty(K)} \le C n^r, n \ge 1,$$

is a subalgebra of $\mathcal{G}(\Omega)$ denoted by $\mathcal{G}^\infty(\Omega)$.

Let $(r_n)_n$ denote a sequence of positive numbers such that

$$\lim r_n = \infty \quad \text{and} \quad \lim r_n^q n^{-1} = 0$$

for every $q \in \mathbb{N}$. Similarly to [5] Lemma 9 or [9] section 1.2.2, we choose a sequence $(\theta_n)_n$ such that:

(A) $(\theta_n)_n \in \mathcal{E}_M(\mathbb{R}^d)$;
(B) $\operatorname{supp} \theta_n \subset B(0, r_n^{-1})$, $n \ge 1$;
(C) $\forall \chi \in \mathcal{D}(\Omega),\ ((\theta_n * \chi - \chi)|_\Omega)_n \in \mathcal{N}(\Omega)$.

This can be done as follows. Take $\psi \in \mathcal{D}(\mathbb{R}^d)$ having the unit closed ball as support such that $\psi = 1$ on a neighborhood of zero and choose $\rho \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \rho(t)dt = 1 \quad \text{and} \quad \int_{\mathbb{R}^d} t^\alpha \rho(t)dt = 0,\ \alpha \in \mathbb{N}^d \setminus \{0\}.$$

Then for $t \in \mathbb{R}^d$ we set

$$\theta_n(t) = n^{p+m} \rho(nt) \psi(r_n t).$$

Moreover it is seen that $(\theta_n)_n$ satisfies the following two properties:

(A_1) $\int_{\mathbb{R}^d} \theta_n(t)dt = 1 + o(n^{-q})$ as $n \to \infty$, $q \in \mathbb{N}$;
(B_1) $\int_{\mathbb{R}^d} t^\alpha \theta_n(t)dt = o(n^{-q})$ as $n \to \infty$, $q \in \mathbb{N}$, $\alpha \in \mathbb{N}^d \setminus \{0\}$

where $t^\alpha = t_1^{\alpha_1} ... t_d^{\alpha_d}$ and $|\alpha| = \alpha_1 + ... + \alpha_d$.

The embedding of the Schwartz distribution space $\mathcal{D}'(\Omega)$ is then realized by the linear map $\mathcal{D}'(\Omega) \ni f \mapsto \mathrm{cl}(f * \theta_n|_\Omega) \in \mathcal{G}(\Omega)$. The image of a distribution through this embedding is called a generalized distribution.

The integral of $f \in \mathcal{G}(\Omega)$ over $L \Subset \Omega$ is defined as the generalized complex number $\mathrm{cl}(\int_L f_n(x)dx)$ and does not depend on the chosen representative $(f_n)_n$. If f has compact support, one defines $\int_\Omega f$ as $\int_L f$ where L is an arbitrary compact set in Ω which contains $\operatorname{supp} f$ in its interior.

If $\mathcal{A}$ and $\mathcal{B}$ are vector spaces, we denote by $\mathcal{L}(\mathcal{A}, \mathcal{B})$ the vector space of linear maps (without continuity) from $\mathcal{A}$ to $\mathcal{B}$.

3. Factorization and extension of linear maps. The aim of this section is to establish some results on factorization and extension of linear maps with a view to application to generalized algebras. It is easily seen from elementary algebra arguments that

PROPOSITION 3.1. *Let $\mathcal{A}, \mathcal{B}, \mathcal{X}_\alpha$ and $\mathcal{X}_\beta$ denote vector spaces or algebras over a field $\mathbb{K}$. Let $\Pi_\alpha : \mathcal{X}_\alpha \to \mathcal{A}$ and $\Pi_\beta : \mathcal{X}_\beta \to \mathcal{B}$ be two linear maps.*

(i) *If* $T : \mathcal{A} \to \mathcal{B}$ *is a linear map and* Π_β *is surjective, then there exists a linear map* $\Phi : \mathcal{X}_\alpha \to \mathcal{X}_\beta$ *such that*

$$(*) \quad T \circ \Pi_\alpha = \Pi_\beta \circ \Phi.$$

(ii) *Conversely, let* $\Phi : \mathcal{X}_\alpha \to \mathcal{X}_\beta$ *be a linear map and assume that* Π_α *is surjective. Then, there exists a linear map* $T : \mathcal{A} \to \mathcal{B}$ *such that* $(*)$ *is fulfilled if and only if* $\Phi(\ker \Pi_\alpha) \subset \ker \Pi_\beta$*. If this condition is fulfilled, then one has* $\Pi_\alpha^{-1}(\ker T) = \ker \Phi$*. Moreover, there exists a linear map* $p_\alpha : \mathcal{A} \to \mathcal{X}_\beta$ *such that*

$$p_\alpha \circ \Pi_\alpha = \Phi \quad \textit{and} \quad \Pi_\beta \circ p_\alpha = T.$$

In particular, with obvious notations, Φ_1 *and* Φ_2 *being given, one has* $T_1 = T_2$ *for the corresponding maps* T_1 *and* T_2*, if and only if* $\Pi_\beta \circ \Phi_1 = \Pi_\beta \circ \Phi_2$*.*

With the notation of Proposition 3.1, we denote by $\mathcal{L}^*(\mathcal{X}_\alpha, \mathcal{X}_\beta)$ the subspace of $\mathcal{L}(\mathcal{X}_\alpha, \mathcal{X}_\beta)$ whose elements Φ satisfy $\Phi(\ker \Pi_\alpha) \subset \ker \Pi_\beta$. Then, we have

Corollary 3.2. *Assume that* Π_α *and* Π_β *are both surjective. The linear map* $\widehat{\Pi}_\beta : \mathcal{L}^*(\mathcal{X}_\alpha, \mathcal{X}_\beta) \to \mathcal{L}(\mathcal{X}_\alpha, \mathcal{B})$ *defined by* $\widehat{\Pi}_\beta(\Phi) = \Pi_\beta \circ \Phi$ *is surjective. Moreover we have the following vector space isomorphisms:*

$$\mathcal{L}(\mathcal{A}, \mathcal{B}) \simeq \frac{\mathcal{L}^*(\mathcal{X}_\alpha, \mathcal{X}_\beta)}{\ker \widehat{\Pi}_\beta} \simeq \mathcal{L}(\mathcal{X}_\alpha, \mathcal{B}).$$

Proof. We show that $\widehat{\Pi}_\beta$ is surjective. Let $q \in \mathcal{L}(\mathcal{X}_\alpha, \mathcal{B})$. Since Π_α is surjective, it follows that there exists $T \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ such that $T \circ \Pi_\alpha = q$. Since Π_β is surjective, from Proposition 3.1(i), there exists $\Phi \in \mathcal{L}(\mathcal{X}_\alpha, \mathcal{X}_\beta)$ such that $T \circ \Pi_\alpha = \Pi_\beta \circ \Phi$. This equality implies that $\Phi(\ker \Pi_\alpha) \subset \ker \Pi_\beta$, that is $\Phi \in \mathcal{L}^*(\mathcal{X}_\alpha, \mathcal{X}_\beta)$. Hence, $q = \widehat{\Pi}_\beta(\Phi)$ proving the surjectivity of $\widehat{\Pi}_\beta$. From a classical elementary theorem of algebra this implies the isomorphism $\mathcal{L}^*(\mathcal{X}_\alpha, \mathcal{X}_\beta)/\ker \widehat{\Pi}_\beta \simeq \mathcal{L}(\mathcal{X}_\alpha, \mathcal{B})$.

We define a linear map $\Gamma : \mathcal{L}(\mathcal{A}, \mathcal{B}) \to \mathcal{L}^*(\mathcal{X}_\alpha, \mathcal{X}_\beta)/\ker \widehat{\Pi}_\beta$ as follows. Let $T \in \mathcal{L}(\mathcal{A}, \mathcal{B})$. As already seen, there exists $\Phi \in \mathcal{L}^*(\mathcal{X}_\alpha, \mathcal{X}_\beta)$ such that $T \circ \Pi_\alpha = \widehat{\Pi}_\beta(\Phi)$. We set $\Gamma(T) = \mathrm{cl}(\Phi)$ (the class of Φ). This definition makes sense because if $\Phi' \in \mathcal{L}^*(\mathcal{X}_\alpha, \mathcal{X}_\beta)$ satisfies $T \circ \Pi_\alpha = \widehat{\Pi}_\beta(\Phi')$, it follows that $\Phi - \Phi' \in \ker \widehat{\Pi}_\beta$ and then $\mathrm{cl}(\Phi) = \mathrm{cl}(\Phi')$. Obviously, Γ is a linear map; we show that it is injective. For, let $T, T' \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ satisfy $\Gamma(T) = \Gamma(T')$. Set $\Gamma(T) = \Phi$ and $\Gamma(T') = \Phi'$. Hence, we have $\Phi - \Phi' \in \ker \widehat{\Pi}_\beta$, which means that $T \circ \Pi_\alpha = T' \circ \Pi_\alpha$. Since Π_α is surjective, it follows that $T = T'$ proving the injectivity of Γ. We now show the surjectivity of Γ. Let $g \in \mathcal{L}^*(\mathcal{X}_\alpha, \mathcal{X}_\beta)/\ker \widehat{\Pi}_\beta$ and set $g = \mathrm{cl}(\Phi)$. Since $\Phi \in \mathcal{L}^*(\mathcal{X}_\alpha, \mathcal{X}_\beta)$, we have $\Phi(\ker \Pi_\alpha) \subset \ker \Pi_\beta$. It follows from Proposition 3.1(ii) that there exists $T \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ such that $T \circ \Pi_\alpha = \widehat{\Pi}_\beta(\Phi)$. Hence $\Gamma(T) = \mathrm{cl}(\Phi)$, that is, $g = \Gamma(T)$. The corollary is then proved ∎

With the previous notation, assume that $\mathcal{A}$ and $\mathcal{B}$ are factor spaces of $\mathcal{X}_\alpha$ and $\mathcal{X}_\beta$ respectively, Π_α and Π_β being the canonical maps. Then, the associated map Φ given by Proposition 3.1 is called a *representative* of T.

Corollary 3.3. *Assume that* Π_α *is surjective and let* $\sigma : \mathcal{X} \to \mathcal{X}_\alpha$ *be an injective linear map such that* $\sigma(\mathcal{X}) \cap \ker \Pi_\alpha = \{0\}$ *(neutrix condition). Then, for every linear map*

$\Psi : \mathcal{X} \to \mathcal{B}$ there exist linear maps $\Phi : \mathcal{X}_\alpha \to \mathcal{B}$ and $T : \mathcal{A} \to \mathcal{B}$ such that $\Phi \circ \sigma = \Psi$ and $T \circ \Pi_\alpha = \Phi$.

In particular, $\mathcal{X}$ can be identified to a subspace of $\mathcal{A}$ and then, every linear map $\Psi : \mathcal{X} \to \mathcal{B}$ can be extended to a linear map $T : \mathcal{A} \to \mathcal{B}$.

Proof. Let E be a subspace of $\mathcal{X}_\alpha$ such that $\mathcal{X}_\alpha = \sigma(\mathcal{X}) \oplus \ker \Pi_\alpha \oplus E$. Define Φ by $\Phi(\sigma(x)) = \Psi(x)$, $\Phi | \ker \Pi_\alpha \oplus E = 0$. It is easily seen that Φ is a well defined a linear map from $\mathcal{X}_\alpha$ to $\mathcal{B}$, and $\Phi \circ \sigma = \Psi$. Since $\ker \Pi_\alpha \subset \ker \Phi$, it follows from Proposition 3.1(ii), that there exists a linear map $T : \mathcal{A} \to \mathcal{B}$ such that $T \circ \Pi_\alpha = \Phi$. We show that $\Pi_\alpha \circ \sigma$ is injective. For, let $(x, y) \in \mathcal{X}^2$ be such that $(\Pi_\alpha \circ \sigma)(x) = (\Pi_\alpha \circ \sigma)(y)$. Then we have $(\Pi_\alpha \circ \sigma)(x - y) = 0$, that is $\sigma(x - y) \in \sigma(\mathcal{X}) \cap \ker \Pi_\alpha$. It follows from the hypothesis that $\sigma(x - y) = 0$, and since σ is injective, then $x = y$. We have $T \circ (\Pi_\alpha \circ \sigma) = \Phi \circ \sigma = \Psi$. Hence, through the mentioned identification T is an extension of Ψ ∎

Notice that if moreover Π_α and σ are both morphisms of algebras, this holds true for $\Pi_\alpha \circ \sigma$ and then, $\mathcal{X}$ may be identified to a subalgebra of $\mathcal{A}$.

EXAMPLE 3.4. Let Y and X denote open sets in $\mathbb{R}^p$ and $\mathbb{R}^m$ respectively. Take $\mathcal{X} = \mathcal{D}(Y)$, $\mathcal{X}_\alpha = \mathcal{E}_{M,c}$ (subalgebra of $\mathcal{E}_M$ whose elements have a compact support), $\mathcal{A} = \mathcal{G}_c(Y)$ (subalgebra of $\mathcal{G}(Y)$ whose element have a compact support). We may take for example, $\mathcal{B} = \mathcal{G}(X)$. We have an embedding of algebras $\sigma : \mathcal{D}(Y) \to \mathcal{E}_{M,c}(Y)$ defined by $\sigma(\varphi) = (\varphi)_n$ (constant sequence $= \varphi$). We take Π_α as the canonical surjective map $i_c : \mathcal{E}_{M,c} \to \mathcal{G}_c(Y)$. It is easily seen that the neutrix condition $\sigma(\mathcal{X}) \cap \ker i_c = \{0\}$ is satisfied. Hence, every linear map from $\mathcal{D}(Y)$ to $\mathcal{G}(X)$ can be extended to a linear map from $\mathcal{G}_c(Y)$ to $\mathcal{G}(X)$.

4. Continuity and moderateness. With the previous notation, if $\mathcal{X}_\alpha$ and $\mathcal{X}_\beta$ are topological spaces, T will be said to be *r-continuous* if T admits a representative Φ which is continuous. In the sequel we shall be concerned with the case where $\mathcal{A} = \mathcal{X}_\alpha = \mathcal{X}$ and where $\sigma = \Pi_\alpha = Id_{\mathcal{X}}$ (the identity map).

Let E denote a topological algebra or vector space over $\mathbb{K}$ and let $\mathcal{X}_\beta$ be a subalgebra or a subspace of $E^{\mathbb{N}}$. A representative Φ of T assumed to be valued in $E^{\mathbb{N}}$ will be written in the sequential form $\Phi = (\Phi_n)_n$ with $\Phi_n : \mathcal{X}_\alpha \to E$. Then the r-continuity of T means that each Φ_n is continuous.

Now, let Y and X denote open sets in $\mathbb{R}^p$ and $\mathbb{R}^m$ respectively. If M denotes a compact set in Y, as usual $\mathcal{D}_M(Y)$ denotes the subset of $\mathcal{D}(Y)$ consisting of elements with support in M. Let $\Phi = (\Phi_n)_n : \mathcal{D}(Y) \to \mathcal{E}(X)^{\mathbb{N}}$ be a linear map. The continuity of Φ means that for each n and each compact set $M \subset Y$, $\Phi_n : \mathcal{D}_M(Y) \to \mathcal{E}(X)$ is continuous. That is for every compact set $N \subset X$ and every $\beta \in \mathbb{N}^m$ there are $j \in \mathbb{N}$ and $C_n > 0$ such that

$$\|\partial^\beta(\Phi_n(\varphi))\|_{L^\infty(N)} \leq C_n \sup_{|\alpha| \leq j} \|\partial^\alpha \varphi\|_{L^\infty(Y)},\ \varphi \in \mathcal{D}_M(Y). \tag{1}$$

If Φ is valued in $\mathcal{E}_M(X)$, it will be called *moderate* which means that: φ being fixed in $\mathcal{D}_M(Y)$, for every compact set $N \subset X$ and every $\beta \in \mathbb{N}^m$, there exist $C > 0$, $r \in \mathbb{R}$ such that

$$\|\partial^\beta(\Phi_n(\varphi))\|_{L^\infty(N)} \leq C n^r,\ n \in \mathbb{N}^*.$$

We have similar definitions for $\Phi = (\Phi_n)_n : \mathcal{D}(Y) \to \mathbb{C}^{\mathbb{N}}$ and $T : \mathcal{D}(Y) \to \overline{\mathbb{C}}$.

In our sequential setting of Colombeau algebras, according to [5] Definition 20, a sequence $(\Phi_n)_n$ of $\mathbb{C}$-linear maps, $\Phi_n : \mathcal{D}(Y) \to \mathcal{E}(X)$ is said to be *continuously moderate* (resp. *negligible*) if it satisfies (1) with a moderate (resp. negligible) sequence $(C_n)_n$. We have the following theorem:

THEOREM 4.1. *A linear map* $\Phi = (\Phi_n)_n : \mathcal{D}(Y) \to \mathcal{E}(X)^{\mathbb{N}}$ (or $\mathbb{C}^{\mathbb{N}}$) *is continuous and moderate if and only if it is continuously moderate.*

Proof. Obviously, if (1) is satisfied as mentioned, then Φ is continuous and moderate. We now prove the converse. To do that it is enough to consider the case of $\mathcal{E}(X)^{\mathbb{N}}$. With the previous notation we set:

$$f_n(\varphi) = \frac{\ln \|\partial^\beta(\Phi_n(\varphi))\|_{L^\infty(N)}}{\ln n},\ \varphi \in \mathcal{D}_M(Y), n \geq 2.$$

From the moderateness of Φ, for each $\varphi \in \mathcal{D}_M(Y)$ there exists $r(\varphi) \in \mathbb{R}$ such that

$$f_n(\varphi) \leq r(\varphi),\ n \geq 2.$$

It is easily seen that the f_n are lower semi-continuous maps from $\mathcal{D}_M(Y)$ to $[-\infty, +\infty]$. It follows that $f = \sup_n f_n$ is also lower semi-continuous and $f(\varphi) \leq r(\varphi)$ for each fixed φ. Since $\mathcal{D}_M(Y)$ is a Baire space, it follows that there exists a non-void open set $U \subset \mathcal{D}_M(Y)$ on which f is uniformly bounded. Hence, there exists $r \in \mathbb{R}$ such that

$$f_n(\varphi) \leq r,\ \varphi \in U, n \geq 2.$$

From the linearity of $\partial^\beta \circ \Phi_n$ and the triangle inequality applied to $\|.\|_{L^\infty(N)}$, there exist a neighborhood V of zero in $\mathcal{D}_M(Y)$ and $s \in \mathbb{R}$ such that $f_n(\varphi) \leq s,\ \varphi \in V, n \geq 2$. Moreover V contains a neighborhood W of zero of the form

$$W = \{\varphi \in \mathcal{D}_M(Y) : \sup_{|\nu| \leq j} \|\partial^\nu \varphi\|_\infty \leq C\}.$$

Let $\varphi \in \mathcal{D}_M(Y), \varphi \neq 0$. We have

$$\mu(\varphi) =: \sup_{|\nu| \leq j} \|\partial^\nu \varphi\|_\infty \neq 0 \text{ and } \frac{C\varphi}{\mu(\varphi)} \in W.$$

It follows that $f_n(\frac{C\varphi}{\mu(\varphi)}) \leq s,\ n \geq 2$; that is

$$\|\partial^\beta(\Phi_n(\varphi))\|_{L^\infty(N)} \leq C^{-1} n^s \sup_{|\nu| \leq j} \|\partial^\nu \varphi\|_\infty,\ n \geq 2.$$

This inequality is also true for $\varphi = 0$. The theorem is thus proved. ∎

5. The Schwartz kernel theorem. Let Y and X denote two open sets of $\mathbb{R}^p$ and $\mathbb{R}^m$. We use notation of section 2 with $\Omega = Y \times X$. The embedding of $\mathcal{D}'(Y)$ (resp. $\mathcal{D}'(X)$) in $\mathcal{G}(Y)$ (resp. $\mathcal{G}(X)$) will be defined with the sequence $(\theta_n^p)_n$ (resp. $(\theta_n^m)_n$) constructed similarly to $(\theta_n)_n$. If $K \in \mathcal{G}(Y \times X)$, then one may define a linear integral operator $\widetilde{K} : \mathcal{G}_c(Y) \to \mathcal{G}(X)$, $u \mapsto \widetilde{K}(u)$ by

$$\widetilde{K}(u) = \mathrm{cl}\Big(\int_Y K_n(y, .) u_n(y) dy \Big)$$

where $(K_n)_n$ and $(u_n)_n$ are arbitrary representatives of K and u respectively (see e.g. [5] and [12], section 4). It is easily seen that if $K \in \mathcal{G}^\infty(Y \times X)$, then $\widetilde{K}(\mathcal{G}_c(Y)) \subset \mathcal{G}^\infty(X)$. The main results of [5] on Schwartz's kernel theorem are given in Proposition 21(i), (ii); Theorem 24 and Proposition 27. Translated in our sequential setting this gives:

(a) *Any continuously moderate sequence $(L_n)_n \in (\mathcal{L}(\mathcal{D}(\mathbb{R}^p), \mathcal{E}(\mathbb{R}^m))^{\mathbb{N}}$ can be extended to a map $L \in \mathcal{L}(\mathcal{G}_c(\mathbb{R}^p), \mathcal{G}(\mathbb{R}^m))$ defined by*

$$L(f) = \mathrm{cl}(L_n(f_n)),$$

where $(f_n)_n$ is any representative of f.
(b) *The extension L depends on the sequence $(L_n)_n$ only in the following way: if $(R_n)_n$ is a negligible sequence of maps, then the extensions of $(L_n)_n$ and $(L_n + R_n)_n$ are equal.*
(c) *If the constant C_n in (1) satisfies $C_n = O(n^{r(|\beta|)})$ as $n \to \infty$ for some sequence r of positive integers satisfying $\limsup_{l\to\infty}(r(l)/l) < 1$, then*

$$\forall f \in \mathcal{G}_c^\infty(\mathbb{R}^p), L(f) = \mathrm{cl}\left(\int_{\mathbb{R}^p} H_{L,n}(.,y) f_n(y)dy\right),$$

where $(H_{L,n})_n$ (resp. $(f_n)_n$) is any representative of H_L (resp. f).
(d) *Let $(\phi_n)_n \in \mathcal{D}(\mathbb{R}^p)^{\mathbb{N}}$ be a sequence of mollifiers such that for all $k \in \mathbb{N}$ and all $\alpha \in \mathbb{N}^p \setminus \{0\}$, $\int_{\mathbb{R}^p} \phi_n(x)dx = 1 + O(n^{-k})$ and $\int_{\mathbb{R}^p} x^\alpha \phi_n(x)dx = O(n^{-k})$ as $n \to \infty$. Let $s \in (0,1)$ and $\Lambda \in \mathcal{L}(\mathcal{D}(\mathbb{R}^p), \mathcal{D}'(\mathbb{R}^m))$ be continuous in the strong topology and consider the family $(L_n)_n$ of linear maps defined by*

$$L_n : \mathcal{D}(\mathbb{R}^p) \to \mathcal{E}(\mathbb{R}^m), f \mapsto \Lambda(f) * \phi_{n^s}.$$

Then $(L_n)_n$ is continuously moderate and $\Lambda(f)$ is equal to $\widetilde{H_L}(f)$ in the generalized distribution sense for any $f \in (\mathcal{D}(\mathbb{R}^p)$, that is

$$\forall \phi \in \mathcal{D}(\mathbb{R}^m), \langle \Lambda(f), \phi \rangle = \int_{\mathbb{R}^m} \widetilde{H_L}(f)\phi \text{ in } \overline{\mathbb{C}}.$$

The proof of the above results follows closely from the classical Schwartz's kernel theorem as given by Hörmander ([8], chap. V) and uses Cartan's approximation lemma ([8], Lemma 4.1.3). Our approach takes advantage of Theorem 4.1 by using the explicit form of the kernel of a continuous linear map from $\mathcal{D}(Y)$ to $\mathcal{D}(X)$ as given by the classical Schwartz's kernel theorem. Unlike [5] we do not need the strong topology of $\mathcal{D}'$. We also use recent results from the embedding of the algebra $\mathcal{G}^\infty(\Omega)$ in the space $\mathcal{L}(\mathcal{D}(\Omega), \overline{\mathbb{C}})$ of $\mathbb{C}$-linear maps from $\mathcal{D}(\Omega)$ to $\overline{\mathbb{C}}$ (see [11] and [12]). In particular our approach enables us to get a strongest relationship with the classical Schwartz's kernel theorem (see Theorem (5.5)). We now state our Schwartz kernel theorem.

THEOREM 5.1. *Let Y and X denote open sets in $\mathbb{R}^p$ and $\mathbb{R}^m$ respectively and let T be an r-continuous linear map from $\mathcal{D}(Y)$ to $\mathcal{G}(X)$. Then, there exists a kernel generalized function $K \in \mathcal{G}(Y \times X)$ such that for every $u \in \mathcal{D}(Y)$ and every $v \in \mathcal{D}(X)$, one has:*

$$\int_{Y\times X} K(u \otimes v) = \int_X T(u)v. \tag{2}$$

Moreover there exists at most one generalized distribution K and at most one generalized function K in $\mathcal{G}^\infty(Y \times X)$ satisfying (2).

Proof. Let $\Phi = (\Phi_n)_n$ denote a continuous representative of T. Since Φ_n is linear continuous from $\mathcal{D}(Y)$ to $\mathcal{E}(X)$, it follows from the Schwartz kernel theorem that it admits a unique distribution kernel $H_n \in \mathcal{D}'(Y \times X)$ defined by

$$H_n(\chi) = \int_X \Phi_n(\chi(.,x))(x)dx,\ \chi \in \mathcal{D}(Y \times X),$$

where $\Phi_n(\chi(.,x))(x)$ is the value at x of the image of $y \mapsto \chi(y,x)$ under Φ_n.

We show that $H = (H_n)_n$ is moderate. To see this, let $M \Subset Y \times X$ and set

$$M_1 = \{y \in Y : \exists x \in X, (y,x) \in M\};\ \ M_2 = \{x \in X : \exists y \in Y, (y,x) \in M\}.$$

Obviously, M_1 and M_2 are compact sets. Moreover for every $x \in X$ and every $\chi \in \mathcal{D}_M(Y \times X)$, one has $\operatorname{supp}\chi(.,x) \subset M_1$. Let $\psi \in \mathcal{D}_{M_1}(Y)$. Since Φ is moderate:

$$\exists C > 0,\ \exists r \in \mathbb{R},\ \|\Phi_n(\psi)\|_{L^\infty(M_2)} \le Cn^r,\ n \ge 1.$$

Hence, we have

$$\|\Phi_n(\chi(.,x))\|_{L^\infty(M_2)} \le Cn^r,\ n \ge 1.$$

Denoting by λ the Lebesgue measure, it follows that

$$|H_n(\chi)| \le \lambda(M_2)Cn^r,\ n \ge 1.$$

Finally, H being continuous and moderate, for every compact set M in $Y \times X$, there are $j \in \mathbb{N}$ and $(D_n)_n \in \mathcal{E}_M$ such that for every $\chi \in \mathcal{D}_M(Y \times X)$,

$$|H_n(\chi)| \le D_n \sup_{|\gamma|\le j} \|\partial_y^\gamma \chi\|_{L^\infty(M)}. \tag{3}$$

First step. We assume that $H = (H_n)_n$ has compact support. We set

$$G_n = H_n * \check{\theta}_n,$$

where θ_n is defined in section 2 and $\check{\theta}(t) = \theta(-t)$. Let $\zeta \in Y \times X$ and let $\nu \in \mathbb{N}^{p+m}$. We have $G_n(\zeta) = H_n(\theta_n(.-\zeta))$ and then $\partial^\nu G_n(\zeta) = (-1)^{|\nu|}H_n(\partial^\nu\theta_n(.-\zeta))$. Let $M \Subset Y \times X$. There exists $M' \Subset Y \times X$ such that: $\forall \zeta \in M$, $\operatorname{supp}\theta_n(.-\zeta) \subset M'$. Hence, with the notation of (3) we have an inequality of the form

$$\|\partial^\nu G_n\|_{L^\infty(M)} \le D_n \sup_{|\gamma|\le j} \|\partial_y^\gamma \partial^\nu \theta_n\|_\infty.$$

It follows that $(G_n)_n \in \mathcal{E}_M(Y \times X)$. We denote by K the class of $(G_n)_n$ in $\mathcal{G}(Y \times X)$. Since the distribution G_n is a smooth function we then have

$$G_n(\chi) = \int_{Y\times X} G_n\chi,\ \chi \in \mathcal{D}(Y \times X). \tag{4}$$

Let $\chi \in \mathcal{D}(Y \times X)$. From (B), there exists $n(\chi) \in \mathbb{N}$ such that $\operatorname{supp}(\theta_n * \chi - \chi) \Subset Y \times X$ for $n \ge n(\chi)$. Then we have:

$$(G_n - H_n)(\chi) = H_n(\theta_n * \chi - \chi),\ n \ge n(\chi).$$

It is easily seen that moderate (resp. negligible) sequences with compact support are sent to moderate (resp. negligible) ones by continuous moderate maps. Hence, using property

(C) above, we get

$$((G_n - H_n)(\chi))_n \in \mathcal{N}_0,\ \chi \in \mathcal{D}(Y \times X). \tag{5}$$

From the definition of H_n, we have

$$H_n(u \otimes v) = \int_X \Phi_n(u)v,\ (u,v) \in \mathcal{D}(Y) \times \mathcal{D}(X).$$

It follows from (4) and (5) that

$$\left(\int_{Y\times X} G_n(u \otimes v) - \int_X \Phi_n(u)v\right)_n \in \mathcal{N}_0,\ (u,v) \in \mathcal{D}(Y) \times \mathcal{D}(X).$$

Since $K = \mathrm{cl}(G_n)$, it follows from the definition of the integral of a generalized function that for every $(u,v) \in \mathcal{D}(Y) \times \mathcal{D}(X)$ we have

$$\int_{Y\times X} K(u \otimes v) = \int_X T(u)v.$$

The existence part of the theorem is then proved when $\operatorname{supp} H$ is a compact set.

Second step. We assume that $\operatorname{supp} H$ is an arbitrary closed set of $Y \times X$. Let $(U_i)_i$ be an open covering of $Y \times X$ where U_i is a relatively compact subset. Let $(\sigma_i)_i$ denote a partition of unity subordinated to this covering. Set

$$H_n^i = \sigma_i H_n,\ \ G_n^i = H_n^i * \check{\theta}_n.$$

Since $\operatorname{supp} H_n^i \Subset U_i$, it follows that for n large enough, $\operatorname{supp} G_n^i \Subset U_i$. We set

$$K^i = \mathrm{cl}(G_n^i | Y \times X) \in \mathcal{G}_c(Y \times X).$$

We then have $\operatorname{supp} K^i \subset U_i$. It follows that the sum $\sum_i K^i$ is locally finite and then we may define $K \in \mathcal{G}(Y \times X)$ as

$$K = \sum_i K^i.$$

Similarly to (5) we have $((G_n^i - H_n^i)(\chi))_n \in \mathcal{N}_0,\ \chi \in \mathcal{D}(Y \times X)$ and then

$$\left(\int_{Y\times X} G_n^i(u \otimes v) - H_n^i(u \otimes v)\right)_n \in \mathcal{N}_0.$$

It follows that

$$\sum_i \left(\int_{Y\times X} G_n^i(u \otimes v) - H_n^i(u \otimes v)\right)_n = \left(\int_{Y\times X} G_n(u \otimes v) - H_n(u \otimes v)\right)_n \in \mathcal{N}_0,$$

which means that $(\int_{Y\times X} G_n(u \otimes v) - \int_X \Phi_n(u)v)_n \in \mathcal{N}_0$. Expressed in term of classes this is equivalent to

$$\int_{Y\times X} K(u \otimes v) = \int_X T(u)v,$$

proving the first part of the theorem.

Third step. We use the following

Theorem ([12], Theorem 4.2). *Let $K \in \mathcal{G}^\infty(Y \times X)$ and let T_K be defined on $\mathcal{G}_c(Y \times X)$ by $T_K(w) = \int_{Y\times X} Kw$. Then, the following conditions are equivalent*

i) $K = 0$;

ii) $T_K|\mathcal{D}(Y)\otimes\mathcal{D}(X)=0$;
iii) $\widetilde{K}|\mathcal{D}(Y)=0$;
iv) ${}^t\widetilde{K}|\mathcal{D}(X)=0$.

Now assume that $L\in\mathcal{G}(Y\times X)$ is a generalized distribution such that $\int_{Y\times X} L.(u\otimes v) = 0$ for all $(u,v)\in\mathcal{D}(Y)\times\mathcal{D}(X)$. Denote by $J\in\mathcal{D}'(Y\times X)$ the associated distribution. If $(L_n)_n$ is a representative of L, then we have $\lim_{n\to\infty} L_n = J$ in $\mathcal{D}'(Y\times X)$. It follows that $0=\lim_{n\to\infty}\int_{Y\times X} L_n(u\otimes v) = J(u\otimes v)$, showing that J vanishes on $\mathcal{D}(Y)\otimes\mathcal{D}(X)$. Hence $J=0$ and thus $L=0$. Also there exists at most one $K\in\mathcal{G}^\infty(Y\times X)$ which satisfies (2). To see this, suppose that $\int_{Y\times X} L(u\otimes v)=0$ for all $(u,v)\in\mathcal{D}(Y)\times\mathcal{D}(X)$ with $L\in\mathcal{G}^\infty(Y\times X)$. Then, it follows from [12], Theorem 4.2 that $L=0$. This proves the theorem ∎

Remark 5.2. It is well-known ([4], section 2.6) that the map Λ defined by:

$$\Lambda:\mathcal{G}(\Omega)\to\mathcal{L}(\mathcal{D}(\Omega),\overline{\mathbb{C}}),\qquad [\Lambda(f)](u)=\int_\Omega fu,$$

is not injective. It follows generally that there is no uniqueness if we remove in Theorem 5.1 the assumption that K is a generalized distribution or a generalized function of class $\mathcal{G}^\infty$.

We give two corollaries of Theorem 5.1 involving properties of $\mathcal{G}^\infty$.

Corollary 5.3. *If $(\widetilde{K}-T)(\mathcal{D}(Y))\subset\mathcal{G}^\infty(X)$, then $\widetilde{K}$ is an r-continuous extension of T to $\mathcal{G}_c(Y)$.*

Proof. From the injectivity of the restriction of Λ to $\mathcal{G}^\infty(X)$ and since (2) may be written as $\int_X \widetilde{K}(u).v=\int_X T(u).v$, it follows that $T=\widetilde{K}|_{\mathcal{D}(Y)}$. Let $(K_n)_n$ denote a representative of K and consider $\Psi:\mathcal{D}(Y)^{\mathbb{N}}\to\mathcal{E}(X)^{\mathbb{N}}$ defined by $\Psi[(\varphi_n)_n]=(\widetilde{K}_n.\varphi_n)_n$. It is easily seen that Ψ is a continuous linear map such that $\Phi(\mathcal{E}_{M,c}(Y))\subset\mathcal{E}_M(X)$ and then $\Phi:\mathcal{E}_{M,c}(Y)\to\mathcal{E}_M(X)$ defined by $\Phi[(\varphi_n)_n]=\Psi[(\varphi_n)_n]$ is a continuous representative of $\widetilde{K}$ proving the corollary. ∎

We denote by $\operatorname{supp}_Y T$ the complement of the set of points $y\in Y$ for which there exists an open neighborhood V_y such that

$$T(u)=0,\ u\in\mathcal{D}(Y),\ \operatorname{supp}u\subset V_y.$$

In the following Proj_1 denotes the first projection $Y\times X\to Y$, $(y,x)\mapsto y$.

Corollary 5.4. *If $K\in\mathcal{G}^\infty(Y\times X)$ then $\mathrm{Proj}_1(\operatorname{supp}K)\subset\operatorname{supp}_Y T$.*

Proof. Let $y\in Y\setminus\operatorname{supp}_Y T$ and let V_y denote an open neighborhood of y such that $T(u)=0$ for all $u\in\mathcal{D}(Y)$ with $\operatorname{supp}u\subset V_y$. It follows that

$$\int_X \widetilde{K}(u).v=\int_X T(u).v=0,\ v\in\mathcal{D}(X).$$

Since $K\in\mathcal{G}^\infty(Y\times X)$ we have $\widetilde{K}(u)\in\mathcal{G}^\infty(X)$ and then $\widetilde{K}|_{\mathcal{D}(V_y)}=0$. From [12] Theorem 4.2, it follows that $K|_{V_y\times X}=0$. Hence $\operatorname{supp}K\subset(Y\setminus V_y)\times X$ that is $\mathrm{Proj}_1(\operatorname{supp}K)\subset Y\setminus V_y$. It follows that $\mathrm{Proj}_1(\operatorname{supp}K)\subset\operatorname{supp}_Y T$. ∎

We now examine the relationship with the classical Schwartz kernel theorem. Let $S:\mathcal{D}(Y)\to\mathcal{D}'(X)$ be a continuous linear map where $\mathcal{D}'(X)$ is equipped with its weak topology. This means that for each compact set $M \Subset Y$, for all sequence $(\varphi_k)_k$ such that $\lim_k \varphi_k = 0$ in $\mathcal{D}_M(Y)$ and for every $v \in \mathcal{D}(X)$, we have $\lim_k [S(\varphi_k)](v) = 0$. Let $L \in \mathcal{D}'(Y \times X)$ denote the Schwartz kernel of S. We denote by $\mathbf{j}$ the embedding of $\mathcal{D}'(Y \times X)$ in $\mathcal{G}(Y \times X)$ and by $\mathbf{i}$ that of $\mathcal{D}'(X)$ in $\mathcal{G}(X)$. The associated linear map to S is then $T = \mathbf{i} \circ S : \mathcal{D}(Y) \to \mathcal{G}(X)$. With this notation we have the following:

Theorem 5.5. *Let $S : \mathcal{D}(Y) \to \mathcal{D}'(X)$ be a continuous linear map where $\mathcal{D}'(X)$ is endowed with its weak topology. Let H denote its Schwartz kernel in $\mathcal{D}'(Y \times X)$ and let $T = \mathbf{i} \circ S$. Then, T is an r-continuous linear map from $\mathcal{D}(Y)$ to $\mathcal{G}(X)$ which has $\mathbf{j}(H)$ as (unique) kernel generalized distribution.*

Proof. Let $(X_i)_i$ denote an open covering of X and let $(\tau_i)_i$ be a partition of unity subordinated to this covering. We define $S_i = \tau_i S$ by

$$[S_i(u)](V) = [S(u)](\tau_i v), \quad (u, v) \in \mathcal{D}(Y) \times \mathcal{D}(X).$$

It follows that S may be written as a locally finite sum $S = \sum_i S_i$ where the S_i are valued in $\mathcal{E}'(X)$. Obviously the S_i's as defined are continuous. If H_i is the Schwartz kernel of S_i, we have $H_i(u \otimes v) = [S_i(u)](v) = [S(u)](\tau_i v)$ for all $(u, v) \in \mathcal{D}(Y) \times \mathcal{D}(X)$. By summing up we find $\sum_i H_i(u \otimes v) = [S(u)](v)$; showing that $\sum_i H_i(u \otimes v) = H(u \otimes v)$. This being true for all $(u, v) \in \mathcal{D}(Y) \times \mathcal{D}(X)$, it follows that $H = \sum_i H_i$ (a locally finite sum). Hence, applying $\mathbf{j}$ on both sides of the previous equality shows that we may assume that S is valued in $\mathcal{E}'(X)$.

Let $(\theta_n^m)_n$ be a sequence defining the embedding of $\mathcal{D}'(X)$ in $\mathcal{G}(X)$ and defined similarly to $(\theta_n)_n$. We define $\tilde{\mathbf{i}} : \mathcal{D}'(X) \to \mathcal{E}_M(X)$ by $\tilde{\mathbf{i}}(f) = (f * \check{\theta}_n^m)_n$. Hence a representative of T is $\Phi = \tilde{\mathbf{i}} \circ S : \mathcal{D}(Y) \to \mathcal{E}_M(X)$. If we write Φ in the form $\Phi = (\Phi_n)_n$, then we have $\Phi_n(u) = S(u) * \check{\theta}_n^m$. Since $[S(u) * \check{\theta}_n^m](v) = [S(u)](\theta_n^m * v)$ and S is continuous, it follows that Φ_n is also continuous. A kernel K of T is already given from the proof of Theorem 5.1 by $K = \mathrm{cl}(K_n)$ with $K_n = H * \check{\theta}_n$, that is $K = \mathbf{j}(H)$. ∎

Remark 5.6. Denote by $(\theta_n^p)_n$ the sequence defining the embedding of $\mathcal{D}'(Y)$ in $\mathcal{G}(Y)$. If $\mathrm{cl}(\theta_n) = \mathrm{cl}(\theta_n^p \otimes \theta_n^m)$, then we can prove that $\mathbf{j}(H)$ is the kernel of T as follows:

Let $(u, v) \in \mathcal{D}(Y) \times \mathcal{D}(X)$. We have

$$\begin{aligned}[\Phi_n(u)](v) &= [S(u) * \check{\theta}_n^m](v) = [S(u)](\theta_n^m * v), \\ &= H(u \otimes (\theta_n^m * v)).\end{aligned}$$

We notice that $\mathrm{cl}[H(u \otimes (\theta_n^m * v))] = \mathrm{cl}[H((\theta_n^p * u) \otimes (\theta_n^m * v))]$. Since we have

$$\begin{aligned}H((\theta_n^p * u) \otimes (\theta_n^m * v)) &= H(\theta_n^p \otimes \theta_n^m) * (u \otimes v)), \\ H((\theta_n^p * u) \otimes (\theta_n^m * v)) &= (H * \check{\theta}_n)(u \otimes v) + \psi_n,\end{aligned}$$

where $(\psi_n)_n \in \mathcal{N}_0$, it follows that

$$(\Phi_n(u).(v) - (H * \check{\theta}_n).(u \otimes v))_n \in \mathcal{N}_0.$$

The above relation means that

$$\int_X T(u)(v) = \int_{Y \times X} \mathbf{j}(H)(u \otimes v),$$

since $T(u) = \mathrm{cl}(\Phi_n(u))$.

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