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SEMI-FORMAL THEORY AND STOKES' PHENOMENON OF NON-LINEAR MEROMORPHIC SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. This article continues earlier work of the author on non-linear systems of ordinary differential equations, published in *Asymptotic Analysis* 15 (1997), MR no. 98g:34015b. There, a completely formal theory was presented, while here we are concerned with a semi-formal approach: Solutions of non-linear systems of ordinary meromorphic differential equations are represented as, in general divergent, power series in several free parameters. The coefficients, aside from an exponential polynomial, a general power and integer powers of the logarithm, contain holomorphic functions that are the multi-sums of formal power series. In J. Écalle's terminology such a semi-formal solution may be regarded as a *transseries*. In the author's opinion, however, they are best understood as power series in several variables. In this setting, we shall define and investigate the non-linear analogues of normal solutions, Stokes multipliers, and central connection coefficients, well known in the linear case. Moreover, we shall briefly address the question of convergence of the semi-formal series occurring. In particular, we wish to point out that in the cases when the series, due to the small denominator phenomenon, fails to converge, it is natural to be content with what shall be called *partial convergence* of the series, meaning that some of the variables are set equal to 0, leaving a power series in fewer variables that then converges.

1. Introduction. Throughout this article, we shall be concerned with a ν-dimensional non-linear system of ordinary differential equations of the form

$$z^{r+1}x' = \check{g}(z,x), \tag{1}$$

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where the *Poincaré rank* r is a positive integer, $x = (x_1, \dots, x_\nu)^T$ is a vector of dimension $\nu \geq 1$, and

$$\check{g}(z,x) = \sum_{|p|\geq 1} g_p(z) x^p = G(z)x + \sum_{|p|\geq 2} g_p(z) x^p \tag{2}$$

is a *ν-dimensional formal power series* in the variables $x_1, \dots, x_\nu$. As usual, the row vector $p = (p_1, \dots, p_\nu)$ denotes a multi-index, $|p| := p_1 + \dots + p_\nu$ denotes the *length of* p, and $x^p := x_1^{p_1} \cdot \ldots \cdot x_\nu^{p_\nu}$. Let $\mathrm{e}^{(j)}$ denote the multi-index with a single 1 in position j and 0's elsewhere. The square matrix

$$G(z) = [g_1(z), \dots, g_\nu(z)], \qquad g_j(z) := g_{\mathrm{e}^{(j)}}(z), \quad 1 \leq j \leq \nu, \tag{3}$$

shall be named *the linear part* of $\check{g}(z,x)$, and $z^{r+1}x' = G(z)x$ will be referred to as *the linear system corresponding to* (1). We emphasize that we assume (formally) $\check{g}(z,0) = 0$, so $x(z) \equiv 0$ is a (formal) solution of (1). While the power series (2) may diverge for every $x \neq 0$, we require the coefficients $g_p(z)$ to be holomorphic functions in a fixed disc $\mathbb{D}_\rho$ of radius $\rho > 0$ about the origin. Therefore, we may expand

$$\forall\, p \in \mathbb{N}_0^\nu \setminus \{0\}: \quad g_p(z) = \sum_{k=0}^{\infty} g_{p,k} z^k, \qquad z \in \mathbb{D}_\rho\,. \tag{4}$$

Systems (1) satisfying these requirements will henceforth be called *semi-formal systems*, in contrast to (purely) *formal* systems which have been studied, e.g., in [2, 3]. There, the notation $\hat{g}(z,x)$, resp. $\hat{g}_p(z)$, instead of $\check{g}(z,x)$, resp. $g_p(z)$, was used, and the coefficients $\hat{g}_p(z)$ were allowed to be *formal power series* in z. If the power series (2), for every fixed $z \in \mathbb{D}_\rho$, converges for some x with $x_k \neq 0$ for every $k = 1, \dots, \nu$, then we shall speak of a *convergent* system. In such a case we shall usually write $g(z,x)$ instead of $\check{g}(z,x)$.

Every convergent or semi-formal system may also be regarded as a formal one, and all formal systems have been shown in [2, 3] to admit a *complete formal solution*, where the adjective *complete* indicates that the formal solution is a (formal) power series in ν free parameters $c = (c_1, \dots, c_\nu)^T \in \mathbb{C}^\nu$ and has invertible linear part; see Lemma 4.1 for more details. In [4] it has been shown that all the formal power series occurring in such a complete formal solution are multi-summable in the sense of J. Écalle [19, 20, 21]—also compare [5] for a convenient reference to the theory of multi-summability and its application to linear systems. Here, we are concerned with *semi-formal solutions*, of which some are obtained from a formal one by replacing all formal power series by their sums. Other semi-formal solutions, however, correspond to solutions of initial value problems, and we shall in particular discuss how the various kinds of semi-formal solutions are interrelated.

Remark 1.1. Since solutions of (1), even in the linear case, in general have a logarithmic branch point at the origin, it is natural to consider the variable z on the universal covering surface, denoted by $\mathbb{S}_\rho$, of the punctured disc $\mathbb{D}'_\rho := \mathbb{D}_\rho \setminus \{0\}$.

For most of the investigations to follow, it shall be convenient to restrict to a system whose linear part has certain additional properties. We shall say that a system (1) is *normalized*, provided that the following condition holds:

(N) The corresponding linear system $z^{r+1}x' = G(z)x$ has a formal fundamental solution $\hat{X}(z) = \hat{F}(z)z^L e^{Q(z)}$, where

(a) $\hat{F}(z)$ is a formal matrix power series whose constant term is the identity matrix.

(b) $Q(z) = \operatorname{diag}[q_1(z), \dots, q_\nu(z)]$ is a diagonal matrix of, not necessarily distinct, polynomials in the variable $w := z^{-1}$ without constant terms. The degrees of all these polynomials then are not larger than r.

(c) $L = \Lambda + N$, with a diagonal matrix $\Lambda = \operatorname{diag}[\lambda_1, \dots, \lambda_\nu]$ and a nilpotent matrix N that commutes with both Λ and $Q(z)$. Moreover, we require that there exists an integer m such that $\operatorname{Re}\lambda_k \in [m, m+1)$ for $k = 1, \dots, \nu$.

It is well known [7, 8, 22, 5] that a *linear* system can always be transformed to a normalized one by means of a change of variable, by replacing z by z^μ with a natural number μ, and a *linear transformation* $x = T(z)\tilde{x}$, where the $\nu \times \nu$ matrix $T(z)$ is holomorphic in a disc about the origin and invertible for $z \neq 0$. However, note that $\det T(z)$ may vanish at the origin, so the transformation $T(z)$ shall be called *meromorphic* to emphasize that $T^{-1}(z)$ will, in general, have a pole at the origin. Such transformations may also be applied to a non-linear system, so that for our purpose we may always assume that (1) is normalized. However, observe that the transformation $x = T(z)\tilde{x}$ may change the Poincaré rank r of our system. Moreover, it may be so that for a normalized system the linear part $G(z)$ vanishes for $z = 0$, in which case all polynomials $q_j(z)$ have degrees strictly less than r. If we would even allow a transformation $T(z)$ that is "truely meromorphic", i.e., has a pole at the origin, then we could also achieve that the integer m occurring in assumption (N) is non-negative. However, such a transformation, applied to a non-linear system, may make the coefficients $g_p(z)$ for $|p| \geq 2$ have poles at the origin, the order of which will, in general, be a multiple of $|p|$. This we do not want to do, and so we will have to allow m to be an arbitrary integer.

Remark 1.2. Note that the theory of linear systems implies uniqueness of the formal solution $\hat{X}(z)$ occurring in (N).

2. Singular vs. non-singular directions. Given a matrix $Q(z)$ as in (N) and a multi-index $p = (p_1, \dots, p_\nu) \neq 0$, we define polynomials $q(z,p)$ in z^{-1} and complex numbers $\lambda(p)$ by

$$q(z,p) = \sum_{j=1}^{\nu} p_j q_j(z), \qquad \lambda(p) = r + \sum_{j=1}^{\nu} p_j(\lambda_j - r). \tag{5}$$

While the polynomials $q(z,p)$ play a role here, the numbers $\lambda(p)$ shall be used in a later section! In case $p = \mathrm{e}^{(k)}$, the polynomial $q(z,p)$ equals $q_k(z)$, for $k = 1, \dots, \nu$. For these multi-indices of length 1, the degrees and leading terms of $q(z,p) - q_j(z)$ are of great importance for linear systems when discussing multi-summability of formal solutions, or the Stokes multipliers. The same applies in the non-linear theory as well, but for arbitrary multi-indices p: For some p and $1 \leq j \leq \nu$, it may happen that $q(z,p) - q_j(z)$ vanishes identically; if not, it has a non-zero degree $k(p,j)$ which is at most equal to r and shall be named the *corresponding level.* One can easily give examples, even in the linear case, showing that the set of levels can be any subset of $\{1,\ldots,r\}$. In order to keep notation as simple as possible, we shall here consider the "worst case" and regard *every natural number $k \in \{1,\ldots,r\}$ as a possible level. So when we shall speak of multi-summability in this article, we shall always mean $(r, r-1, \ldots, 1)$-summability* (observe that it is standard notation to label the levels of summability in decending order). However, note that for some of the levels k the set of singular directions which we are going to define below may be empty, because there may not be any polynomial $q(z,p) - q_j(z)$ of degree equal to k. In such a case we shall speak of an *irrelevant level.* Compare this to the discussion of *optimal summability types* in [5, p. 173] to see that such irrelevant levels can then be disregarded.

Given a natural number μ and a level $k \in \{1,\ldots,r\}$, we say that a number $\tau \in \mathbb{R}$ is *singular of level* (k,μ), provided that a multi-index p of length $|p| \le \mu$ and a $j \in \{1,\ldots,\nu\}$ exist, so that $\deg(q(z,p) - q_j(z)) = k$, and such that for positive real ξ we have

$$\lim_{\xi \to 0+} \xi^k \big(q(\xi e^{i\tau}, p) - q_j(\xi e^{i\tau})\big) \in \mathbb{R}_+ \,. \tag{6}$$

In other words, if q denotes the highest coefficient of $q(z,p) - q_j(z)$, then τ is a possible value for $\arg q^{1/k}$. Occasionally, we shall also speak of a singular direction of level k, provided that a μ exists so that this direction is singular of level (k,μ). The set of all singular directions of level k can be empty; if not, then it is a countable set of *period* $2\pi/k$, meaning that τ is singular of that level whenever $\tau \pm 2\pi/k$ is so, too. Note that in the linear theory only singular directions of level $(k,1)$ play a role. The set of singular directions of level (k,μ) is *a discrete subset* of $\mathbb{R}$, but the same may not be the case for *all* singular directions of level k, as the following examples show:

- Suppose that $\nu = 2$ and $q_1(z) = z^{-1}$, $q_2(z) \equiv 0$. Then $q(z,p) - q_j(z)$ either is $\equiv 0$ or has a highest coefficient which is either -1 or a positive integer. Consequently, the only level is $k = 1$, and the singular directions are integer multiples of π.
- Suppose that $\nu = 2$ and $q_1(z) = (1+i)z^{-1}$, $q_2(z) = -\sqrt{2}\, z^{-1}$. Again, the only level is $k = 1$, but the highest coefficient of $q(z,p) - q_j(z)$, in case $j = 2$, equals $p_1 - (p_2 - 1)\sqrt{2} + ip_1$. The real part of this number never vanishes, but will get arbitrarily small for some multi-indices $p = (p_1, p_2)$, while its imaginary part goes off to $+\infty$. Hence singular directions accumulate at the values $\pi/2$ modulo 2π, but not at the values $-\pi/2$ modulo 2π. The same can be seen to hold for $j = 1$.
- Suppose that $\nu = 4$ and $q_1(z) = 1/z$, $q_2(z) = -1/z$, $q_3(z) = i/z$, $q_4(z) = -i/z$. Then the set of highest coefficients of $q(z,p) - q_j(z)$ consists of all numbers $a + ib$ with arbitrary $a, b \in \mathbb{Z}$. Therefore, the singular directions here are a dense subset of $\mathbb{R}$.

Remark 2.1. Given $\mu \in \mathbb{N}$, we shall say that a direction τ is *singular of order* μ, provided that a $k \in \{1,\ldots,r\}$ exists so that τ is singular of level (k,μ). While the set of *all* singular directions can be dense in $\mathbb{R}$, observe that the set singular directions of order μ always is discrete. A vector $d \in \mathbb{R}^r$, which we shall here (matching the decreasing order of the levels of summability) conveniently write as

$$d = (d_r, d_{r-1}, \ldots, d_1)^T,$$

shall be named a *non-singular multi-direction of order* $\mu \in \mathbb{N}$, provided that each d_k is not among the singular directions of level (k,μ), and in addition

$$|d_k - d_{k-1}| \le \frac{\pi}{2}\Big(\frac{1}{k-1} - \frac{1}{k}\Big) = \frac{\pi}{2}\,\frac{1}{k(k-1)}\,, \quad 2 \le k \le r. \tag{7}$$

For the meaning of (7), refer to the theory of multi-summation [5, p. 161]. If a multi-direction d is non-singular of order μ *for every* μ, we shall simply call it non-singular. Since the sets of singular directions of level k are countable, we conclude that the set of non-singular multi-directions is not empty. More precisely, for every open interval $J \subset \mathbb{R}$ we can always find a non-singular multi-direction d with $d_r \in J$. *Also, please bear in mind for later that here we have modified the notation commonly used in the theory of multi-summability and enumerated the coordinates of a multi-direction in reverse order.*

3. Log-exponential expressions. In this section, we shall consider a fixed multi-direction $d = (d_r, d_{r-1}, , \dots, d_1)^T$ satisfying (7). Moreover, we shall also consider a polynomial in z^{-1} with vanishing constant term and of degree at most r, denoted as $q(z)$, and a complex numbers λ. In the application to a system (1), $q(z)$ and λ shall be related to the polynomials $q(z,p)$, resp. the numbers $\lambda(p)$ defined above, but this is not important at the moment.

We say that a function $x(z)$, holomorphic on $\mathbb{S}_\rho$, is a *log-exponential expression*, abbreviated as *l-ee*, *in the multi-direction* d *of degree* $m \in \mathbb{N}_0$, whenever it is of the form

$$x(z) = z^\lambda \mathrm{e}^{q(z)} \sum_{j=0}^{m} f_j(z)(\log z)^j, \qquad f_m(z) \not\equiv 0, \tag{8}$$

where each $f_j(z)$ is the sum in the multi-direction d of a formal power series $\hat f_j(z) = \sum_k f_k^{(j)} z^k$ (recall that in this article the type of multi-summability is always equal to $(r, r-1, \dots, 1)$). *Since we shall always consider a fixed multi-direction d, we shall also speak of a l-ee for short, but observe that such an expression always depends upon d.* Hence a l-ee is a polynomial of degree m in $\log z$, with coefficients that either vanish identically or are l-ee of degree $m = 0$, and we also consider the zero polynomial $x(z) \equiv 0$ as a l-ee. For $x(z) \not\equiv 0$, we shall refer to the pair $(q(z), \lambda)$ as *the type* of the l-ee, bearing in mind that λ can be replaced by $\lambda - \mu$ for any $\mu \in \mathbb{N}$, with a corresponding change of the coefficient functions $f_j(z)$. If the type $(q(z), \lambda)$ vanishes, we shall occasionally speak of a *logarithmic expression.* To every l-ee $x(z)$ we have a corresponding *formal l-ee*, namely

$$\hat x(z) = z^\lambda \mathrm{e}^{q(z)} \sum_{j=0}^{m} \hat f_j(z)(\log z)^j, \tag{9}$$

obtained by replacing the functions $f_j(z)$ by the corresponding formal power series $\hat f_j(z)$. Such formal expressions have been introduced, e.g., in Coddington and Levinson's classical book [14]. However, note that to an arbitrary formal l-ee we may *not* have a corresponding l-ee $x(z)$ in the multi-direction d, since some (or all) $\hat f_j(z)$ may fail to be multi-summable in this multi-direction. Even if they are so multi-summable, their sums being denoted by $f_j(z)$, then $x(z)$ as in (8) may not admit holomorphic continuation to $\mathbb{S}_\rho$ and thus will not be a l-ee in the above sense. However, in the application we have in mind here, all $x(z)$ will satisfy linear inhomogeneous first order ODE from which holomorphy on $\mathbb{S}_\rho$ will follow.

It has been shown in [14] that, if we restrict $\operatorname{Re}\lambda$ to any half-open interval of unit length, then with respect to a natural interpretation of equality of formal l-ees, every formal l-ee $\hat{x}(z)$ has a *unique* representation in the form (9), and therefore in particular its type is uniquely defined. The same is true for proper (i.e., not formal) l-ees, as we show now:

Proposition 3.1. *Let J be any half-open interval of unit length, and restrict to l-ee with* $\operatorname{Re}\lambda \in J$. *Then every l-ee $x(z) \not\equiv 0$ has a unique representation of the form* (8), *and therefore in particular its type is uniquely defined.*

Proof. Suppose that (8) holds. The theory of multi-summability implies existence of a sector S of opening more than π/r and bisecting direction d_r in which $f_j(z) \cong \hat{f}_j(z) = \sum_{k=k_j}^{\infty} f_k^{(j)} z^k$ as $z \to 0$ in S. Without loss of generality we may assume that $f_{k_j}^{(j)} \neq 0$ for all $j = 0, \ldots, m$, except when all $f_k^{(j)}$ vanish, in which case we may choose k_j as large as we please. Let $k := \min\{k_0, \ldots, k_m\}$, and choose the maximal value of $j \in \{0, \ldots, m\}$ so that $k_j = k$. Then we conclude that

$$x(z)\mathrm{e}^{-q(z)} z^{-\lambda-k_j} (\log z)^{-j} \longrightarrow f_{k_j}^{(j)} (\neq 0) \qquad (S \ni z \to 0).$$

From this observation uniqueness of $q(z)$ and λ follows. Moreover, since integer powers of z times a finite number of integer powers of $\log z$ form an *asymptotic scale*, the theory of asymptotic expansions implies that all $\hat{f}_j(z)$ are uniquely determined. The theory of multi-summability then gives uniqueness of their sums $f_j(z)$. ∎

In view of the above proposition, to every l-ee $x(z)$ there corresponds a unique formal l-ee $\hat{x}(z)$ which we shall name *its asymptotic expansion in the multi-direction d*. Conversely, if $\hat{x}(z)$ is given, if all $\hat{f}_j(z)$ are multi-summable in the multi-direction d, and if the sums $f_j(z)$ are such that $x(z)$, given by (8), can be continued onto $\mathbb{S}$, then we say that $\hat{x}(z)$ is *multi-summable in the multi-direction* d, and we call $x(z)$ *the sum of* $\hat{x}(z)$ *in this multi-direction.* Every d for which this is so shall also be referred to as a *non-singular multi-direction* for $\hat{x}(z)$.

For a fixed multi-direction d, the set of l-ees of a given type can be easily seen to be closed with respect to addition and differentiation, and we now wish to show the same for indefinite integration. More precisely, given a l-ee $x(z)$ of type $(q(z), \lambda)$, we shall investigate whether there exists another such expression of the same type, so that

$$zy'(z) = x(z). \tag{10}$$

That this is always so for formal l-ees $\hat{x}(z)$ resp. $\hat{y}(z)$ has been shown in [22, p. 132], where uniqueness of $\hat{y}(z)$ has been discussed as well. The same can be done for proper expressions under some natural condition linking d to $q(z)$. This proposition possibly exists in the literature on multi-summability, and certainly can be deduced from other, slightly more restricted results, e.g., from a theorem in an article of Balser and Tovbis [10, Theorem 3], but we choose to include its proof for the sake of the reader.

REMARK 3.2. Concerning the proof of the next proposition, observe that a function $y(z)$ is a solution of (10) if and only if we have

$$y(z) = c + \int_{z_0}^{z} x(w) w^{-1}\, dw,$$

for arbitrary $z_0 \neq 0$ and $c \in \mathbb{C}$, and any such $y(z)$ is holomorphic in $\mathbb{S}_\rho$, since the same holds by assumption for $x(z)$. For the uniqueness part of proof it suffices to observe that a non-zero constant is a log-exponential expression of a given type $(q(z), \lambda)$ (and of degree $m = 0$) if and only if $q(z) \equiv 0$ and $-\lambda \in \mathbb{N}_0$.

PROPOSITION 3.3. *For $q(z)$, λ, and d as above, let a l-ee $x(z)$ in the multi-direction d of type $(q(z), \lambda)$ and degree $m \geq 0$ be given. Then the following statements are true*:

(a) *Assume that $q(z) \not\equiv 0$. Moreover, let $\mu := \deg q(z)$, and let q denote the highest coefficient of $q(z)$. If $\mu\, d_\mu \not\equiv \arg q$ modulo 2π, then there exists a unique l-ee $y(z)$ in the multi-direction d such that (10) holds, and $y(z)$ is of the same degree as $x(z)$, but may be rewritten so that it is of type $(q(z), \lambda + \mu)$.*

(b) *Assume that $q(z) \equiv 0$ but $-\lambda \notin \mathbb{N}_0$. Then there exists a unique l-ee $y(z)$ in the multi-direction d such that (10) holds, and $y(z)$ is of the same type and degree as $x(z)$.*

(c) *Assume that $q(z) \equiv 0$ and $k_0 := -\lambda \in \mathbb{N}_0$. Then there exists a l-ee $y(z)$ in the multi-direction d such that (10) holds. The $y(z)$ is of the same type as $x(z)$, but in general of degree $m+1$, and $y(z)$ is unique up to an additive constant term.*

Proof. For the question of uniqueness of $y(z)$, compare Remark 3.2. Note that in all three cases, if such a $y(z)$ exists, it is of the form

$$y(z) = z^\lambda e^{q(z)} \sum_{j=0}^{m+1} g_j(z) (\log z)^j,$$

allowing that $g_{m+1}(z)$ may be identically zero. Inserting into (10) and comparing coefficients of powers of $\log z$, we obtain

$$\Big(z q'(z) + \lambda + z\, \frac{d}{dz} \Big) g_{m+1}(z) = 0,$$

while for the remaining terms we have

$$\Big(z q'(z) + \lambda + z\, \frac{d}{dz} \Big) g_j(z) = f_j(z) - (j+1) g_{j+1}(z), \quad 0 \leq j \leq m.$$

Since we want each $g_j(z)$ to be the multi-sum of some power series $\hat{g}_j(z) = \sum_k g_k^{(j)} z^k$, these series must be formal solutions of the above equations, with $f_j(z)$ replaced by $\hat{f}_j(z) = \sum_k f_k^{(j)} z^k$. In case (a), let $\mu (\leq r)$ and q be as above. Then the first equation does not have a non-trivial formal power series solution, while the other ones hold if and only if the coefficients $g_k^{(j)}$ satisfy identities of the form

$$q g_k^{(j)} + \ldots = f_{k-\mu}^{(j)} - (j+1) g_{k-\mu}^{(j)} \qquad \forall\, k \geq 0,$$

with the right hand side vanishing for $k < \mu$. These identities determine the coefficients $g_k^{(j)}$ uniquely, and in particular $g_0^{(j)} = \ldots = g_{\mu-1}^{(j)} = 0$ for every $j = 0, \ldots, m$. Therefore, we see that $y(z)$ indeed may be rewritten as a l-ee of type $(q(z), \lambda + \mu)$. In case (b), we also see $\hat{g}_{m+1}(z) = 0$, and the recursion formula for $j = 0, \dots, m$ simplifies to

$$(\lambda + k) g_k^{(j)} = f_k^{(j)} - (j+1) g_k^{(j)} \qquad \forall\, k \ge 0, \tag{11}$$

determining all the coefficients, since $\lambda + k$ never vanishes. In the last case, the first equation holds if and only if $g_{m+1}(z) = z^{k_0} c$, with arbitrary $c \in \mathbb{C}$. Inserting into the identities (11) which still hold in this case, we can select c such that for $j = m$ and $k = k_0$ the right hand side vanishes, and then the coefficients $g_k^{(m)}$ are uniquely determined except for $g_{k_0}^{(m)}$ which may be chosen arbitrarily. Proceeding to $j = m - 1$ and further, the same arguments apply, showing existence of the formal series $\hat{g}_j(z)$ plus the fact that they are uniquely determined except for $j = 0$ for which $g_{k_0}^{(0)}$ may have any value. Thus, formal power series solutions always exist, and the proof can be completed using results from the theory of multi-summation as in [5, p. 181] to ensure the summability of these formal solutions. ∎

Remark 3.4. We emphasize that in all cases except the last, there is *only one way* to choose a solution of (10) so that the function $y(z)$ is again a l-ee. The exceptional case occurs in what is occasionally named the *resonant case*. In this case we shall always *choose the integral so that we do not have an "unnecessary integration constant"*, or in other words: We shall choose the coefficient of the power z^{k_0} in $g_0(z)$ to be equal to zero. For this unique solution of (10) we shall in all cases use the notation

$$y(z) = \int x(z) z^{-1}\, dz.$$

It is not obvious, however, whether our choice of the indefinite integral in the resonant case is the most natural one, but we wish to make *some unique* choice in order to later have a unique definition of sfn-solutions. Also, observe that the condition in case (a) may be understood as follows: Consider the formal l-ee $\hat{x}(z)$ corresponding to $x(z)$, and let $\hat{y}(z)$ denote the (unique) formal solution of (10). Then all singular multi-directions for $\hat{x}(z)$ also are singular for $\hat{y}(z)$. In addition, all d which are non-singular for $\hat{x}(z)$ may become singular for $\hat{y}(z)$ if $\mu\, d_\mu \equiv \arg q$ modulo 2π. Observe that this fact is in accordance with the definition of singular directions given in Section 2 whenever $q(z) = q(z, d) - q_j(z)$ for suitable j.

Remark 3.5. Regarding the condition in part (a) of Proposition 3.3, we observe the following for later reference: The general theory of multi-summability ensures existence of a (small) number $\varepsilon > 0$ such that, for all multi-directions $\tilde{d}$ with $\tilde{d}_j = d_j$ for $j \ne \mu$ and $|\tilde{d}_\mu - d_\mu| < \varepsilon$, the function $x(z)$ is a l-ee in the multi-direction $\tilde{d}$, too. If $\mu\, d_\mu \not\equiv \arg q$ modulo 2π holds, then we may make ε so small that the condition remains valid when d_μ is replaced by $\tilde{d}_\mu$. So in this case, the function $y(z)$ shall be a l-ee in all the multi-directions $\tilde{d}$. On the other hand, if $\mu\, d_\mu \equiv \arg q$ modulo 2π, and ε is small enough, then $\mu\, \tilde{d}_\mu \not\equiv \arg q$ modulo 2π holds for $\tilde{d}_\mu \ne d_\mu$. Hence for all multi-directions $\tilde{d}$ with $\tilde{d}_\mu < d_\mu$, resp. $\tilde{d}_\mu > d_\mu$, we shall have a corresponding l-ee $y_-(z)$, resp. $y_+(z)$, but they may not coincide! This, in a way, is the origin of Stokes' phenomenon for l-ee.

Remark 3.6. For later application, note that the product of two l-ees of possibly different types is another l-ee whose type is *the sum* of the types of the two factors!

4. Semi-formal series in several variables. In this section we shall present a few basics on formal power series in ν variables. One example of such a series already occurred as the right hand side of (1), but since we wish to apply the identities and results derived here to what shall be called semi-formal solutions of (1), we shall use a different notation:

By $\check{x}(z,c)$ we are going to denote a power series in ν variables $c = (c_1, \ldots, c_\nu)$, with vanishing constant term and coefficients that are functions in the variable z. In more detail, such a series shall be written as

$$\check{x}(z,c) = \sum_{|p|\geq 1} x_p(z) c^p = X(z)c + \sum_{|p|\geq 2} x_p(z) c^p, \tag{12}$$

with a multi-index p as in the introduction, and $X(z) = [x_1(z), \ldots, x_\nu(z)]$, $x_j(z) := x_{\mathrm{e}(j)}(z)$, denoting its linear part. The *coefficients* $x_p(z)$ will always be vectors of length ν denoted as $x_p(z) = (x_1^{(p)}(z), \ldots, x_\nu^{(p)}(z))^T$, where each $x_j^{(p)}(z)$ is a function which is holomorphic on $\mathbb{S}_\rho$—the algebra of all such functions shall be denoted by $\mathcal{O}(\mathbb{S}_\rho)$. So in other words, the coefficients are holomorphic functions for $0 < |z| < \rho$ but may have a logarithmic branch point at the origin. Thus we may also write

$$\check{x}(z,c) = (\check{x}_1(z,c), \ldots, \check{x}_\nu(z,c))^T, \qquad \check{x}_j(z,c) = \sum_{|p|\geq 1} x_j^{(p)}(z) c^p. \tag{13}$$

The set of all such series shall be denoted by $\mathcal{O}(\mathbb{S}_\rho)_0^\nu[[c]]$, omitting the superscript ν in case of $\nu = 1$. The following special cases for series $\check{x}(z,c)$ shall be important later on:

(a) If the coefficients $x_p(z) =: x_p$ do not depend upon z for each multi-index p, we shall denote the series by $\check{x}(c)$, speaking of a *constant series*. The set of all constant series shall be denoted as $\mathbb{C}_0^\nu[[c]]$.
(b) If the coefficients $x_p(z)$ all are holomorphic (and single-valued) at the origin, we shall call $\check{x}(z,c)$ a *holomorphic series*. For the set of all holomorphic series we shall write $\mathcal{O}(\mathbb{D}_\rho)_0^\nu[[c]]$. Observe that the right hand side of (1) is such a series, but with the variables denoted by $x_1, \ldots, x_\nu$. In short hand notation, this may be expressed by writing $g(z,x) \in \mathcal{O}(\mathbb{D}_\rho)_0^\nu[[x]]$.
(c) As the most important situation, the coefficients $x_p(z)$ shall be l-ee, and in this case we shall speak of a *log-exponential series*.

Occasionally we shall consider more general series, denoted as $\hat{x}(z,c)$, with coefficients $\hat{x}_p(z)$ that, instead of functions, are allowed to be *formal logarithmic-exponential expressions*. Such series shall be called *formal* ones, while the previous kind are named *semi-formal* series to indicate that their coefficients are functions instead of formal expressions. The reader may verify that most of the formulas below hold for formal series as well!

For semi-formal power series of the above form, an addition is always well defined, and $\mathcal{O}(\mathbb{S}_\rho)_0^\nu[[c]]$ is a vector space over $\mathbb{C}$. The space $\mathcal{O}(\mathbb{S}_\rho)$ even is an algebra with unit element $e(z) \equiv 1$. Therefore $\mathcal{O}(\mathbb{S}_\rho)_0[[c]]$ is an algebra, too, but does not have a unit element with respect to multiplication, since it only contains series with vanishing constant term. However, we may define for any $\nu \in \mathbb{N}$ and any ν-dimensional multi-index $q = (q_1, \ldots, q_\nu) \neq 0$

$$\forall\, \check{x}(z,c) \in \mathcal{O}(\mathbb{S}_\rho)_0^\nu[[c]] : \quad \check{x}^q(z,c) = \check{x}_1^{q_1}(z,c) \cdot \ldots \cdot \check{x}_\nu^{q_\nu}(z,c) \in \mathcal{O}(\mathbb{S}_\rho)_0[[c]],$$

setting $\check{x}_j^0(z,c) \equiv 1$. Since we restrict ourselves to series with vanishing constant term, we have

$$\check{x}^q(z,c) = \sum_{|p|\geq|q|} x_{qp}(z)c^p,$$

with (scalar) coefficients $x_{qp}(z)$. One may check that these coefficients satisfy the identities

$$x_{q^{(1)}+q^{(2)},p}(z) = \sum_{\substack{|p^{(j)}|\geq|q^{(j)}|\\ p^{(1)}+p^{(2)}=p}} x_{q^{(1)}p^{(1)}}(z)x_{q^{(2)}p^{(2)}}(z) \tag{14}$$

for all p with $|p| \geq |q^{(1)}| + |q^{(2)}|$, where $q^{(j)}$ are multi-indices of length $|q^{(j)}| \geq 1$. This formula allows for recursive computation of $x_{qp}(z)$ with $|q| \geq 2$, while for $q = \mathrm{e}^{(k)}$ we have $\check{x}^q(z,c) = \check{x}_k(z,c)$, implying that

$$x_{\mathrm{e}^{(k)}p}(z) = x_k^{(p)}(z) \qquad \forall\, p \quad \forall\, k = 1,\dots,\nu.$$

As in [2, 3], we can define a *formal composition*: Let $\check{x}(z,c), \check{g}(z,c) \in \mathcal{O}(\mathbb{S}_\rho)_0^\nu[[c]]$ be given, and let $G(z)$ denote the linear part of $\check{g}(z,c)$. Then

$$\check{g}(z,\check{x}(z,c)) = \sum_{|q|\geq 1} g_q(z)\check{x}^q(z,c) = \sum_{|p|\geq 1} h_p(z)c^p, \tag{15}$$

with coefficients $h_p(z)$ which are obtained, through insertion of the expansion for $\check{x}^q(z,c)$ and a formal interchange of the order of summation, as

$$h_p(z) = G(z)x_p(z) + \sum_{\substack{q\\ 2\leq|q|\leq|p|}} x_{qp}(z)g_q(z) \qquad \forall\, p \in \mathbb{N}_0^\nu, \tag{16}$$

which clearly is a finite sum. In particular we conclude that the linear part of the formal composition equals the product, in this order, of the linear parts of $\check{g}(c)$ and $\check{x}(c)$.

Obviously, the formal power series with linear part I and $x_p = 0$ whenever $|p| \geq 2$ acts as an identity element with respect to formal composition. We say that a formal series $\check{x}(z,c)$ is *invertible* with respect to formal composition, provided that another series $\check{y}(z,c)$ exists so that $\check{y}(z,\check{x}(z,c))$ equals the identity element. The following lemma can be easily verified and is in parts analogous to [3, Lemma 1], so we here omit its proof. Note, however, that $\mathcal{O}(\mathbb{S}_\rho)_0^\nu[[c]]$ with respect to addition and formal composition is not a ring but only a near-ring, meaning that one of the two distributional laws fails. This fact shall not be of importance in this article and is included for the sake of completeness only!

LEMMA 4.1. *With the notation introduced above, the following holds*:

(a) $\mathcal{O}(\mathbb{S}_\rho)_0^\nu[[c]]$, *with respect to termwise addition and formal composition, is a right near-ring with unit element, and addition is commutative.*
(b) *A formal series $\check{x}(z,c) \in \mathcal{O}(\mathbb{S}_\rho)_0^\nu[[c]]$ is invertible with respect to formal composition if and only if its linear part $X(z)$ is invertible. This, in turn, is so if and only if* $\det X(z) \neq 0$ *for every $z \in \mathbb{S}_\rho$.*
(c) *The set of all invertible formal series is a non-abelian group with respect to formal composition.*

For later reference, we shall write $\mathbb{G}^\nu(\mathcal{O}(\mathbb{S}_\rho), c)$ for the group of all invertible formal series from $\mathcal{O}(\mathbb{S}_\rho)_0^\nu[[c]]$. An important subgroup of this group is $\mathbb{G}^\nu(\mathbb{C}, c)$, i.e. the invertible series whose coefficients are independent of the variable z.

5. Semi-formal solutions. Let a, not necessarily normalized, semi-formal system (1) be given. For every semi-formal series as in (12) we may form the formal composition $\check g(z, \check x(z,c))$, given by the identities (15) and (16). Comparing coefficients in (1), we see that $\check x(z,c)$ formally is a solution if and only if for every multi-index p we have

$$z^{r+1} x_p'(z) = G(z) x_p(z) + \sum_{2\le|q|\le|p|} x_{qp}(z) g_q(z). \tag{17}$$

For $|p| = 1$, the sum on the right of (17) is empty, showing that in this case $x_p(z)$ will be an arbitrary solution vector of the linear system $z^{r+1}x' = G(z)x$. For $2 \le |q| \le |p|$, the recursion equations (14) imply that the functions $x_{qp}(z)$ only depend upon "earlier" coefficients $x_{\tilde q}(z)$ with $|\tilde q| \le |p| - 1$, and therefore (17) is an inhomogeneous linear system from which we wish to compute the coefficient $x_p(z)$, assuming that the earlier ones are known. So we note that the problem of finding a semi-formal solution is intimately related to finding solutions of lower triangular systems—except for the fact that the set of multi-indices is infinite and not well-ordered. Nonetheless we shall be able to develop a theory which is analogous to that of *reduced systems* investigated in [9].

Suppose for the moment that for every p with $|p| \ge 1$ we have *selected* a solution $x_p(z)$ of (17). Then all coefficients $x_p(z)$ are holomorphic on the universal covering of the disc $D'_\rho = \{0 < |z| < \rho\}$, and altogether we obtain *one selected semi-formal solution* $\check x(z,c) \in \mathcal{O}(\mathbb{S}_\rho)_0^\nu[[c]]$. This solution shall be called *complete*, provided that its linear part $X(z)$ is invertible, i.e., is a fundamental solution of the corresponding linear system. In other words, a complete semi-formal solution belongs to the group $\mathbb{G}^\nu\big(\mathcal{O}(\mathbb{S}_\rho), c\big)$. The following lemma may be thought of as justifying the term *complete semi-formal solution*:

LEMMA 5.1. *Let a semi-formal solution $\check x(z,c)$ of* (1) *be given. If we choose any $v(c) \in \mathbb{C}_0^\nu[[c]]$, the formal composition $\check x(z, v(c))$ is again a semi-formal solution of* (1). *If $\check x(z,c)$ even is a complete semi-formal solution of* (1), *then any other semi-formal solution is of the form $\check x(z, v(c))$, with suitable $v(c)$ as above.*

Proof. The series $\check x(z,c)$ is a semi-formal solution of (1) if and only if the two semi-formal series $z^{r+1}\check x'(z,c)$ and $g(z, \check x(z,c))$ are equal, and this equality remains unchanged if we substitute $v(c)$ for c. For $v(c) = \sum_{|p|\ge1} v_p c^p$, with $v_p \in \mathbb{C}^\nu$ and $c \in \mathbb{C}^\nu$, we conclude from results in Section 4 that $v^q(c) = \sum_{|p|\ge|q|} v_{qp} c^p$, with numbers v_{qp} which may be recursively obtained using an identity analogous to (14). Inserting $v(c)$ into (12) and formally interchanging the order of summation, we obtain in analogy to (16) that

$$\check x(z, v(c)) = \sum_{|p|\ge1} \tilde x_p(z) c^p, \quad \tilde x_p(z) = X(z) v_p + \sum_{\substack{q \\ 2\le|q|\le|p|}} v_{qp} x_q(z). \tag{18}$$

Let $\check y(z,c) = \sum y_p(z) c^p$ be any semi-formal solution of (1), then the equation $\check y(z,c) = \check x(z, v(c))$ is equivalent to $y_p(z) = \tilde x_p(z)$ for every $p \ne 0$, and we intend to investigate whether we can choose the vectors v_p so that this is correct. Assume this being done for all multi-indices $p \ne 0$ of a length strictly less than μ, with some given $\mu \in \mathbb{N}$—this assumption is void for $\mu = 1$, since then the set of such p is empty. Then, for any p of length μ, the coefficients $y_p(z)$ and $\tilde{x}_p(z)$ both satisfy the same inhomogeneous linear system. Completeness of $\check{x}(z,c)$ is equivalent with its linear part $X(z)$ being a fundamental solution of the corresponding homogeneous equation. Defining $\tilde{x}_p(z)$ by (18), with arbitrarily chosen v_p, we obtain the general solution of the inhomogeneous equation. Consequently, it is possible to choose v_p such that $y_p(z) = \tilde{x}_p(z)$ holds. ∎

As an application of the last lemma, we observe that for any semi-formal solution $\check{x}(z,c)$ its coefficients $x_p(z)$ all can be analytically continued from a given point $z \in \mathbb{S}_\rho$ to the corresponding point on the next sheet of the surface—this point being denoted by $z\mathrm{e}^{2\pi i}$. Thus, the series $\check{x}(z\mathrm{e}^{2\pi i}, c)$ also is a semi-formal solution. If $\check{x}(z,c)$ is complete, we conclude existence of $m(c) \in \mathbb{C}_0^\nu[[c]]$ so that

$$\check{x}(z\mathrm{e}^{2\pi i}, c) = \check{x}(z, m(c)). \tag{19}$$

The formal series $m(c)$ shall be referred to as the *monodromy function* corresponding to $\check{x}(z,c)$. Writing M for the linear part of $m(c)$, we find the monodromy relation $X(z\mathrm{e}^{2\pi i}) = X(z)M$ for the linear part of $x(z,c)$. So we see that the matrix M is what in the linear theory usually is called the *monodromy factor* for $X(z)$, while the name *monodromy matrix* is used for a different entity.

One particular type of semi-formal complete solutions is as follows: For some $z_0 \in \mathbb{S}_\rho$ we choose the (unique) fundamental solution $X(z)$ of the linear system corresponding to (1) which satisfies the restriction $X(z_0) = I$. Then, we define for every multi-index p with $|p| \geq 2$

$$x_p(z) = X(z) \int_{z_0}^{z} X^{-1}(w) \Big[\sum_{2 \leq |q| \leq |p|} x_{qp}(w) g_q(w) \Big] \frac{dw}{w^{r+1}}\,. \tag{20}$$

These functions clearly satisfy (17), and $x_p(z_0) = 0$ for every such p. Hence for the corresponding semi-formal solution $\check{x}(z,c)$ we find (formally)

$$\check{x}(z_0, c) = c.$$

Thus we consider this semi-formal solution as *formally solving initial value problems at the point* z_0. As a convenient notation for this kind of solution we choose to write $\check{x}(z,c;z_0) = \sum x_p(z;z_0)c^p$. If the system (1) is convergent, then it is well known that the (unique) solution satisfying the initial condition $x(z_0) = c$, with given $c \in \mathbb{C}^\nu$, is a holomorphic function of the initial data, for z in a disc about z_0 of sufficiently small radius. So in this case, we even have that $\check{x}(z,c;z_0)$, for every fixed z close to z_0, converges in some polydisc about the origin!

For $\check{x}(z,c;z_0)$ as above, let $m(c;z_0)$ denote the corresponding monodromy function. Writing $X(z;z_0)$, resp. $M(z_0)$, for the linear part of $\check{x}(z,c;z_0)$, resp. $m(c;z_0)$, and observing that $X(z_0;z_0) = I$, $x_p(z_0;z_0) = 0$ for all p of length ≥ 2, we obtain $M(z_0) = X(z_0\mathrm{e}^{2\pi i};z_0)$, while for all p of length ≥ 2 we have

$$m_p(z_0) = M(z_0) \int_{z_0}^{z_0\mathrm{e}^{2\pi i}} X^{-1}(w;z_0) \Big[\sum_{2 \leq |q| \leq |p|} x_{qp}(w;z_0) g_q(w) \Big] \frac{dw}{w^{r+1}}\,. \tag{21}$$

While semi-formal solutions $\check{x}(z, c; z_0)$ occur naturally when studying (1) near a non-singular point z_0, the type we introduce next is of much greater interest when investigating the behaviour of solutions as z approaches the origin in some given sector: Let us consider a normalized system (1). The formal power series part $\hat{F}(z)$ of the fundamental solution $\hat{X}(z)$ is known to be multi-summable in every non-singular multi-direction d of order 1, and we denote its sum by $F(z, d)$. The matrix $X(z, d) = F(z, d)z^L e^{Q(z)}$ then is a fundamental solution of the linear system corresponding to (1). In [8], so-called *normal solutions* have been introduced via a restriction of the form of their Stokes multipliers, and in [6] they have been shown to be equal to some of the matrices $X(z, d)$. In [1] other families of normal solutions have been investigated, all of which coincide with the matrices $X(z, d)$ for particular choices of d. *Here, we shall use the term normal solution for* all *the matrices $X(z, d)$, with arbitrary non-singular multi-directions d of order* 1, *and we shall now introduce their non-linear analogues.*

THEOREM 5.2. *Let a normalized system* (1) *be given. For every non-singular multi-direction d there exists a semi-formal solution*

$$\check{x}(z, c; d) = \sum_{|p|\geq 1} x_p(z; d) c^p$$

whose linear part is the normal solution $X(z, d)$ of the corresponding linear equation. The coefficients $x_p(z; d)$ all are l-ees of type $(q(z, p), \lambda(p))$ which are recursively obtained by means of the identity

$$x_p(z; d) = X(z, d) \int X^{-1}(z, d) \Big[\sum_{2\leq |q| \leq |p|} x_{qp}(z; d) g_q(z) \Big] \frac{dz}{z^{r+1}} , \tag{22}$$

and they are uniquely defined if we choose the indefinite integral in the resonant case according to the rule explained in Remark 3.4.

Proof. In the terminology introduced in Section 3, the entries in the jth column of $X(z, d)$ are l-ees of type $(q_j(z), \lambda_j) = (q(z, e^{(j)}), \lambda(e^{(j)}))$. Assume for some μ that we have chosen $x_{\tilde{p}}(z)$ for all multi-indices $\tilde{p}$ of length strictly less than μ, so that they are such expressions of corresponding types $(q(z, \tilde{p}), \lambda(\tilde{p}))$. Then we conclude from (14), using Remark 3.6, that for any multi-index p of length $|p| = \mu$ the integrand on the right hand side of (22) is a vector whose jth component is a l-ee $x_j(z)$ of type $(q(z, p) - q_j(z), \lambda(p) - \lambda_j)$. Hence, we obtain from Proposition 3.3 existence of a corresponding l-ee $y_j(z)$, which in all three cases may be regarded as being of the same type, so that (10) holds. Combining these $y_j(z)$ into a vector and multiplying with $X(z, d)$ then implies that $x_p(z, d)$ is a l-ee of type $(q(z, p), \lambda(p))$. This vector either is unique anyway, or there is one choice which we prefer since it does not introduce unnecessary free constants in the asymptotic expansion of $x_p(z, d)$. ■

REMARK 5.3. The semi-formal solution $\check{x}(z, c; d)$, whose existence and uniqueness was shown above, will be called the *semi-formal normal solution*, or for short: the *sfn-solution*, of (1) in the multi-direction d. The coefficients $x_p(z)$ are l-ees of type $(q(z, p), \lambda(p))$. Therefore, in view of (5), we can also write $\check{x}(z, c; d)$ in the form (observe that we follow the convention and write $\Lambda - r$ instead of $\Lambda - rI$)

$$\breve{x}(z,c;d) = z^r \sum_{|p|\ge 1} \ell_p(z;d)\big(\mathrm{e}^{Q(z)} z^{\Lambda - r} c\big)^p, \tag{23}$$

with logarithmic expressions $\ell_p(z)$, i.e. l-ee of type $(0,0)$. Written in this form, $x(z,c;d)$ indeed is what is (unfortunately) called a *transseries*. So, roughly speaking the sfn-solution is a formal power series, not in the variables c_j, but in $\mathrm{e}^{q_j(z)} z^{\lambda_j - r} c_j$, with $1 \le j \le \nu$, and coefficients that are logarithmic expressions.

6. Connection problems. For any two non-singular multi-directions d and $\tilde{d}$, there is a unique invertible matrix $V(\tilde{d},d)$ such that $X(z,d) = X(z,\tilde{d})V(\tilde{d},d)$, and we shall refer to this matrix as the corresponding *Stokes multiplier*. This notion here is used in a wider sense than in other articles, since the two multi-directions can be completely arbitrary, while in the linear theory one usually restricts them so that the matrix has as few non-trivial elements as possible.

Since sfn-solutions are always complete, we conclude from Lemma 5.1:

- Given two non-singular multi-directions d and $\tilde{d}$, there exists a unique formal expression $\breve{v}(c;\tilde{d},d) = \sum_p v_p(\tilde{d},d)c^p$ for which
$$\breve{x}(z,c;d) = \breve{x}(z,\breve{v}(c;\tilde{d},d);\tilde{d}).$$
This $\breve{v}(c;\tilde{d},d)$ will be referred to as the (formal) *Stokes series* corresponding to d and $\tilde{d}$.

As a simple consequence from the definition of normal solutions and Stokes series, we obtain the following result:

Theorem 6.1. *For any normalized system* (1)*, all Stokes series are invertible with respect to formal composition, and for any three non-singular multi-directions d, $\tilde{d}$, $\hat{d}$ we have $\breve{v}(c;d,d) = c$, i.e. is the identity element of $\mathbb{G}^\nu(\mathbb{C},c)$, while $\breve{v}(c;\tilde{d},d)$ and $\breve{v}(c;d,\tilde{d})$ are inverse to one another. Moreover,*

$$\breve{v}(c;\hat{d},d) = \breve{v}(\breve{v}(c;\tilde{d},d);\hat{d},\tilde{d}).$$

Altogether, this shows that the Stokes series form a subgroup of the group $\mathbb{G}^\nu(\mathbb{C},c)$.

Proof. Follows directly from the definitions with help of Lemma 4.1. ∎

Given a normalized system (1), the set of corresponding Stokes series, which is a subgroup of $\mathbb{G}^\nu(\mathbb{C},c)$ according to the last theorem, shall be named *the Stokes group* corresponding to (1). For an analysis of the structure of $\breve{v}(c;\tilde{d},d)$ for the case of two singular directions that are *close to another*, refer to Theorem 6.3.

While the determination of the Stokes multipliers in the linear theory occasionally is referred to as the *lateral connection problem*, one speaks of a *central connection problem* to indicate that one wishes to link an arbitrary fundamental solution to any one of the normal ones. Correspondingly, let us consider the semi-formal solutions corresponding to an initial value problem, denoted as $\breve{x}(z,c;z_0)$. Given any non-singular multi-direction d, we obtain existence (and uniqueness) of $\breve{\omega}(c;d,z_0) \in \mathbb{G}^\nu(\mathbb{C},c)$ with

$$\breve{x}(z,c;z_0) = \breve{x}(z,\breve{\omega}(c;d,z_0);d).$$

We shall name $\check{\omega}(c; d, z_0)$ the *central connection series* in the multi-direction d. It is clear that, for two non-singular multi-directions d and $\tilde{d}$, the central connection series $\check{\omega}(c; d, z_0)$ and $\check{\omega}(c; \tilde{d}, z_0)$ are related via the corresponding Stokes series:

THEOREM 6.2. *For arbitrary $z_0 \in \mathbb{S}$ and any two non-singular multi-directions d and $\tilde{d}$, we have*

$$\check{\omega}(c; \tilde{d}, z_0) = \check{v}(\check{\omega}(c; d, z_0); \tilde{d}, d).$$

Proof. By definition, $\check{x}(z, c; z_0) = \check{x}(z, \check{\omega}(c; d, z_0); d) = \check{x}(z, \check{\omega}(c; \tilde{d}, z_0); \tilde{d})$, and $\check{x}(z, c; d) = \check{x}(z, \check{v}(c; \tilde{d}, d); \tilde{d})$. By substituting in the second identity $\check{\omega}(c; d, z_0)$ for c and comparing with the first one, the statement of the theorem follows. ∎

In the linear theory, as presented in [8], normal solutions are a countable set of fundamental solutions, indexed according to a corresponding enumeration of Stokes rays—these are not the same, but closely related to, what here are called singular directions. In the non-linear case, however, the set of all singular directions is countable, but may be dense in $\mathbb{R}$. Therefore, in general, no enumeration of this set may exist for which the sector bounded by the singular directions number $\nu - 1$ and ν is free of other singular directions. On the other hand, for any fixed multi-index p only singular directions of order $\mu = |p|$ are relevant for the definition of $x_p(z; d)$, and these directions are a discrete set! As a consequence, the dependence of $x_p(z; d)$ on d, or in other words: the structure of the coefficients of the Stokes series $v(c; \tilde{d}, d)$, can be analysed as follows:

THEOREM 6.3. *Given $k \in \{1, \ldots, r\}$ and $\mu \in \mathbb{N}$, let d and $\tilde{d}$ be two multi-directions whose coordinates d_j and $\tilde{d}_j$ are non-singular of order μ, and so that*

$$d_j = \tilde{d}_j \quad \forall\, j \neq k, \qquad d_k < \tilde{d}_k.$$

Without loss of generality, also assume that the open interval $(d_k, \tilde{d}_k)$ is small, and so that it contains exactly one singular direction $\hat{d}_k$ of level (k, μ). Then, for every p with $|p| = \mu$, a non-zero element in the vector $v_p(\tilde{d}, d)$ can only occur in a position j with $\deg(q(z, p) - q_j(z)) = k$ and $k\hat{d}_k \equiv \arg q$ modulo 2π, where $q = q(p, j)$ denotes the highest coefficient of $q(z, p) - q_j(z)$.

Proof. In view of (22), we conclude from Proposition 3.3 for all values $\delta \in (d_k, \tilde{d}_k)$ with $\delta \neq \hat{d}_k$ that

$$x_q(z; d(\delta)) = \begin{cases} x_q(z; d) & (d_k < \delta < \hat{d}_k) \\ x_q(z; \tilde{d}) & (\hat{d}_k < \delta < \tilde{d}_k) \end{cases} \qquad \forall\, q \text{ with } |q| \leq \mu.$$

On the other hand, for a given p with $|p| = \mu$, the functions $x_p(z; d)$ and $x_p(z; \tilde{d})$ will be different, in general: On one hand, it can occur that $\hat{d}_k$ even is a singular direction of order $\mu - 1$, then the integrand on the right hand side of (22) shall, in general, change when d is replaced by $\tilde{d}$. Since this integrand only involves terms $x_{\tilde{p}}(z; d)$ with $|\tilde{p}| < \mu$, the functions $x_{qp}(z; \tilde{d})$ can be expressed as combinations of the functions in the multi-direction d, with help of the coefficients $v_q(\tilde{d}, d)$ for multi-indices q of length at most $\mu - 1$. On the other hand, $\hat{d}_k$ may not be singular of level $\mu - 1$, and then $x_{qp}(z; \tilde{d}) = x_{qp}(z; d)$ follows. Therefore, in both cases we may say that for the integrand on the right hand side of (22) the transition from d to $\tilde{d}$ can be made in terms of functions, resp. coefficients, of lower order. Consequently, a non-zero element in $v_p(\tilde{d}, d)$ can only result from the fact that we may need to change the interpretation of the integral in (22) when switching from d to $\tilde{d}$. According to Remark 3.5, this occurs only in a position j with $\deg q(z,p) - q_j(z) = k$ and $k\hat{d}_k \equiv \arg q$ modulo 2π, where q denotes the highest coefficient of $q(z,p) - q_j(z)$. In particular, if no such j exists, then $v_p(\tilde{d}, d) = 0$. ∎

Observe that the last theorem in the linear case corresponds to the conditions on the support of the Stokes multipliers given in [8, 22].

7. Convergence of sfn-solutions. In this final section we very informally address the question of convergence of sfn-solutions, which is not the main point of this paper but of great importance nonetheless:

It is a well known fact that the terms of a sfn-solution can grow so rapidly that the series fails to converge for all values of $z \in \mathbb{S}_\rho$. The reason for this lies in what is called the *small denominator phenomenon*, and to avoid this, some extra conditions on the leading terms of the polynomials $q_j(z)$ have been used. From the list of articles and books dealing with these and/or related questions we only mention a few relatively recent ones by Ovidiu Costin [15, 16, 17, 18], and Braaksma and Stolovitch [13]. Boele Braaksma [12] studied the corresponding situation for difference equations.

In the author's opinion, the convergence of a sfn-solution *in toto* is not a very natural question: Let a non-singular multi-direction $d = (d_r, \ldots, d_1)$ be given. Then the coefficients of $\check{x}(z,c;d)$ are holomorphic in a sector S_d with bisecting direction d_r and opening larger than π/r, and their behaviour as $z \to 0$ in S is known. So the series $\check{x}(z,c;d)$ is constructed to represent a family of solutions that have a clear behaviour as the variable z approaches the origin in S_d. For this reason, it does not help if the series only converges at such points in S_d that keep a distance from the origin. So instead it is natural to require convergence of the series, if possible, for all z in a sectorial region $G \subset S_d$, i.e. a region that has the origin as a boundary point. As we shall illustrate by examples, this in general cannot occur for all values of c in some polydisc about the origin of $\mathbb{C}^\nu$, even when its radii are small! Therefore, it is more natural to investigate *partial convergence* in the following sense:

- Given a subset $J \subset \{1, \ldots, \nu\}$, we say that $\check{x}(z,c;d)$ is partially convergent of type J in a sectorial region $G \subset S_d$ (for the definition of sectorial regions, compare [5]—here it suffices to say that they always have the origin as a boundary point), provided that for every closed subsector $\overline{S}$ of G there exists an $r > 0$ so that $\check{x}(z,c;d)$ converges uniformly (in z) for $z \in \overline{S}$, for every $c = (c_1, \ldots, c_\nu)^T$ with

$$c_j = 0 \quad (j \notin J), \qquad |c_j| < r \quad (j \in J).$$

As an example motivating the above definition, we consider a very simple *decoupled system* of the form

$$z^{r+1} x_j' = a_j(z)(x_j + x_j^2), \qquad 1 \le j \le \nu,$$

with polynomials $a_j(z)$ of degree at most r. One can verify that this system has the (complete) solution $x(z,c)=(x_1(z,c),\dots,x_\nu(z,c))^T$ with

$$x_j(z,c)=\frac{c_j e^{q_j(z)}z^{\lambda_j}}{1-c_j e^{q_j(z)}z^{\lambda_j}}\,,\quad z^r(\lambda_j+zq_j'(z))=a_j(z),\qquad 1\le j\le\nu.$$

Each $x_j(z,c)$ can be expanded into a (convergent) power series in the variable c_j, converging if and only if

$$\left|c_j e^{q_j(z)}z^{\lambda_j}\right|<1\qquad(1\le j\le\nu). \tag{24}$$

For every non-singular multi-direction d, this is the sfn-solution of our system, which in this decoupled case is independent of d. In order to have partial convergence in some sectorial region G, in the sense defined above, we have to choose the type J so that in this region the inequalities (24) hold for sufficiently small values of $|c_j|$. This is the case if and only if we choose J so that for each $j\in J$ all $e^{q_j(z)}$ remain bounded as $z\to 0$ in G, and in case that $q_j(z)$ vanishes identically, we have that $\operatorname{Re}\lambda_j\ge 0$.

The above example is very simple, but still depicts the general situation to a large degree: a general sfn-solution can always be written in the form (23). Since the coefficients $\ell_p(z;d)$ are of *moderate behaviour* at the origin (when restricting to a corresponding sector S_d), the exponential polynomials $q(z;p)$ will dominate the behaviour of the terms of the series. Therefore, in order to have partial convergence in a sectorial region G, it will again be necessary to choose J so that no exponential polynomials remain that grow as $z\to 0$ in G. However, observe that in general one may be able to relax the condition on the numbers λ_j, since a possible growth of the term $z^{\lambda(p)}$ may be compensated by the corresponding coefficient $\ell_p(z;d)$. Nonetheless, note that in particular when all polynomials vanish identically (which is an excellent situation in the linear theory, since then no divergent power series enter the formal solution), it may occur that $\check{x}(z,c;d)$ is not partially convergent for any J but the empty set—this can also be observed from the example given above! On the other hand, if all polynomials are non-zero, and if we select J accordingly, then the corresponding series is convergent.

Note that if $\check{x}(z,c;d)$ is partially convergent in a sectorial region G, then it represents a family of solutions that are bounded at the origin (when approaching inside of G). Hence, a natural question is whether this family includes *all* bounded solutions. Unfortunately, the answer is negative: In the simple example of a decoupled system, assume that for some j_0 we have $q_{j_0}(z)\equiv 0$ and $\operatorname{Re}\lambda_{j_0}<0$. If we take $c_j=0$ for $j\ne j_0$, the corresponding sfn-solution is bounded at the origin, but the series is convergent *outside* of a disc around the origin. Therefore, $\check{x}(z,c;d)$ is not partially convergent for any J containing j_0, hence we cannot represent all bounded solutions by this series!

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