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ON THE THEOREM OF GÉZA GRÜNWALD AND JÓZEF MARCINKIEWICZ

LÁSZLÓ SZILI

Department of Numerical Analysis, Eötvös Loránd University
Budapest, Pázmány P. sétány 1/C, H–1117, Hungary
E-mail: szili@ludens.elte.hu

PÉTER VÉRTESI

Alfréd Rényi Mathematical Institute of the Hungarian Academy of Sciences
Budapest, Reáltanoda u. 13–15, H–1053, Hungary
E-mail: vertesi@renyi.hu or *veter@renyi.hu*

Abstract. This survey is a tribute to Géza Grünwald and Józef Marcinkiewicz dealing with the so called Grünwald–Marcinkiewicz Theorem.

1. Preface. In 1910, exactly hundred years ago, two outstanding mathematicians were born: (in alphabetical order) the Hungarian Géza Grünwald and the Polish Józef Marcinkiewicz. But we must note some other similarities. They proved in the same year (1935) the first version of their famous result (as today called) the Grünwald–Marcinkiewicz Theorem (G–M Theorem, for short). Both improved this first version and got the final form already in the next year; the theorem was a part of their PhD dissertations. And, finally, both were killed during the second world war: Géza Grünwald became a holocaust victim in 1942 while Józef Marcinkiewicz died in 1940 in the Katyn massacre.

Paraphrasing what Antoni Zygmund (who was a teacher, a friend and a collaborator of Józef Marcinkiewicz) wrote, we may say that the short period of their mathematical activity left a definite imprint on mathematics. Considering what they might have done one may view their early death as a great blow to mathematics (cf. [21, p. 1]).

In this survey which is a tribute to their memories, we deal with the above mentioned Grünwald–Marcinkiewicz Theorem. First we consider its preliminaries (Part 2), then the theorem, some parts of the proof (Part 3), finally we see some related results, developments and problems in Part 4.

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For the interested reader we may suggest three survey papers and their references written by A. Zygmund [21], Pál Turán [17] and P. Vértesi [20].

2. Preliminaries. Interpolation. Lagrange interpolation. Lebesgue function. Lebesgue constant. Optimal Lebesgue constant. Divergence of interpolation. What is interpolation? "Perhaps it would be interesting to dig to the roots of the theory and to indicate its historical origin. Newton, who wanted to draw conclusions from the observed location of comets at equidistant times as to their location at arbitrary times arrived at the problem of determining a 'geometric' curve passing through arbitrarily many given points. He solved this problem by the interpolation polynomial bearing his name" (P. Turán [18]).

2.1. Let us begin with some definitions and notation. Let $C = C(I)$ denote the space of continuous functions on the interval $I := [-1, 1]$, and let $\mathcal{P}_n$ denote the set of algebraic polynomials of degree at most n. $\|\cdot\|$ stands for the usual maximum norm on C. Let X be an *interpolatory matrix* (*array*), i.e.,

$$X = \big\{x_{kn} = \cos\vartheta_{kn};\ k = 1, \dots, n;\ n = 1, 2, \dots\big\},$$

with

$$-1 \le x_{nn} < x_{n-1,n} < \dots < x_{2n} < x_{1n} \le 1 \tag{1}$$

and $0 \le \vartheta_{kn} \le \pi$, and consider the corresponding *Lagrange interpolation polynomial*

$$L_n(f, X, x) := \sum_{k=1}^{n} f(x_{kn})\ell_{kn}(X, x), \quad n \in \mathbf{N}. \tag{2}$$

Here, for $n \in \mathbf{N}$,

$$\ell_{kn}(X, x) := \frac{\omega_n(X, x)}{\omega_n'(X, x_{kn})(x - x_{kn})}, \quad 1 \le k \le n,$$

with

$$\omega_n(X, x) := \prod_{k=1}^{n}(x - x_{kn}),$$

are polynomials of exact degree $n - 1$. They are called the *fundamental polynomials* associated with the *nodes* $\{x_{kn} : k = 1, \dots, n\}$.

The main question is for what choices of the interpolation array X we can expect that (uniformly, pointwise, etc.) $L_n(f, X) \to f \quad (n \to \infty)$.

Since, by the Chebyshev alternation theorem, the best uniform approximation $P_n(f)$ to $f \in C$ from $\mathcal{P}_n$ interpolates f in at least $n + 1$ points, there exists, for each $f \in C$, an interpolation matrix Y for which

$$\|L_{n+1}(f, Y) - f\| = E_n(f) := \min_{P \in \mathcal{P}_n} \|f - P\|$$

goes to 0 as $n \to \infty$. However, for the *whole class* C, the situation is different.

To formulate the corresponding negative result we quote some estimates and introduce further definitions.

By the classical Lebesgue estimate,

$$\begin{aligned} |L_n(f,X,x)-f(x)| &\le |L_n(f,X,x)-P_{n-1}(f,x)|+|P_{n-1}(f,x)-f(x)| \\ &\le |L_n(f-P_{n-1},X,x)|+E_{n-1}(f) \\ &\le \Big(\sum_{k=1}^{n}|\ell_{k,n}(X,x)|+1\Big)E_{n-1}(f), \end{aligned} \tag{3}$$

therefore, with the notation

$$\lambda_n(X,x) := \sum_{k=1}^{n}|\ell_{kn}(X,x)|, \quad n\in\mathbf{N}, \tag{4}$$

$$\Lambda_n(X) := \|\lambda_n(X,x)\|, \quad n\in\mathbf{N}, \tag{5}$$

(*Lebesgue function* and *Lebesgue constant* (of Lagrange interpolation), respectively) we have for $n\in\mathbf{N}$

$$|L_n(f,X,x)-f(x)| \le \{\lambda_n(X,x)+1\}E_{n-1}(f) \tag{6}$$

and

$$\|L_n(f,X)-f\| \le \{\Lambda_n(X)+1\}E_{n-1}(f). \tag{7}$$

"After ... the approximation theorem of Karl Weierstrass, it was hoped that there exists a (non-equidistant) system of nodes for which the Lagrange interpolation polynomials converge uniformly for every function continuous in $[-1,1]$. The mathematical world was awakened from this dream in 1914 by Georg Faber [4] who showed that there is *no* such system." (P. Turán [18, p. 25]).

Namely, he proved the then rather surprising lower bound

$$\Lambda_n(X) \ge \frac{1}{12}\log n, \quad n\ge 1, \tag{8}$$

for any interpolation array X. Based on this result he obtained

THEOREM 2.1. *For any fixed interpolation array X there exists a function $f\in C$ for which*

$$\limsup_{n\to\infty}\|L_n(f,X)\| = \infty. \tag{9}$$

2.2. The previous estimates show clearly the importance of the Lebesgue function $\lambda_n(X,x)$ and the Lebesgue constant $\Lambda_n(X)$. During the last about 100 years, there were proved very general relations concerning their behaviour and applied to obtain divergence theorems for $L_n(f,X)$.

First, we state the counterpart of (8). Namely, using an estimate of L. Fejér

$$\Lambda_n(T) = \frac{2}{\pi}\log n + O(1),$$

one can see that the order $\log n$ in (8) is best possible (here T is the Chebyshev matrix, i.e. $x_{kn} = \cos\frac{2k-1}{2n}\pi)$.

The next statement, the more or less complete pointwise estimation is due to P. Erdős and P. Vértesi from 1981.

THEOREM 2.2. *Let $\varepsilon > 0$ be given. Then, for any fixed interpolation matrix $X \subset [-1,1]$ there exist sets $H_n = H_n(\varepsilon, X)$ of measure $\le \varepsilon$ and a number $\eta = \eta(\varepsilon) > 0$ such that*

$$\lambda_n(X,x) > \eta \log n$$

if $x \in [-1,1] \setminus H_n$ and $n \ge 1$.

2.3. Let us say some words about the *optimal Lebesgue constant*. In 1961, P. Erdős, improving an earlier result of P. Turán and himself, proved that

$$\Big|\Lambda_n^* - \frac{2}{\pi}\log n\Big| \le c,$$

where

$$\Lambda_n^* := \min_{X \subset I} \Lambda_n(X), \quad n \ge 1,$$

is the *optimal Lebesgue constant.* As a consequence of this result, the closer investigation of Λ_n^* attracted the attention of many mathematicians.

In 1978, Ted Kilgore, Carl de Boor and Alan Pinkus proved the so–called Bernstein-Erdős conjectures concerning the optimal interpolation array X.

Using this result, P. Vértesi in 1990 obtained the value of Λ_n^* *within the error* $o(1)$. Namely,

$$\Lambda_n^* = \frac{2}{\pi}\log n + \chi + O\Big(\Big(\frac{\log\log n}{\log n}\Big)^2\Big),$$

where $\chi = \frac{2}{\pi}(\gamma + \log\frac{4}{\pi}) = 0.521251\ldots$ and $\gamma = 0.577215\ldots$ is the Euler constant.

3. The Theorem

3.1. In Part 2, we saw that the Lebesgue function is at least is order $\log n$ on a "big" set (cf. Theorem 2.2). However, for the matrix

$$E = \Big\{x_{kn} = -1 + \frac{k}{n},\ k = 0,1,2,\ldots,2n;\ n = 1,2,3,\ldots\Big\}$$

the above order is much bigger, namely it grows exponentially with n. Using this fact S. Bernstein in 1918 proved as follows.

THEOREM 3.1. *For the function $f_1(x) = |x|$*

$$\limsup_{n\to\infty} |L_n(f_1, E, x)| = \infty \quad \textit{if} \quad x \in (-1,1),\ x \ne 0.$$

But an analogous result for the matrix T, where the order of the Lebesgue function is the smallest possible remained open until 1935.

3.2. The so called Grünwald–Marcinkiewicz Theorem says as follows

THEOREM 3.2 (Grünwald–Marcinkiewicz). *There exists a function $f \in C$ for which*

$$\limsup_{n\to\infty} |L_n(f, T, x)| = \infty$$

for every $x \in [-1,1]$.

3.3. Remarks

A. Józef Marcinkiewicz considered the *trigonometric* interpolation based on the equidistant nodes $\frac{2k\pi}{2n+1}$ but it differs only formally from the Lagrange interpolation based on the Chebyshev matrix $T = \left\{\cos \frac{2k-1}{2n}\pi\right\}$.

B. Originally both Géza Grünwald and Józef Marcinkiewicz proved the existence of $g \in C$ for which the sequence $\{L_n(T, g, x)\}$ diverges *almost everywhere* (see [7], [9]). A. Zygmund writes [21, p. 16]: "It is curious that a year later (in 1936) both authors could, independently of each other, strengthen their examples by constructing continuous functions whose Chebyshev interpolating polynomials diverge *everywhere*." (cf. [8], [11]).

C. Theorem 3.2 is important for two reasons.

First, it shows, dramatically, that Lagrange interpolation polynomials may be very poor approximating tools even for the "very good" matrix T. Actually, this theorem clearly shows the limitations of Lagrange interpolation even on these nodes.

Second, it is well known that there are many similarities between the approximation properties of

(i) the partial sums of the Fourier-Chebyshev expansion of $f \in C$ and
(ii) the interpolating polynomials $L_n(f, T)$.

These similarities can be used with great effect. For example, a result known for (i) will suggest an analogous result for (ii) if we consider continuous functions with bounded variation. However, the analogy is not perfect. The famous result of Carleson implies that if $f \in C$ then the partial sums of the Fourier-Chebyshev expansion of f converge to f a.e. in $[-1, 1]$. On the other hand, Theorem 3.2 shows that this is certainly not the case for the interpolating polynomials $L_n(f, T)$.

3.4. Let say some words *on the proofs*. We quote some parts of the structure given by Józef Marcinkiewicz [11] (cf. I. P. Natanson [13, Chapter 2, §3]).

3.4.1.

LEMMA 3.3. *If* $S_n = \left\{\frac{2k-1}{2n}\pi,\ k = 1, 2, \dots, n\right\}$ *then* $S_n \cap S_{n+1} = \emptyset$. *Moreover, let* $S = \left\{\frac{p}{q}\pi,\ (p, q) = 1,\ \text{otherwise arbitrary}\right\}$. *If* $Q \subset S$ *and* Q *is finite, then there exist arbitrarily large values of* n *for which*

$$(S_n \cap Q) \setminus \{0\} = (S_{n+1} \cap Q) \setminus \{0\} = \emptyset.$$

This simple and (at first sight) innocently-looking statement is fundamental. Whenever one may prove a similar relation for a pointsystem, there is a good chance to get a G–M type *everywhere* divergence theorem.

3.4.2. Another observation is a slight generalization of the Weierstrass theorem. Namely

LEMMA 3.4. *If* $\varphi \in C$ *and* $\{t_1, t_2, \dots, t_s\}$ *are* s *fixed distinct points in* $[-1, 1]$, *then there is a polynomial* R *with*

$$\begin{cases} R(t_k) = \varphi(t_k), & 1 \le k \le s \\ \|R - \varphi\| \le 2. \end{cases}$$

Notice that the lemma does not say anything on the degree of R!

3.4.3. Here is another fundamental statement.

Lemma 3.5. *If $p > 2$ is an integer, then there exists a polynomial $R_p(x)$ with $\|R_p\| \le 2$ but for arbitrary $x \in \left[-\cos\frac{\pi}{p}, \cos\frac{\pi}{p}\right] = I_p$ one can find a $u = u(x) > p$ such that*

$$|L_u(R_p, T, x)| > p.$$

The final part of the proof of this lemma strongly uses another simple observation. Namely if $x = \cos\vartheta \in I_p$, then

$$\frac{2}{p} \le \sin\vartheta = \sin(n+1)\vartheta\cos n\vartheta - \sin n\vartheta\cos(n+1)\vartheta \le |\cos n\vartheta| + |\cos(n+1)\vartheta|,$$

or in other words,

$$\max(|T_n(x)|, |T_{n+1}(x)|) \ge \frac{1}{p};$$

this bound will be used at the lower estimation of $\max(\lambda_n(T,x), \lambda_{n+1}(T,x))$.

3.4.4. The other parts, lemmas and their combinations are left to the interested reader; we emphasize that the proof of Géza Grünwald is different; however, of course, his steps solve problems analogous to the above ones (cf. [8]).

4. Some related results, developments and problems

4.1. As the Part 3.3.C shows the similarities between the partial sums of Fourier–Chebyshev expansion of $f \in C$ and $L_n(f,T)$ interpolating polynomials are quite limited. Another dramatic example of this distinction is given by the next statements which is a reformulation of a statement of Józef Marcinkiewicz [10]:

For any fixed $x \in [-1,1]$ there exists a function $f \in C$ for which

$$\limsup_{n\to\infty} \left|\frac{1}{n}\sum_{k=1}^{n} L_k(f,T,x)\right| = \infty.$$

In other words, the arithmetic means of Lagrange interpolating polynomials of a continuous function can diverge at any given point. This is in marked contrast to the celebrated theorem of L. Fejér about the arithmetic means of the partial sums of Fourier series [5].

Remark. Many-many years later, in 1991, P. Erdős and G. Halász [1] proved as follows.

Given a positive sequence $\{\varepsilon_n\}$ converging to zero however slowly, one can construct a function $f \in C$ such that for almost all $x \in [-1,1]$

$$\frac{1}{n}\left|\sum_{k=1}^{n} L_k(f,T,x)\right| \ge \varepsilon_n \log\log n$$

for infinitely many n.

4.2. While it is quite straightforward to obtain the analogue of the Grünwald–Marcinkiewicz Theorem for the interpolation based on the roots of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ (cf. [16, Chapter 4]) whenever $|\alpha| = |\beta| = \frac{1}{2}$, the general case, i.e. when $\alpha, \beta > -1$, arbitrary, was settled only in 1976 by A.A. Privalov [15] but only the "almost everywhere" version. Namely, he proved

THEOREM 4.1. *If $\alpha, \beta > -1$ are fixed, then there is an $f \in C$ with*

$$\limsup_{n\to\infty} \left|L_n(f, X^{(\alpha,\beta)}, x)\right| = \infty \text{ a.e. in } [-1, 1].$$

(*Above,* $L_n(f, X^{(\alpha,\beta)})$ *is the corresponding Lagrange interpolatory polynomial.*)

One of the main difficulties is that we *do not know* the exact place of the nodes $\{x_{kn}^{(\alpha,\beta)}\}$, i.e. *we do not have the complete analogue* of Lemma 3.3.

4.3. For an arbitrary point system it was proved by P. Erdős and P. Vértesi [3] in 1981.

THEOREM 4.2. *Let $X \subset [-1, 1]$ be any point group [interpolatory array]. Then there exists a continuous function $f(x)$ so that for almost all x*

$$\limsup_{n\to\infty} |L_n(f, X, x)| = \infty.$$

It is easy to see that (generally) "a.e." cannot be replaced by "everywhere".

4.4. For higher order Hermite–Fejér interpolation one can also get the corresponding G–M theorem. First we give the corresponding definitions.

Let $m \in \mathbf{N}$ and let $f \in C$. Then, for each $n \in \mathbf{N}$ there is a unique polynomial $H_{mn}(f, T, x)$ such that

- $\deg H_{mn}(f, T, x) \le mn - 1$,
- $H_{mn}(f, T, x_{kn}) = f(x_{kn})$ $(k = 1, 2, \ldots, n)$,
- $H_{mn}^{(j)}(f, T, x_{kn}) = 0$ $(k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m - 1)$,

where $H_{mn}^{(j)}(f, T, x)$ denotes the j-th derivative of $H_{mn}(f, T, x)$ with respect to x.

We refer to $H_{mn}(f, T, x)$ as a higher order Hermite-Fejér interpolation polynomial (the order being $mn - 1$) corresponding to the function f and the n-th row of the matrix of nodes T. Recently there has been considerable interest in such polynomials.

In the case $m = 1$ the interpolation polynomials $H_{mn}(f, T, x)$ are merely the Lagrange interpolation polynomials $L_n(f, T, x)$. In the case $m = 2$, the interpolation polynomials $H_{mn}(f, T, x)$ become the well known Hermite–Fejér interpolation polynomials which behave quite differently from Lagrange interpolation polynomials. Namely, as it was proved by L. Fejér (see [6])

$$\lim_{n\to+\infty} \|H_{2n}(f, T, x) - f(x)\| = 0 \text{ for any } f \in C.$$

What happens when $m > 2$? It appears that, in many ways, the behaviour of $H_{mn}(f, T, x)$ is determined by the *parity* of m. If m is *even*, then $H_{mn}(f, T, x)$ behave like $H_{2n}(f, T, x)$ (cf. [19]):

$$\lim_{n\to+\infty} \|H_{mn}(f, T, x) - f(x)\| = 0 \text{ for any } f \in C \ (m = 2, 4, 6, \ldots).$$

However for *odd* m we can prove (cf. T. M. Mills, R. Sakai, P. Vértesi; [12], [19]) as follows.

Let $m = 1, 3, 5, \ldots$ be fixed. Then there exists a function $f \in C$ such that, for all $x \in [-1, 1]$,

$$\limsup_{n\to\infty} |H_{mn}(f, T, x)| = +\infty.$$

That means, we got a G–M type theorem for $H_{mn}(f, T)$ (m is odd).

4.5. In the paper F. Pintér, P. Vértesi [14], the analogue of the G–M theorem was obtained for Lagrange interpolation by entire functions of exponential type.

4.6. Finally we mention some problems.

A. Try to get the G–M theorem for $X^{(\alpha,\beta)}$.

B. Find those arrays X for which G–M theorem holds true.

C. Extend the G–M theorem on infinite intervals using weighted norm.

D. Prove the theorem of Erdős and Halász for every $x \in [-1,1]$ (see 4.1). Prove it for other arrays X.

E. Prove the above problems for higher order Hermite–Fejér interpolation.

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